数理解析研究所講究録 1461 巻 2005 年 70-84

Robust Solution of Stochastic Linear Complementarity Problems

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1 Introduction

The linear complementarity problem (LCP) is to find a vector $x \in R^n$ such that

$$Ax + p \geq 0, \ x \geq 0, \ x^T(Ax+p) = 0,$$

where $A \in R^{n\times n}$ and $p \in R^n$. This problem is generally denoted as LCP(A,p). The LCP has a significant number of applications in engineering and economics [4, 5, 8]. In practice, due to several types of uncertainties such as weather, material, trade, loads, supply, demand, cost, etc., the data in the LCP can only be estimated based on limited information. Suppose that $M(\omega) \in R^n, q(\omega) \in R^n$, for $\omega \in \Omega \subset R^m$, are random quantities on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where the probability distribution $\mathcal{P}$ is known. In order to take the stochastic uncertainty into account appropriately, deterministic formulations of the *stochastic linear complementarity problem* (SLCP)

$$M(\omega)x + q(\omega) \geq 0, \ x \geq 0, \ x^T(M(\omega)x + q(\omega)) = 0, \qquad \omega \in \Omega \tag{1.1}$$

have been studied. In this paper, we consider two existing deterministic formulations. Let us denote

$$y(x,\omega) := M(\omega)x + q(\omega).$$

Let $\phi : R^2 \to R$ be a function, called an *NCP function*, which satisfies

$$\phi(a,b) = 0 \quad \Longleftrightarrow \quad a \geq 0, \ b \geq 0, \ ab = 0. \tag{1.2}$$

Then it is easy to verify that for each $\omega \in \Omega$, x_ω is a solution of (1.1) if and only if it is an optimal solution of the following minimization problem with zero objective value:

$$\min_{x \in R^n_+} \|\Phi(x,\omega)\|^2, \tag{1.3}$$

where $R^n_+ := \{x \in R^n \,|\, x \geq 0\}$ and

$$\Phi(x,\omega) := \begin{pmatrix} \phi(y_1(x,\omega), x_1) \\ \vdots \\ \phi(y_n(x,\omega), x_n) \end{pmatrix}.$$

In the literature of linear complementarity problems, $\|\Phi(x,\omega)\|$ is called a residual for LCP$(M(\omega),q(\omega))$, since x_ω solves LCP$(M(\omega),q(\omega))$ if and only if it solves $\Phi(x,\omega)=0$. On the other hand, from the literature of stochastic optimization, $\|\Phi(x,\omega)\|^2$ can be regarded as a random cost function for LCP$(M(\omega),q(\omega))$. In this sense, a deterministic formulation for the SLCP called the *expected residual minimization problem* in [3] may be regarded as an *expected total cost minimization problem* [1, 10, 14] for (1.3).

- **Expected Residual Minimization (ERM) Formulation** [3]:

Find a vector $x \in R^n_+$ that minimizes the *expected total residual* defined by an NCP function:

$$\min_{x\in R^n_+} f(x) := E[\|\Phi(x,\omega)\|^2], \tag{1.4}$$

where $E[\|\Phi(x,\omega)\|^2$ is the expectation function of the random function $\|\Phi(x,\omega)\|^2$.

The expectation function of the random function $y(x,\omega)$ yields another deterministic formulation [9] for SLCP, which may be called the *expected value formulation.*

- **Expected Value (EV) Formulation** [9]:

Find a vector $x \in R^n$ such that

$$\bar{y}(x) := E[y(x,\omega)] \geq 0,\ x \geq 0,\ x^T\bar{y}(x) = 0. \tag{1.5}$$

Let

$$\bar{M} = E[M(\omega)] \quad \text{and} \quad \bar{q} = E[q(\omega)]$$

be the expectation matrix and vector of the random matrix $M(\cdot)$ and vector $q(\cdot)$, respectively. Then $\bar{y}(x) = \bar{M}x + \bar{q}$ and the EV formulation (1.5) is to find a solution of the LCP$(\bar{M},\bar{q})$.

Let S_{ERM} and S_{EV} be the solution sets of the ERM formulation (1.4) and EV formulation (1.5), respectively. It is shown in [6] that if S_{EV} is bounded for any $\bar{q}$, then S_{ERM} is bounded for any $q(\cdot)$. However, the converse is not true in general.

The LCP has been studied for more than a half century. We have rich theoretical results on the existence of solutions for the LCP, which provide a powerful framework for developing efficient algorithms to solve the LCP. In particular, because of many important applications, the monotone LCP has been studied most extensively. In this paper, we focus our attention on the SLCP (1.1) with the expectation matrix $\bar{M}$ being a positive semi-definite matrix, i.e.,

$$x^T\bar{M}x \geq 0 \ \text{for all}\ x \in R^n.$$

We call (1.1) *a monotone SLCP* if $\bar{M}$ is a positive semi-definite matrix.

Obviously, if $M(\omega)$ is a positive semi-definite matrix for all $\omega \in \Omega$, then $\bar{M}$ is a positive semi-definite matrix. However, the expectation matrix $\bar{M}$ being a positive semi-definite matrix does not implies that

$$\mathcal{P}\{\omega \in \Omega \mid M(\omega) \text{ is positive semi-definite}\} > 0.$$

Although the positive definiteness of $\bar{M}$ does not ensure the existence of an $\omega \in \Omega$ such that $M(\omega)$ is positive semi-definite, we find that the monotone LCP$(\bar{M}, \bar{q})$ serves as an important tool in the study of the monotone SLCP with the ERM formulation..

This paper is organized as follows: In Section 2, we study the existence of solutions for the ERM formulation of the monotone SLCP based on the monotone LCP$(\bar{M}, \bar{q})$. In Section 3, we investigate the robustness of the ERM formulation. In Section 4, we give a procedure to generate a test problem of monotone SLCP, which allows the user to specify the size of the problem, the condition number of the expectation matrix $\bar{M}$ and the number of active constraints at a global solution of the ERM formulation. We report numerical results for hundreds of test problems by using a semismooth Newton-type method with a descent direction line search.

In this paper, $\|\cdot\|$ denotes the Euclidean norm $\|\cdot\|_2$. For any positive integer s and a vector $z \in R^s$, we denote $[z]_+ = \max(0, z)$, where the maximum is taken component-wise. For a subset $J \subseteq \{1, 2, \dots, s\}$, z_J denotes the subvector of z with components $z_j, j \in J$.

2 Existence of solution

In this section, we study the relation between the EV formulation LCP$(\bar{M}, \bar{q})$ and the ERM formulation of the monotone SLCP. First, we summarize some results on the existence of a solution for a deterministic monotone LCP. Recall that a square matrix A is called an R_0 matrix if the solution set of LCP$(A, 0)$ consists of the origin only.

Lemma 2.1 *Suppose that A is a positive semi-definite matrix.*

1. *[4] If the LCP(A, b) is feasible, i.e., there is a vector $x \geq 0$ such that $Ax + b \geq 0$, then it has a solution.*

2. *[4] The LCP(A, b) has a nonempty and bounded solution set for any b if and only if A is in addition an R_0 matrix.*

3. *[2] The solution set of LCP(A, b) is nonempty and bounded if and only if LCP(A, b) has a strictly feasible point, i.e., there is a vector $x > 0$ such that $Ax + b > 0$.*

Recall that $M(\cdot)$ is called a *stochastic R_0 matrix* if

$$x \geq 0,\ M(\omega)x \geq 0,\ x^T M(\omega)x = 0,\ \ a.e. \quad \Longrightarrow \quad x = 0.$$

The ERM formulation (1.4) utilizes an NCP function that possesses the property (1.2). There are a variety of functions that satisfy (1.2). Among them, the most popular NCP functions are the "min" function ϕ_1 and the Fischer-Burmeister (FB) function ϕ_2, which are defined by

$$\phi_1(a, b) := \min(a, b)$$

and

$$\phi_2(a,b) := a + b - \sqrt{a^2 + b^2},$$

respectively.

The FB function has a number of nice properties. Among others, a distinctive property from the "min" function is that $\|\Phi(\cdot,\omega)\|^2$ defined by the FB function is continuously differentiable everywhere. However, the FB function lacks flexibility in dealing with the monotone LCP. Some other merit functions and NCP functions have nice properties in dealing with monotone LCP [2, 11]. Here, we consider a version of the penalized FB NCP function given in [2]

$$\phi_3(a,b) := \lambda(a + b - \sqrt{a^2 + b^2}) + (1-\lambda)a_+b_+, \tag{2.1}$$

where $\lambda \in (0,1)$.

The ERM formulation (1.4) defined by the "min" function and the penalized FB function has different properties in regard to smoothness and boundedness. When we discuss their different properties, we use $\Phi_1(x,\omega), f_1(x)$, and $\Phi_3(x,\omega), f_3(x)$ to distinguish the functions $\Phi(x)$ and $f(x)$ defined by the "min" function ϕ_1 and the penalized FB function ϕ_3, respectively. When we discuss the ERM formulation (1.4) defined by any of the NCP functions, we use the notations $\Phi(x,\omega)$ and $f(x)$.

Assumption I. $f(x)$ is finite and continuous at any $x \in R^n_+$.

This assumption holds if $M(\omega)$ and $q(\omega)$ are measurable functions of ω with the following property

$$E[(\|M(\omega)\| + \|q(\omega)\|)^2] < \infty.$$

2.1 "min" function

In this subsection, we consider the ERM formulation (1.4) defined by the "min" function.

Lemma 2.2 *If $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$, then for any random matrix $M(\cdot)$ and vector $q(\cdot)$, the solution set S_{ERM} of the ERM formulation (1.4) defined by the "min" function is nonempty.*

Theorem 2.1 *Assume that $\bar{M}$ is a positive semi-definite matrix. If there are $\bar{x} \geq 0$ and $\hat{x} > 0$ such that*

$$\min_{1\leq i\leq n} \{\hat{x}_i, (\bar{M}\hat{x} + \bar{q})_i\} > \sqrt{f_1(\bar{x})} := \bar{\gamma}, \tag{2.2}$$

then the level set

$$D_1(\bar{\gamma}) := \{x \mid f_1(x) \leq \bar{\gamma}^2\}$$

is nonempty and bounded.

Corollary 2.1 *Under the assumptions of Theorem 2.1, the solution set S_{ERM} of the ERM formulation (1.4) defined by the "min" function is nonempty and bounded.*

Remark 2.1 *If $\bar{M}$ is positive definite, then $\bar{M}$ is an R_0 matrix. From Lemma 2.1, there is an $\hat{x} > 0$ such that $\bar{M}\hat{x} > 0$. This implies that for any $\gamma > 0$, there is a $\lambda > 0$ such that* $\min\{\lambda\hat{x}_i, (\lambda\bar{M}\hat{x} + \bar{q})_i\} \geq \gamma$. *Hence by Theorem 2.1, $\bar{M}$ being positive definite implies that the level set $D_1(\gamma)$ is bounded for any $\gamma > 0$ and thus the solution set S_{ERM} of the ERM formulation (1.4) defined by the "min" function is nonempty and bounded.*

2.2 Penalized FB function

In this subsection, we consider the ERM formulation (1.4) with the penalized FB NCP function ϕ_3 defined by (2.1).

Theorem 2.2 *If the monotone LCP($\bar{M}, \bar{q}$) has a nonempty and bounded solution set, then for any $\gamma \geq 0$, the level set*

$$D_3(\gamma) := \{x \,|\, f_3(x) \leq \gamma\}$$

is bounded.

Corollary 2.2 *If the monotone LCP($\bar{M}, \bar{q}$) has a nonempty and bounded solution set, then the ERM formulation (1.4) defined by the penalized FB function ϕ_3 has a nonempty and bounded solution set.*

Remark 2.2 *Let $\Omega_0 \subseteq \Omega$, $M_0 = E[M(\omega)1_{\{\omega\in\Omega_0\}}]$ and $q_0 = E[q(\omega)1_{\{\omega\in\Omega_0\}}]$. From*

$$E[\|\Phi(x,\omega)1_{\{\omega\in\Omega_0\}}\|] \leq E[\|\Phi(x,\omega)\|], \tag{2.3}$$

we can weaken the assumption (2.2) in Theorem 2.1 by assuming that M_0 is positive semi-definite and there are $\bar{x} \geq 0$, $\hat{x} > 0$ such that

$$\min_{1\leq i\leq n} \{\hat{x}_i, (M_0\hat{x} + q_0)_i\} > \sqrt{f_1(\bar{x})}.$$

Moreover, we can weaken the assumption of Theorem 2.2 by assuming that the monotone LCP(M_0, q_0) has a nonempty and bounded solution set.

2.3 Regularization

To establish the solvability of the ERM formulation (1.4) for the monotone SLCP without assuming the boundedness of the solution set of the monotone LCP($\bar{M}, \bar{q}$), we consider a regularized version of (1.4). For $\epsilon > 0$, let

$$y(x,\omega,\epsilon) := (M(\omega) + \epsilon I)x + q(\omega)$$

and

$$\Phi(x,\omega,\epsilon) := \begin{pmatrix} \phi(y_1(x,\omega,\epsilon), x_1) \\ \vdots \\ \phi(y_n(x,\omega,\epsilon), x_n) \end{pmatrix}.$$

The regularized problem for (1.4) is defined as

$$\min_{x\in R^n_+} f(x,\epsilon) := E[\|\Phi(x,\omega,\epsilon)\|^2]. \tag{2.4}$$

We will study the behavior of the sequence $\{x_{\epsilon_k}\}$ of solutions to (2.4) for an arbitrarily chosen positive sequence $\{\epsilon_k\}$ tending to zero. In the following, to simplify the notation, we will denote $\{\epsilon\}$ and $\{x_\epsilon\}$ for $\{\epsilon_k\}$ and $\{x_{\epsilon_k}\}$, respectively.

Theorem 2.3 *Suppose $\bar{M}$ is positive semi-definite. Then for any $\epsilon > 0$, the regularized problem (2.4) has a nonempty and bounded solution set S_{ERM_ϵ}. Let $x_\epsilon \in S_{ERM_\epsilon}$ for each $\epsilon > 0$. Then every accumulation point of the sequence $\{x_\epsilon\}$ is contained in the set S_{ERM}.*

We should clarify the meaning of the conclusion of Theorem 2.3. The result applies regardless of whether the sequence $\{x_\epsilon\}$ has an accumulation point or not. In the case where $\{x_\epsilon\}$ has an accumulation point, the ERM formulation has a solution. In the opposite case, we do not know if it has a solution. Now, we show that if the monotone LCP$(\bar{M},\bar{q})$ has a solution, then $\{x_\epsilon\}$ has an accumulation point, and thus the ERM formulation has a nonempty solution set S_{ERM} and every accumulation point of $\{x_\epsilon\}$ is contained in S_{ERM}. To establish this result, we use Li's error bound [12] for the monotone LCP.

Lemma 2.3 *[12] Suppose that A is positive semi-definite. Then there is a constant $c > 0$ such that*

$$\|x - \bar{x}(x)\| \leq c(\|\min(x, Ax+p)\| + [x^T(Ax+p)]_+), \tag{2.5}$$

where $\bar{x}(x)$ is a closest solution of LCP(A,p) to x under the norm $\|\cdot\|$.

Theorem 2.4 *Suppose the monotone LCP$(\bar{M},\bar{q})$ has a solution. Then the sequence $\{x_\epsilon\}$ is bounded.*

3 Robust solution

The EV formulation and the ERM formulation take into account all random events and give decisions under uncertainty. In general, the decisions may not be the best or may be even infeasible for each individual event. However, in many cases, we have to take risk to make a priori decision based on limited information of unknown random events. Naturally, one wants to know how good or how bad the decision given by a deterministic formulation can be. In this section, we study the robustness of solutions of the ERM formulation (1.4) for the monotone SLCP.

Let $\bar{\Phi} := E[\Phi(x,\omega)]$. For any x, by taking expectation in

$$\|\Phi(x,\omega)\|^2 = \|\bar{\Phi}(x)\|^2 + 2\bar{\Phi}(x)^T(\Phi(x,\omega) - \bar{\Phi}(x)) + \|\Phi(x,\omega) - \bar{\Phi}(x)\|^2,$$

we find

$$f(x) = E[\|\Phi(x,\omega)\|^2] = \|\bar{\Phi}(x)\|^2 + E[\|\Phi(x,\omega) - \bar{\Phi}(x)\|^2].$$

Note that the second term

$$\begin{aligned} E[\|\Phi(x,\omega) - \bar{\Phi}(x)\|^2] &= E[\mathrm{tr}(\Phi(x,\omega) - \bar{\Phi}(x))(\Phi(x,\omega) - \bar{\Phi}(x))^T] \\ &= \mathrm{tr}E[(\Phi(x,\omega) - \bar{\Phi}(x))(\Phi(x,\omega) - \bar{\Phi}(x))^T] \end{aligned}$$

is the trace of the covariance matrix of the random function $\Phi(x,\omega)$.

Since $\Phi(x,\omega) = 0$ if and only if x solves LCP$(M(\omega), q(\omega))$, and the ERM formulation (1.4) is equivalent to

$$\min_{x\in R^n_+} \|\bar{\Phi}(x)\|^2 + E[\|\Phi(x,\omega) - \bar{\Phi}(x)\|^2], \tag{3.1}$$

an optimal solution of the ERM formulation (1.4) yields a high mean performance of the SLCP and has a minimum sensitivity with respect to random parameter variations in SLCP. Therefore, the ERM formulation (1.4) can be regarded as a robust formulation for SLCP.

Now, we investigate the relation between a solution of the ERM formulation and a solution of LCP$(M(\omega), q(\omega))$ for $\omega \in \Omega$. First, we give a new error bound for monotone LCP which uses the sum of the "min" function $\phi_1(a,b)$ and the penalized FB function $\phi_3(a,b)$. The idea comes from the error bound given by Mangasarian and Ren [13]. Let SOL(A,p) denote the solution set of LCP(A,p), and define the distance from a point x to the set SOL(A,p) by dist$(x, \mathrm{SOL}(A,p)) := \|x - \bar{x}(x)\|$, where $\bar{x}(x)$ is a closest solution of LCP(A,p) to x under the norm $\|\cdot\|$. Let

$$\Psi_1(x) := \|\min(x, Ax+p)\|$$

and

$$s(x) := \|[-Ax-p, -x, x^T(Ax+p)]_+\|.$$

Lemma 3.1 *[13] Suppose that A is positive semi-definite and* SOL$(A,p) \neq \emptyset$*. Then there is a constant $c > 0$ such that*

$$\mathrm{dist}(x, \mathrm{SOL}(A,p)) \le c(\Psi_1(x) + s(x)), \quad x \in R^n.$$

Lemma 3.2 *Let $\psi(a,b) = [-b, -a, ab]_+$. Then we have $\|\psi(a,b)\| \le |\phi_3(a,b)|$ for any $a \ge 0$ and $b \in R$.*

From Lemma 3.2, it is easy to see that for any $x \ge 0$,

$$s(x) \le \Psi_3(x) := \|(\phi_3(x_1, (Ax+p)_1), \ldots, \phi_3(x_n, (Ax+p)_n))\|.$$

Moreover, it is known that there is a constant $\kappa > 0$ such that

$$\Psi_1(x) \le \kappa\Psi_3(x), \quad x \in R^n.$$

Using these inequalities with Lemma 3.1, we obtain the following new global error bounds for the monotone LCP(A,p).

Theorem 3.1 *Let the monotone LCP(A,p) have a nonempty solution set SOL(A,p). Then both $\Psi_1+\Psi_3$ and Ψ_3 provide global error bounds for the monotone LCP on R^n_+, that is, there are positive constants α_1 and α_2 such that*

$$\mathrm{dist}(x, \mathrm{SOL}(A,p)) \le \alpha_1(\Psi_1(x)+\Psi_3(x)) \le \alpha_2\Psi_3(x), \quad x \in R^n_+.$$

To give error bounds for SLCP, we assume that $M(\omega)$ is a positive semi-definite matrix and LCP($M(\omega), q(\omega)$) has a nonempty solution set for every $\omega \in \Omega$. This assumption holds in many applications. For instance, consider the stochastic quadratic program

$$\begin{array}{ll} \min & \frac{1}{2}z^TQz + c^Tz \\ \text{s.t} & A(\omega)z \ge b(\omega),\ z \ge 0, \end{array}$$

where Q is a positive definite matrix. The KKT conditions for this quadratic program yield the SLCP involving the random matrix

$$M(\omega) = \begin{pmatrix} Q & -A(\omega)^T \\ A(\omega) & 0 \end{pmatrix}.$$

Clearly this is a positive semi-definite matrix for each ω.

Theorem 3.2 *Assume that $\Omega = \{\omega_1, \omega_2 \dots, \omega_N\} \subset R^m$ and, for every $\omega \in \Omega$, $M(\omega)$ is a positive semi-definite matrix and LCP($M(\omega), q(\omega)$) has a nonempty solution set. Then there are positive constants β_1 and β_2 such that*

$$E[\mathrm{dist}(x, \mathrm{SOL}(M(\omega), q(\omega)))] \le \beta_1(\sqrt{f_1(x)} + \sqrt{f_3(x)}) \le \beta_2\sqrt{f_3(x)}, \quad x \in R^n_+.$$

Theorem 3.2 particularly shows that for $x^* \in S_{ERM}$,

$$E[\mathrm{dist}(x^*, \mathrm{SOL}(M(\omega), q(\omega)))] \le \beta_2\sqrt{f_3(x^*)} = \beta_2 \min_{x \in R^n_+} \sqrt{f_3(x)}. \tag{3.2}$$

Unlike an error bound for the deterministic LCP, the left-hand side of (3.2) is in general positive at a solution of the ERM formulation (1.4). Nevertheless, the inequality (3.2) suggests that the expected distance to the solution set SOL($M(\omega), q(\omega)$) for $\omega \in \Omega$ is also likely to be small at $x^* \in S_{ERM}$. In other words, we may expect that a solution of the ERM formulation (1.4) has a minimum sensitivity with respect to random parameter variations in SLCP. In this sense, solutions of (1.4) can be regarded as robust solutions for SLCP.

4 Numerical experiments

We have conducted some numerical experiments to investigate the properties of solutions of the ERM formulation (1.4) for monotone SLCP. In particular, we have made comparison of the ERM formulation with the EV formulation (1.5) in terms of the measures of optimality and feasibility as well as that of reliability, which are defined through a quadratic programming formulation of SLCP.

We start with some preliminary materials about calculations of gradients and Hessian matrices of functions f_1 and f_3 in the ERM formulation (1.4).

4.1 Gradient and Hessian

If the strict complementarity condition holds with probability one at x, then f_1 is twice continuously differentiable at x. In this case, the gradient $g_1(x)$ of f_1 is given by

$$g_1(x) = E[M(\omega)^T(I - D(x,\omega))(M(\omega)x + q(\omega)) + (I + D(x,\omega))x]$$

and the Hessian matrix $G_1(x)$ of f_1 is given by

$$G_1(x) = E[M(\omega)^T(I - D(x,\omega))M(\omega) + I + D(x,\omega)],$$

where $D(x,\omega) = \mathrm{diag}(\mathrm{sign}(M(\omega)x + q(\omega) - x))$.

The function f_3 defined by (2.1) with $\lambda \in (0,1)$ is continuously differentiable at any point $x \in R^n$, and twice continuously differentiable at point x where $P\{\omega \mid x_i = y_i(x,\omega) = 0,\ i = 1,\dots,n\} = 0$. The gradient $g_3(x)$ of f_3 is given by

$$g_3(x) = E[\nabla\|\Phi_3(x,\omega)\|^2] = 2E[V(x,\omega)^T\Phi_3(x,\omega)],$$

where $V(x,\omega) \in R^{n\times n}$ can be computed by Algorithm 1 in [2]. If f_3 is twice continuously differentiable at x, then the Hessian matrix $G_3(x)$ is given by

$$G_3(x) = E[\nabla^2\|\Phi_3(x,\omega)\|^2] = 2E[V(x,\omega)^TV(x,\omega) + \sum_{i=1}^{n} U_i(x,\omega)(\Phi_3(x,\omega))_i],$$

where $U_i(x,\omega) \in R^{n\times n}$. For each i, $U_i(x,\omega)$ can be computed as follows: Let $\xi_i = (x_i^2 + y_i(x,\omega)^2)^{-\frac{3}{2}}$, $\eta_i = \mathrm{sign}([x_i]_+[y_i(x,\omega)]_+)$, and m_{ij} be the (i,j) element of $M(\omega)$. Then we put

$$(U_i(x,\omega))_{kl} = \begin{cases} -\lambda m_{ik}m_{il}x_i^2\xi_i & k \neq i,\ l \neq i \\ -\lambda m_{ik}(m_{ii}x_i^2 - x_iy_i(x,\omega))\xi_i + (1-\lambda)m_{ik}\eta_i & k \neq i,\ l = i \\ -\lambda m_{il}(m_{ii}x_i^2 - x_iy_i(x,\omega))\xi_i + (1-\lambda)m_{il}\eta_i & k = i,\ l \neq i \\ -\lambda(m_{ii}x_i - y_i(x,\omega))^2\xi_i + 2(1-\lambda)m_{ii}\eta_i & k = i,\ l = i. \end{cases}$$

4.2 Measure of optimality and feasibility

Different deterministic formulations of SLCP have different optimal solutions. To help decision makers to select a proper solution, we introduce some measure of optimality and feasibility for a given point $x \in R^n_+$.

As stated in the introduction, the function value $f(x)$ can be regarded as an expected total cost. Let x^* be a solution of (1.4) with $\Omega = \{\omega_1,\dots,\omega_N\}$. By the definition of ERM formulation, there is no $x \in R^n_+$ such that

$$\mathcal{P}\{\omega \mid \|\Phi(x,\omega)\| < \|\Phi(x^*,\omega)\|\} = 1.$$

Hence x^* is a *weak Pareto optimal solution* of the SLCP in the sense of multi-objective optimization

$$\min_{x\in R^n_+} \begin{pmatrix} \|\Phi(x,\omega_1)\| \\ \vdots \\ \|\Phi(x,\omega_N)\| \end{pmatrix}.$$

Now we define some measure of optimality and feasibility for a given point x, without using an NCP function. For a fixed ω, LCP$(M(\omega), q(\omega))$ is equivalent to the quadratic program

$$\begin{array}{ll} \min & y(x,\omega)^T x \\ \text{s.t} & y(x,\omega) := M(\omega)x + q(\omega) \geq 0,\ x \geq 0 \end{array} \qquad (4.1)$$

in the sense that (4.1) has an optimal solution with zero objective value if and only if LCP$(M(\omega), q(\omega))$ has a solution. We adopt some ideas of loss functions from the literature of stochastic programming [1, 10, 14] to problem (4.1). For $x \in R^n_+$, let

$$\gamma(x,\omega) := \|\min(0, y(x,\omega))\| + x^T[y(x,\omega)]_+. \qquad (4.2)$$

It is easy to verify that x_ω is a solution of (4.1) if and only if $\gamma(x_\omega, \omega) = 0$ and $x_\omega \geq 0$, provided LCP$(M(\omega), q(\omega))$ has a solution. In (4.2), the first term evaluates violation of the nonnegativity condition and the second term evaluates the loss in the objective function of (4.1). For a fixed $x \in R^n_+$, the expected total loss is defined by $E[\gamma(x,\omega)]$. For two points $x^*, \bar{x} \in R^n_+$, we define the measure of dominance of x^* over $\bar{x}$ by

$$\pi(x^*, \bar{x}) := \mathcal{P}\{\omega \,|\, \gamma(x^*,\omega) < \gamma(\bar{x},\omega)\}. \qquad (4.3)$$

If $\pi(x^*, \bar{x}) > 0.5$, then x^* has more chance to dominate $\bar{x}$, and so x^* may be regarded as a better point than $\bar{x}$ in the multi-objective optimization problem

$$\min_{x\in R^n_+} \begin{pmatrix} \gamma(x,\omega_1) \\ \vdots \\ \gamma(x,\omega_N) \end{pmatrix}.$$

In many engineering and economic applications of SLCP, the inequality $y(x,\omega) \geq 0$ describes the safety of the system, and the guarantee of safety is critically important. Under those circumstances, we may judge that a failure occurs if and only if there is an index i such that $y_i(x,\omega) < 0$. Let

$$y^{\min}(x,\omega) := \min_{1\leq i\leq n} y_i(x,\omega).$$

The reliability of x with a tolerance $\epsilon > 0$ is then defined by

$$rel_\epsilon(x) := \mathcal{P}\{\omega \,|\, y^{\min}(x,\omega) \geq -\epsilon\}.$$

4.3 Test problems

We give a procedure to generate a test problem of the ERM formulation for discretized monotone SLCP,

$$\min_{x \in R^n_+} f(x) := \frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{n} \phi(x_i, (M^j x + q^j)_i)^2, \tag{4.4}$$

where $M^j = M(\omega^j)$ and $q^j = q(\omega^j)$ for $j = 1, \ldots, N$ and $\Omega = \{\omega^1, \ldots, \omega^N\}$.

Let $\hat{x}$ be a nominal point chosen in R^n_+, which is used as a seed of constructing a set of test problems and becomes a solution of the ERM formulation (1.4) in some special cases (see below for the detail). Moreover, the user is required to specify the following parameters:

- n: the number of variables
- N: the number of random matrices and vectors
- μ^2 ($\mu \geq 1$): the condition number of the expectation matrix $\bar{M}$
- n_x: the number of elements in the index set $\mathcal{J} = \{i \mid \hat{x}_i > 0\}$
- $(0, \tau)$: the range of $\hat{x}_i$ for $i \in \mathcal{J}$
- $\#I_j$: the number of elements in the index set $\mathcal{I}_j = \{i \mid \hat{x}_i = 0, (M^j\hat{x} + q^j)_i > 0\}$ for each j
- $\#K_j$: the number of elements in the index set $\mathcal{K}_j = \{i \mid \hat{x}_i = 0, (M^j\hat{x} + q^j)_i = 0\}$ for each j
- $(0, \nu)$: the range of $(M^j\hat{x} + q^j)_i$ for $i \in \mathcal{I}_j$ and each j
- $[0, \beta)$: the range of $(M^j\hat{x} + q^j)_i$ for $i \in \mathcal{J}$
- $(-\sigma, \sigma)$: the range of elements of matrix $\bar{M} - M^j$ for each j

Procedure for generating a test problem of monotone SLCP

1. Randomly generate a vector $\hat{x} \in R^n_+$ that has n_x positive elements in $(0, \tau)$.

2. Generate a diagonal matrix D whose diagonal elements are determined as

$$D_{ii} = \begin{cases} 1/\mu & i = 1 \\ \mu^{\lambda_i} & i = 2, \ldots, n-1 \\ \mu & i = n, \end{cases}$$

   where λ_i, $i = 2, \ldots, n-1$ are uniform variates in the interval $(-1, 1)$.

3. Generate a random orthogonal matrix $U \in R^{n \times n}$ and let $\bar{M} = UDU^T$.

4. Generate N random matrices $B^j \in R^{n\times n}$, $j = 1, 2, \ldots, N$ whose elements are in the interval $(0, 1)$. Set

$$M^j = \bar{M} + \sigma(B^j - B^{N-j+1}), \qquad j = 1, 2, \ldots, N.$$

5. For each $j = 1, 2, \ldots, N$, set

$$q_i^j = \begin{cases} (-M^j\hat{x})_i & i \in \mathcal{K}_j \\ (-M^j\hat{x} + \beta z^j)_i & i \in \mathcal{J} \\ (-M^j\hat{x} + \nu z^j)_i & i \in \mathcal{I}_j, \end{cases}$$

where $z^j \in R^n$ is a random vector whose elements are in the interval (0,1).

Some aspects of the test problem

- The expectation matrix $\bar{M} = UDU^T$ is symmetric positive definite. Its condition number is μ^2 and its eigenvalues are distributed on the interval $[1/\mu, \mu]$.
- If $\sigma = 0$, then all M^j are equal to $\bar{M} = UDU^T$, which is positive definite. For $\sigma > 0$, M^j may not be a positive semi-definite matrix, but $|(\bar{M} - M^j)_{il}| = \sigma|(B^j - B^{N-j+1})_{il}| \leq \sigma$ for all $i, l = 1, \ldots, n$.
- If $\#K_j = 0$ for all $j = 1, \ldots, N$, then f_1 is continuously differentiable at x.
- If $\beta = 0$, then $\hat{x}$ is a solution of LCP(M^j, q^j) for all $j = 1, 2, \ldots, N$. In this case, $\hat{x}$ becomes a global solution of (4.4) with $f(\hat{x}) = \min_{x\in R^n_+} f(x) = 0$.
- $n - n_x$ is the number of active constraints at $\hat{x}$.
- If $\beta > 0$, then we have in general $f(\hat{x}) > 0$. In this case, $\hat{x}$ is not necessarily a solution of (4.4). However, by Theorem 2.1 and Theorem 2.2, the positive definiteness of $\bar{M}$ guarantees that the solution set of (4.4) is nonempty and bounded.

4.4 Numerical results

We used the program of Lemke's method [7] to get a solution $\bar{x}$ of the EV formulation (1.5). To solve the ERM formulation (4.4), we used a semismooth Newton method with descent direction line search [5]. In particular, we first applied a global descent line search with the gradient $\nabla f(x)$ to make the function value sufficiently decrease and get a rough approximate solution. Next, we used a local semi-smooth Newton method with the rough approximate solution as an initial point to get an approximate local optimal solution. As the ERM problem defined by the "min" function is nonsmooth, in a few occasions, the method failed to decrease the function value. When it happened, we restarted the method. All computations were carried out by using MATLAB on a PC.

We first tested our program on hundreds of random problems with $\beta = 0$ generated by the procedure in the last subsection with different parameters $(n, N, \mu, n_x, \nu, \sigma)$ and starting points $x^0 = \ell e$ where $\ell = 0, 10, \ldots, 50$ and e is the n-dimensional vector of ones. Since $\beta = 0$, the solution x^* of (4.4) coincides with the nominal point $\hat{x}$. We have observed that the average function values and relative errors at computed solutions $\tilde{x}$ of (4.4) satisfy

$$f(\tilde{x}) \leq 10^{-26}, \qquad \frac{\|x^* - \tilde{x}\|}{\|x^*\|} \leq 10^{-17},$$

which indicates that our method works successfully in finding a global solution of (4.4).

Next, for each fixed (n, n_x, β, σ) with $\beta > 0$, we used the procedure described in the previous subsection to generate 100 test problems with the following parameters:

$$\tau = 20, \ \mu = 10, \ \nu = 15, \ N = 10^3.$$

The number of elements in the index set $\mathcal{K}_j$ was determined by using a random number as $\#K_j = \text{floor}((n - n_x)\text{rand}(1, N))$. The numbers shown in Tables 4.1 and 4.2 are average values for the 100 problems.

In these tables, x^i is the computed solution, where the index $i = 1$ stands for the "min" function, and $i = 3$ stands for the penalized FB function.

For any $x, \tilde{x} \in R^n_+$, we define $\Gamma(x) := E[\gamma(x, \omega)]$, $\pi(x, \tilde{x})$ and $rel_\epsilon(x)$ as follows:

$$\begin{aligned}
\Gamma(x) &:= \frac{1}{N}\sum_{i=1}^{n} \gamma^j(x), & \gamma^j(x) &= \|\min(0, y^j(x))\| + x^T[y^j(x)]_+, \\
\pi(x, \tilde{x}) &:= \sum_{j=1}^{N} p_j, & p_j &= \begin{cases} \frac{1}{N} & \text{if } \gamma^j(x) < \gamma^j(\tilde{x}) \\ 0 & \text{otherwise,} \end{cases} \\
rel_\epsilon(x) &:= \sum_{j=1}^{N} p_j, & p_j &= \begin{cases} \frac{1}{N} & \text{if } \min_{1 \leq i \leq n} y_i^j(x) \geq -\epsilon \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

where $y^j(x) = M^j x + q^j, j = 1, \ldots, N$.

Table 4.1 Function values and rel_ϵ with $\epsilon = 0$ (left) and $\epsilon = 1$ (right).

(n, n_x, β, σ)	$f_1(x^1)$	$f_1(\bar{x})$	$f_3(x^3)$	$f_3(\bar{x})$	$rel_\epsilon(\bar{x})$	$rel_\epsilon(x^1)$	$rel_\epsilon(x^3)$
20, 10, 10, 20	254.87	2.13e6	447.82	1.05e7	0, 0	0.55, 0.91	0.55, 0.92
20, 10, 10, 10	241.89	4.47e5	448.99	2.13e6	0, 0	0.55, 0.91	0.55, 0.92
20, 10, 5, 10	69.41	2.62e5	131.64	1.34e6	0, 0	0.54, 0.96	0.52, 0.93
20, 10, 5, 0	18.89	75.78	32.69	154.36	0.31, 0.37	0.27, 0.60	0.21, 0.51
40, 20, 10, 20	527.19	6.83e6	998.75	3.01e7	0, 0	0.52, 0.97	0.52, 0.97
40, 20, 10, 10	510.84	1.90e6	999.39	8.52e6	0, 0	0.49, 0.85	0.49, 0.84
40, 20, 5, 10	144.06	1.14e6	270.48	4.65e6	0, 0	0.52, 0.99	0.50, 0.98
40, 20, 5, 0	39.92	154.24	69.48	311.33	0.17, 0.38	0.21, 0.47	0.16, 0.42
60, 30, 10, 20	759.46	1.56e7	1.39e3	6.59e7	0, 0	0.49, 0.97	0.50, 0.98
60, 30, 10, 10	752.87	3.44e6	1.38e3	1.43e7	0, 0	0.45, 0.83	0.45, 0.84
60, 30, 5, 10	219.64	2.64e6	424.33	1.09e7	0, 0	0.48, 1.00	0.46, 1.00
60, 30, 5, 0	58.29	281.16	100.56	576.09	0.51, 0.58	0.37, 0.56	0.28, 0.48

Table 4.2 Relative dominance of solutions based on the stochastic QP formulation

(n, n_x, β, σ)	$\pi(x^1, \bar{x})$	$\pi(x^3, \bar{x})$	$\pi(x^1, x^3)$	$\pi(x^3, x^1)$	$\Gamma(\bar{x})$	$\Gamma(x^1)$	$\Gamma(x^3)$
20, 10, 10, 20	1	1	0.49	0.51	3.67e4	518.13	517.91
20, 10, 10, 10	1	1	0.49	0.51	1.56e4	491.21	490.64
20, 10, 5, 10	1	1	0.42	0.57	1.14e4	241.04	239.05
20, 10, 5, 0	0.50	0.55	0.32	0.60	139.36	84.66	71.00
40, 20, 10, 20	1	1	0.47	0.51	8.69e4	1.08e3	1.08e3
40, 20, 10, 10	1	1	0.42	0.47	4.61e4	1.04e3	1.04e3
40, 20, 5, 10	1	1	0.42	0.58	3.03e4	493.10	490.95
40, 20, 5, 0	0.56	0.62	0.36	0.52	277.82	167.82	148.54
60, 30, 10, 20	1	1	0.48	0.51	1.59e5	1.52e3	1.52e3
60, 30, 10, 10	1	1	0.48	0.52	6.93e4	1.50e3	1.50e3
60, 30, 5, 10	1	1	0.42	0.58	5.80e4	770.57	768.24
60, 30, 5, 0	0.57	0.58	0.42	0.58	552.59	276.76	222.37

As to the reliability $rel_\epsilon(x)$ and the expected total loss $\Gamma(x)$, the solutions x^1 and x^3 exhibit significantly better performance than $\bar{x}$ as shown in Tables 4.1 and 4.2. Moreover, as to the measure of optimality and feasibility $\pi(\cdot, \cdot)$ which is defined through the stochastic quadratic program (4.1), the solutions x^1 and x^3 dominate $\bar{x}$ in most cases. From these results, we may conclude that the ERM formulation yields a solution that has desirable properties in regard to the performance measures related to optimality, feasibility, and reliability.

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