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A survey on asymptotic evaluations of Wiener functional expectations

Shinzo Watanabe, Ritsumeikan University

1 Introduction.

Since the *Wiener measure* was established by N. Wiener as a mathematical model of Brownian motion in 1923, a rigorous theory of integrations on a function space started. In this report, we would review on the problem of evaluating the behavior of the Wiener measure expectation $E(F_\varepsilon(w))$ as $\varepsilon \to 0$, where $\{F_\varepsilon(w),\ \varepsilon > 0\}$ is a family of Wiener functionals parametrized by $\varepsilon > 0$. There have been so many problems of this kind and many important methods have been introduced. Here, we would choose a topic related to *Schilder's theorem* in 1966 (cf. [Sc]), which is concerned with a *Laplace method* on Wiener space.

First, we recall a Laplace method in a finite dimensional Gaussian measure integration, of which a proof can be easily provided by an application of elementary differential and integral calculus. Let $\mu(dx) = (2\pi)^{-d/2}\exp\{-\frac{|x|^2}{2}\}dx$, $x \in \mathbf{R}^d$, be the d-dimensional standard Gaussian distribution. We consider the following integral parametrized by $\varepsilon > 0$:

$$I(\varepsilon) = \int_{\mathbf{R}^d} g(\varepsilon x)\exp\left\{\frac{f(\varepsilon x)}{\varepsilon^2}\right\}\mu(dx).$$

We assume the following conditions on functions f and g: $\mathbf{R}^d \to \mathbf{R}$.

(A.1.1) $f(x)$ is continuous and $\limsup_{|x|\to\infty} |f(x)|/|x|^2 < \frac{1}{2}$.

$g(x)$ is continuous and $|g(x)| = O(e^{K|x|^2})$ as $|x| \to \infty$ for some $K > 0$.

(A.1.2) Setting $F(x) = |x|^2/2 - f(x)$ and $M_F = \{x \in \mathbf{R}^d | F(x) = \min_{y\in\mathbf{R}^d} F(y)\}$, M_F is a singleton; $M_F = \{x_0\}$, $f(x)$ is C^2 at x_0, $g(x_0) \neq 0$ and $\det(I - \partial^2 f(x_0)) = \det \partial^2 F(x_0)) > 0$. Here, $\partial^2 f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)$ is the Hessian of f.

Then we have

$$I(\varepsilon) \sim \frac{1}{\sqrt{\det(I - \partial^2 f(x_0))}} \cdot g(x_0) \cdot \exp\left\{-\frac{F(x_0)}{\varepsilon^2}\right\} \quad \text{as} \quad \varepsilon \to 0.$$

Furthermore, if $f(x)$ and $g(x)$ are smooth near x_0, then, for any $n = 1, 2, \ldots$,

$$\exp\left\{\frac{F(x_0)}{\varepsilon^2}\right\} I(\varepsilon) = c_0 + c_1\varepsilon^2 + \cdots c_n\varepsilon^{2n} + O(\varepsilon^{2n+2}) \quad \text{as} \quad \varepsilon \to 0$$

with $c_0 = \frac{1}{\sqrt{\det(I - \partial^2 f(x_0))}}$, and c_n can be computed explicitly in compoments of $\partial^k f(x_0)$ and $\partial^l g(x_0)$.

A proof can be carried out by writing $I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon)$, where

$$I_1(\varepsilon) = \int_{|\varepsilon x - x_0| \le \delta} g(\varepsilon x) \exp\left\{\frac{f(\varepsilon x)}{\varepsilon^2}\right\} \mu(dx)$$

and

$$I_2(\varepsilon) = \int_{|\varepsilon x - x_0| > \delta} g(\varepsilon x) \exp\left\{\frac{f(\varepsilon x)}{\varepsilon^2}\right\} \mu(dx)$$

for $\delta > 0$. Then $I_2(\varepsilon)$ can be estimated as

$$|I_2(\varepsilon)| \le K \exp\left\{-\frac{F(x_0)}{\varepsilon^2} - \frac{\alpha}{\varepsilon^2}\right\} \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0)$$

for some $K := K(\delta) > 0$, $\alpha := \alpha(\delta) > 0$ and $\varepsilon_0 := \varepsilon_0(\delta) > 0$. We have

$$I_1(\varepsilon) = (2\pi)^{-d/2} e^{-F(x_0)/\varepsilon^2} \int_{|\varepsilon x| \le \delta} g(\varepsilon x + x_0) \exp\left\{-\frac{F(\varepsilon x + x_0) - F(x_0)}{\varepsilon^2}\right\} dx$$

and the asymptotic expansion of the integral in the RHS can be easily obtained.

2 Schilder's asymptotic formula on Wiener functional expectations

The asymptotic formula in the finite dimensional case as given above has been extended by Schilder ([Sc]) to the case of integrals on Wiener space. Let (W, H, P) be an abstact Wiener space (AWS); W is a real separable Banach space, $H \subset W$ is the Cameron-Martin subspace and P is the Borel probability on W such that the family $\{\ l(w) \mid l \in W^* \subset H^*\ \}$ is a Gaussian system with mean 0 and covariance $E(l(w)l'(w)) = \langle l, l' \rangle_{H^*}$. Thus, we may think of P a standard Normal distribution on the Hilbert space H realized on a suitable enlarged space W. As usual, the dual H^* is identified with H by the Riesz theorem. A typical example is the d-dimensional Wiener space in which

$$W_0(\mathbf{R}^\mathrm{d}) := \{\ w;\ [0,T] \ni t \mapsto w(t) \in \mathbf{R}^d,\ \text{continuous},\ w(0) = 0\ \},$$

with the usual maximum norm $||w|| = \max_{0 \le t \le T} |w(t)|$,

$$H = \left\{\ h \in \mathbf{W}_0(\mathbf{R}^\mathrm{d}) \mid h(t) = \int_0^t \dot{h}(s)ds,\ \dot{h} \in L^2\left([0,T] \to \mathbf{R}^d\right)\ \right\}, \quad ||h||_H = ||\dot{h}||_{L^2}$$

and P is the d-dimensional Wiener measure on it. Here T is a positive constant; sometimes, the time interval is taken to be $[0, \infty)$ and then $W_0(\mathbf{R}^d)$ is a Fréchet space with a family of maximum (semi)norms on subintervals.

Scilder's theorem on the AWS (W, H, P) can be stated in the same way as the formula given in Introduction if the differential calculus is understood in the sense of Fréchet differential calculus on the Banach space W.

Let $f = f(w)$ and $g = g(w)$ be real-valued continuous functions on the Banach space W and consider the following Wiener functional integral parametrized by $\varepsilon > 0$:

$$I(\varepsilon) = E\left[g(\varepsilon w)\exp\left\{\frac{f(\varepsilon w)}{\varepsilon^2}\right\}\right] := \int_W g(\varepsilon w)\exp\left\{\frac{f(\varepsilon w)}{\varepsilon^2}\right\} P(dw).$$

We assume the following conditions on functions f and g: $W \to \mathbf{R}$.

(A.2.1) $\limsup_{||w||\to\infty} |f(w)|/||w||^2 < \alpha$ for some $\alpha > 0$ such that $E(e^{\alpha||w||^2}) < \infty$.
$|g(w)| = O(e^{K||w||^2})$ as $||w|| \to \infty$ for some $K > 0$.

(A.2.2) Setting
$F(h) = |h|_H^2/2 - f|_H(h)$, $h \in H$, and $M_F = \{h \in H | F(h) = \min_{h' \in H} F(h')\}$, M_F is a singleton; $M_F = \{h_0\}$, $f(w)$ is C^2 at h_0, $g(h_0) \neq 0$ and $\det(I - A_2[h_0]) > 0$.

Here, $A_2[h_0] : H \to H$ is a linear trace class operator defined as follows: By (A.2.1), we have

$$f(h_0 + \varepsilon w) = f(h_o) + \frac{\varepsilon^2}{2} D^2 f(h_0)[w] + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0$$

and, if $D^2 f(h_0)[w] = c_0 \oplus c_1(w) \oplus c_2(w)$ is the Wiener chaos decomposition of the quadratic form $D^2 f(h_0)[w]$ on W, then $c_1 = 0$, $c_2(w) = (A_2[h_0]w, w)$ where $A_2[h_0] : H \to H$ is a linear trace class operator and $c_0 = \mathrm{tr}(A_2[h_0])$. Note that

$$E\left[e^{\frac{1}{2}D^2 f(h_0)[w]}\right] = e^{\frac{1}{2}\mathrm{tr}(A_2[h_0])}(\det{}_2(I - A_2[h_0]))^{-1/2} = (\det(I - A_2[h_0]))^{-1/2}$$

where $\det_2$ denotes the Carleman-Fredholm modified determinant. Now Scilder's theorem can be stated as follows

$$I(\varepsilon) \sim \exp\left\{-\frac{F(h_0)}{\varepsilon^2}\right\} g(h_0) E\left[e^{\frac{1}{2}D^2 f(h_0)[w]}\right] \quad \text{as} \quad \varepsilon \to 0.$$

Also, the asymptotic expansion can be obtained when f and g are smooth on W in the sense of Fréchet differential calculus on W.

A crucial point in the proof is to justify the localization : $I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon)$ as in the case of finite dimension. The justification is based on *Schilder's large deviation*: If $F \subset W$ is closed and $G \subset W$ is open, then we have

$$\limsup_{\varepsilon \to 0} 2\varepsilon^2 \log P(\varepsilon w \in F) \leq -\inf\{\ ||h||_H^2\ ;\ h \in F\ \}$$

and

$$\limsup_{\varepsilon \to 0} 2\varepsilon^2 \log P(\varepsilon w \in G) \geq -\inf\{\ ||h||_H^2\ ;\ h \in G\ \}.$$

Cf. e.g., [DS].

3 Asymptotic formula on "conditional" Wiener functional expectations

Thanks to the Malliavin calculus, we can discuss the conditional expectations for *smooth* functionals, smooth in the sense of Malliavin, as a *surface integral* on a *hypersurface* imbedded in Wiener space. Formally, we consider the following integral parametrized by $\varepsilon > 0$;

$$I(\varepsilon) = E\left[g(\varepsilon w)\exp\left\{\frac{f(\varepsilon w)}{\varepsilon^2}\right\}\delta_x(\Phi(\varepsilon w))\right]$$

where f and g: $W \to \mathbf{R}$, Φ: $W \to \mathbf{R}^n$ are Wiener functionals and $\delta_x(\cdot)$ is the Dirac delta function with pole at $x \in \mathbf{R}^n$. This may be regarded as a surface integral of $g(\varepsilon w)\exp\left\{\frac{f(\varepsilon w)}{\varepsilon^2}\right\}$ on the hypersurface $\{w \in W | \Phi(\varepsilon w) = x\}$ in W. We would particularly like to include the case where Wiener functionals involved are *Itô functionals*; the case which is important in applications to heat kernels, security price models in a financial market, etc. Itô functionals are usually not continuous on W so that Schlder's theorem as given in Section 2 is not applicable. Indeed, since, the map $w \in W \to \varepsilon w \in W$ is singular with respect to the Wiener measure, the functionals like $f(\varepsilon w)$ and $g(\varepsilon w)$ are usually meaningless. Also, since $P(H) = 0$, the restictions to the Cameron-Martin space like $f|_H$ of $g|_H$ are meaningless, as well.

For Itô functionals $f(w)$, we can define $f(\varepsilon w)$ and $f(h)$ for $h \in H$ in a natural way; just replace the Wiener path $t \to w(t)$ by $t \to \varepsilon w(t)$ and $t \to h(t)$ which are semimartingale paths, anyway. Also, Itô functionals are smooth in the sense of Malliavin-Sobolev differential calculus on Wiener space and the expression like $\delta_x(\Phi(\varepsilon w))$ can be justified as an element in the Sobolev space with negative differentiability index, in other words, someting like Schwartz distributions on Wiener space. The surface integral is well defined by quasi-sure analysis and disintegration theory as are developed in e.g. [M], [AM], [Su], [L]. In this way, we can develop a rather satisfactory asymptotic theory for conditional Wiener functional expectations as above. We would formulate an asymptotic formula in the following way; we refer to [IW] the notions and notations in the Malliavin calculus, Sobolev spaces of (generalized) Wiener functionals and asymptotic expansions, in particular.

Let $f(w;\varepsilon)$ be a real valued Wiener functional parametrized by $\varepsilon > 0$, smooth in the sense $f \in \mathbf{D}^\infty$ for every $\varepsilon > 0$ such that it has the following asymptotic expansion for every $h \in H$,

$$f(w + \frac{h}{\varepsilon};\varepsilon) = f_0[h] + \varepsilon f_1[h](w) + \varepsilon^2 f_2[h](w) + \cdots \varepsilon^n f_n[h](w) + \cdots,$$

where $f_n[h](w) \in \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_n$. Here, $\mathcal{C}_n$ denotes the space of homogeneous Wiener chaos (Itô's multiple Wiener integrals) of order n in the Wiener chaos expansion $L_2 = \oplus\sum_{n=0}^{\infty}\mathcal{C}_n$ of square-inegrable Wiener functionals. In particular, $f_0[h]$ is constant in w so that it is a function of $h \in H$. We assume that it is continuous in $h \in H$. We assume also that $h \in H \to f_n[h](w) \in L_2$ is continuous for every n.

Consider another real functional $g(w;\varepsilon)$ and also an $\mathbf{R}^n$-valued Wiener functional $\Phi(w;\varepsilon)$, both parametrized by $\varepsilon > 0$, smooth and have the same asymptotic expansion as $f(w;\varepsilon)$. We assume

(A.3.1) Letting $X(w;\varepsilon) = (f(w;\varepsilon), g(w;\varepsilon), \Phi(w;\varepsilon))$ and $X_0[h] = (f_0[h], g_0[h], \Phi_0[h])$, the following principle of large deviations holds: If $F \subset \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n$ is closed and and $G \subset \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n$ is open, then we have

$$\limsup_{\varepsilon \to 0} 2\varepsilon^2 \log P(X(w;\varepsilon) \in F) \leq -\inf\{ \, ||h||_H^2 \, ; \; X_0[h] \in F \, \}$$

and

$$\limsup_{\varepsilon \to 0} 2\varepsilon^2 \log P(X(w;\varepsilon) \in G) \geq -\inf\{ \, ||h||_H^2 \, ; \; X_0[h] \in G \, \}.$$

(A.3.2) The Malliavin covariance $\langle D\Phi^i(w;\varepsilon), D\Phi^j(w;\varepsilon)\rangle_H$ of $\Phi(w;\varepsilon)$ satisfies, for some neighborhood U_x of $x \in \mathbf{R}^n$,

$$\mathbf{1}_{U_x}(\Phi(w;\varepsilon)) \cdot \det\langle D\Phi^i(w;\varepsilon), D\Phi^j(w;\varepsilon)\rangle_H^{-1} \in \bigcap_{1<p<\infty} L_p \quad \text{for every} \quad \varepsilon > 0.$$

The assumpion (A.3.2) guarantees that $\delta_x(\Phi(w;\varepsilon))$ can be defined as a generalized Wiener functional in some Sobolev space of negative differentiability index.

(A.3.3) We set $K_x = \{ \, h \in H \mid \Phi_0[h] = x \, \}$ and assume that it is not empty. Assume further that

$$K_x^{min} := \{ \, h \in K_x \mid \frac{1}{2}||h||_H^2 - f_0[h] = \min_{h' \in K_x} \frac{1}{2}||h'||_H^2 - f_0[h'] \, \}$$

is a compact manifold of dimension, $\dim K_x^{min} = m$, regularly imbedded in H.

Remark 3.1. *By the Hilbertian inner product of H, K_x^{min} is a Riemannian manifold. The assumption (A.3.3) admits the case that K_x^{min} consists of a finite number of points. We refer this as the case of $\dim K_x^{min} = 0$. In Section 1 and Section 2, we stated the results in the case when the minimizing manifold is a singleton; of course, a generalization as in this section is possible, as well.*

(A.3.4) $\det\langle \Phi_1^i[h], \Phi_1^j[h]\rangle_H > 0$ for every $h \in K_x^{min}$. Here, $\Phi_1^i[h] \in H$ represents the first order Wiener chaos $\Phi_1^i[h](w) \in \mathcal{C}_1$.

By the Lagrange multipier principle, we see that for every $h \in K_x^{min}$, there exists a unique $p(h) \in \mathbf{R}^n$ such that $h - f_1[h] = (p(h), \Phi_1[h])_{\mathbf{R}}^n$.

(A.3.5) The following holds for every $h \in K_x^{min}$: for any $k \in H$, $k \neq 0$ such that $\langle \Phi_1^i[h], k\rangle_H = 0$, $i = 1, \ldots, n$, and $k \perp K_x^{min}$,

$$\frac{1}{2}||k||_H^2 - \langle f_{2,2}[h], k \otimes k\rangle_{H\otimes H} - (p(h), \langle \Phi_{2,2}[h], k \otimes k\rangle_{H\otimes H})_{\mathbf{R}^n} > 0.$$

Here, $f_{2,2}[h] \in H \otimes H$ and $\Phi_{2,2}[h] \in H \otimes H \otimes \mathbf{R}^n$ represent the $\mathcal{C}_2$-component of $f_2[h](w)$ and $\Phi_2[h](w)$, respectively.

Theorem 3.1.

$$I(\varepsilon) := E\left[g(w;\varepsilon)\exp\left\{\frac{f(w;\varepsilon)}{\varepsilon^2}\right\}\delta_x(\Phi(w;\varepsilon))\right] = \varepsilon^{-(n+m)}e^{-\frac{\alpha}{\varepsilon^2}}(c_0 + c_1\varepsilon + c_2\varepsilon^2 + \cdots)$$

where $\alpha = \frac{1}{2}||h||^2 - f_0(h) = \min\{\frac{1}{2}||h||^2 - f_0(h); h \in K_x\}$, $h \in K_x^{min}$, *and* c_0 *is given by the integral:*

$$\int_{K_x^{min}} g_0[h]E\left[\exp\left\{\frac{1}{2}f_2[h](w) + \frac{1}{2}(p(h), \Phi_2[h](w))_{\mathbf{R}^n}\right\}\delta_0(\Phi_1[h](w), \mathbf{i}[h](w))\right]\omega(dh).$$

Here, $\omega(dh)$ *is the Riemannian volume on* K_x^{min} *(which degenerates to the counting measure when* $\dim K_x^{min} = 0$*) and* $\mathbf{i}[h](w) := \sum e_i(w)\cdot e_i$, *the sum being taken over an ONB* $\{e_i\}$ *in the tangent space* $T_h(K_x^{min})(\subset H)$, *so that* $\delta_0(*,\cdot)$ *is the Dirac delta function at* $(0,0)$ *on* $\mathbf{R}^n \times T_h(K_x^{min})$.

For details and examples, cf. [TW].

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