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Stochastic integral equations of Fredholm type

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1 Introduction

Let us consider the boundary value problem of a formal stochastic differential equation as follows;

$$\begin{aligned}&\left\{\frac{d^2}{dt^2}+k^2(\omega)\right\}X(t,\omega)=\{a(t,\omega)\frac{dZ}{dt}+b(t,\omega)\}X(t,\omega)+c(t,\omega),\\&X(0,\omega)=x_0,\quad X(1,\omega)=x_1\ \ (x=0,x_1\in R^1).\end{aligned}\tag{1}$$

where the $Z(t,\omega),a(t,\omega),b(t,\omega),c(t,\omega)$ $t\geq 0$ are real square integrable stochastic processes defined on a probability space $(\Omega,\mathcal{F},P)$, and $k(\omega)$ is a random variable. We do not necessarily suppose that the process Z_t is differentiable in t, so that we need to understand this formal problem via an integral equation form as we did for the SDE (stochastic differential equation) in Itô's theory of stochastic calculus. So by taking the Green's function corresponding to the above problem, we may in a formal way rewrite the problem into the stochastic integral equation of the following form;

$$X(t)=f(t,\omega)+\int_0^1 L(t,s,\omega)X(s)ds+\int_0^1 K(t,s,\omega)X(s)d_\varphi Z_s,\tag{2}$$

Here the term $\displaystyle\int d_\varphi Z_t$ represents the noncausal stochastic integral with respect to a properly chosen orthonormal basis $\{\varphi_n\}$ in the real Hilbert space $L^2(0,1)$. Notice that this procedure of transformation is formal but the integral equation of Fredholm type (2) thus derived has a concrete meaning in the theory of noncausal stochastic calculus introduced by the author (cf. the introductory lecture given in [4]).

As a natural generalization of such integral equation, we can think of the integral equation for the random fields, namely the stochastic integral equation for the processes with multi-dimensional parameter $t\in J=[0,1]^d\subset\mathbf{R}^d$.

$$X(t,\omega)=f(t,\omega)+\int_J L(t,s,\omega)X(s)ds+\int_J K(t,s,\omega)X(s)d_\varphi Z(s).\tag{3}$$

Notice that in such case the notion of *Causality* looses its sound meaning because there is no natural order of parameters in the multi-dimensional space, but the stochastic integral with respect to the random field $Z(t,\omega),\ t \in \mathbf{R}^d$ can be treated in the framework of the noncausal theory without any difficulty in causal measurability of the integrands. Following the author's earlier articles [1], [2], [3], we like to show in this note some important results concerning the basic questions of existence and that of uniqueness of solutions, for these two noncausal integral equations (2),(3). To each of these equations we will apply different methods to develop the discussion. In the next paragraph 2 we will discuss the case of the equation (2), and in the paragraph 3 the case of SIEs like (3) for the random fields.

Throughout the discussions, we fix a probability space $(\Omega, \mathcal{F}, P)$ and an underlying driving process with d-dimensional parameter $Z(t,\omega)\ \ t \in [0,1]^d \subset \mathbf{R}^d\ (p \geq 1)$ defined on $(\Omega, \mathcal{F}, P)$, measurable in (t,ω) with respect to the σ field $\mathcal{B}_{R^d} \times \mathcal{F}$. Given these, we understand by the random functions or random fields, those real functions $f(t,\omega),\ t \in [0,1]^d\ (d \geq 1)$, measurable in (t,ω) and almost surely square integrable in t over the interval $[0,1]^d$. Throughout the whole discussions, we also assume that all kernels $L(t,s,\omega),\ K(t,s,\omega)\ \ (s,t \in J = [0,1]^d,\ d \geq 1)$ are almost surely of Hilbert-Schmidt type, that is;

$$P\left[\int_{J\times J}\{L^2(t,s,\omega)+K^2(t,s,\omega)\}dsdt < \infty\right] = 1$$

A random kernel $G(t,s,\omega)$ will introduce the following integral operators acting on the set of random fucntions $X(t)$;

$$\begin{aligned}(GX)(t) &= \int_J G(t,s,\omega)X(s)ds,\\ ((GX))(t) &= \int_J G(t,s,\omega)X(s)d_\varphi Z(s).\end{aligned} \tag{4}$$

2 Uni-dimensional case

We like to begin our study with the noncausal SIE of one dimensional parameter, that is the SIE for the random functions $X(t),\ t \in [0,1]$.

$$X(t) = f(t,\omega) + \alpha\int_0^1 L(t,s,\omega)X(s)ds + \int_0^1 K(t,s,\omega)X(s)d_\varphi Z_s, \tag{5}$$

where α is a real constant.

Following the articles [1], [2] we will show in this paragraph some results, especially the fact that by virtue of the nice properties of our noncausal integral the above SIE can be solved in a very elementary manner.

2.1 Assumptions and notations

As we have already remarked, the integral $\int \acute{d}_\varphi Z_t$ should be understood in the sense of noncausal integral (i.e. the Ogawa integral) with respect to a properly chosen orthonormal basis $\{\varphi_n\} \in L^2([0,1])$. This means that, first of all the fundamental pair $(Z_t, \ \varphi_n\})$ should be nice enough so that the term

$$(\varphi_n, \dot{Z}_t) = \int_0^1 \varphi_n(t) dZ_t$$

is well defined and the definition of the noncausal integral

$$\int_0^1 f(t) d_\varphi Z_t := \sum_n^\infty (f, \varphi_n)(\varphi_n, \dot{Z}) \tag{6}$$

becomes meaningful.

Remark 1 *This constraint can be realized in a natural way when we take as the basis $\{\varphi_n\}$ the system of Haar functions or those orthonormal bases with smooth elements.*

If this random series in (6) converges in probability, we will say that the function f is integrable with respect to the basis $\{\varphi_n\}$ (or φ-integrable for short). Moreover, if the sequence of random functions $\int_0^t f(s) dZ_n^\varphi(s), \ 0 \le t \le 1$ converges (in probability) to the limit $\int_0^t f(s) d_\varphi Z(s)$ in the $L^2(0,1)$-sense,

$$\lim_{n \longrightarrow \infty} \int_0^1 dt \{ \int_0^t f(s) d_\varphi Z(s) - \int_0^t f(s) dZ_n^\varphi(s) \}^2 = 0,$$

we will say that the f is strongly integrable and will denote by $\mathbf{S}$ the totality of all such strongly integrable functions.

For any φ-integrable (in t) random function $f(t,\omega)$ or a kernel $G(t,s,\omega)$, we will use the following convenient notations to denote their noncausal integrals in the parameter t;

$$\tilde{f}(t,\omega) = \int_0^t f(s) d_\varphi Z(s), \quad \tilde{G}(t,s,\omega) = \int_0^t G(r,s,\omega) d_\varphi Z(r).$$

About the choice of the pair $(Z_t, \{\varphi_n\})$ and the regularity of the functions $f, \ K, \ L,$ we put the following assumptions (H);

Assumption (H)

(H,1) $\lim_{n \to \infty} Z_n^\varphi(\cdot)(Z_n^\varphi(1) = Z(\cdot)(Z(1))$ *in proba., where,*

$$Z_n^\varphi(t) = \sum_{k \le n} (\dot{Z}, \varphi_k) \int_0^t \varphi_k(s) ds.$$

Once fixed such pair $(Z_t, \{\varphi_n\})$, we will understand by the noncausal integral the noncausal integral with respect to this pair and denote it simply by the notation, $\int d_* Z(t)$.

(H,2) $f(\cdot), K(\cdot, 1) \in \mathbf{S}$ *and* $(LX), (KX),\ (K_s X) \in \mathbf{S}$ $({}^\forall X \in L^2)$, *where* $K_s(t,s) = \dfrac{\partial}{\partial s} K(t,s)$.

(H,3) $P[\tilde{K}(1,1) \neq 1] = 1$.

2.2 Simple case

We begin with the following simpler equation,

$$X(t) = f(t) + ((KX))(t), \quad ((KX))(t) = \int_0^1 K(t,s)X(s)d_*Z(s) \tag{7}$$

Theorem 2.1 *Under the assumptions (H,1)-(H,3), there is a one-to-one correspondence between the $S-$solution X of (7) and the L^2-solution Y of the random integral equation (8) below;*

$$Y(t) = (\tilde{A}f)(t) + (\tilde{g}Y)(t), \tag{8}$$

where the term (Af) and the random kernel g are as follows,

$$\begin{aligned} (Af)(t) &= f(t) - \frac{\tilde{f}(1)K(t,1)}{\tilde{K}(1,1) - 1} \\ g(t,s) &= \frac{\tilde{K}(t,1)\tilde{K}_s(1,s)}{\tilde{K}(1,1) - 1} - K_s(t,s). \end{aligned} \tag{9}$$

(Proof) This can be easily verified by applying the integration by parts technique to the term $((KX))(t) = \int_0^1 K(t,s)X(s)d_*Z(s)$. In fact, taking the assumption (H) into account, we get

$$((KX))(t) = K(t,1)\tilde{X}(1) - \int_0^1 K_s(t,s)\tilde{X}(s)ds,$$

where, $\tilde{X}(t) = \int_0^t X(r)d_*Z(r)$. Hence,

$$X(t) = f(t) + K(t,1)\tilde{X}(1) - \int_0^1 K_s(t,s)\tilde{X}(s)ds. \tag{10}$$

Substituting this relation into the equation (7) and taking the stochastic integral over $[0,1]$ of the both sides, we get the following equality for the $\tilde{X}(t)$,

$$\tilde{X}(t) = \tilde{f}(t) + \tilde{K}(t,1)\tilde{X}(1) - \int_0^1 K_s(\tilde{t},s)\tilde{X}(s)ds. \tag{11}$$

In particular putting $t = 1$ we get the following expression for $\tilde{X}(1)$,

$$\tilde{X}(1) = \frac{\tilde{f}(1)}{1 - \tilde{K}(1,1)} - \int_0^1 \frac{K_s(1,s)}{1 - \tilde{K}(1,1)} \tilde{X}(s) ds.$$

Substituting this into the equation (11) we find,

$$\tilde{X}(t) = (\tilde{A}f)(t) + (\tilde{g}\tilde{X})(t),$$

which shows that the $Y(t) = \tilde{X}(t)$ is a solution of the random integral equation (8).

Conversely given the L^2-solution $Y(\cdot)$ of the RIE (8), we put

$$X(t) = f(t) + K(t,1)Y(1) - \int_0^1 K_s(t,s)Y(s)ds.$$

Now taking the stochastic integral of both sides of the equation, and comparing with the equation (8) we see that $Y(t) = \tilde{X}(t)$. Hence we find that the X defined as above satisfies the equation (10), and tracing back the integration by parts procedure we confirm that the X is the solution of the SIE (7). □

Since the RIE (8) is a family of the integral equations parametrized by ω and since the kernel $\tilde{g}(t,s,\omega)$ is of Hilbert-Schmidt type for almost all ω, we get the following result by a simple application of the Riesz-Schauder theory.

Corollary 2.1 *The SIE (7) has the unique* **S**-*solution X, provided that the homogeneous equation*

$$X(t) = \int_0^1 K(t,s)X(s)d_*Z(s)$$

does not have a nontrivial **S**-*solution.*

2.3 General case

Let us go back to the general case (5),

$$X(t) = f(t,\omega) + \alpha \int_0^1 L(t,s,\omega)X(s)ds + \int_0^1 K(t,s,\omega)X(s)d_*Z_s.$$

We continue to suppose the same assumptions (H). Then by following a similar argument that we have done for the simpler case (7), we see that to find the **S**-solution is equivalent to find the L^2-solution of the next random integral equation,

$$X(t) = (\mathcal{C}f)(t) + \alpha(\mathcal{B}X)(t), \tag{12}$$

where

$$(\mathcal{C}f)(t) = f(t) + \{(I-G)^{-1}(\tilde{A}f)\}(1)K(t,1) - \{K_s(I-G)^{-1}(\tilde{A}f)\}(t),$$

$$\mathcal{B}(t,s,\omega) = L(t,s) + K(t,1)\{(I-G)^{-1}(\tilde{A}L(\cdot,s))\}(1) - \{K_s(I-G)^{-1}(\tilde{A}L(,s))\}(t),$$

$$G(t,s,\omega) = \tilde{g}(t,s,\omega).$$

Notice that the kernel $\mathcal{B}(t,s,\omega)$ is again of Hilbert-Schmidt type for almost all ω. Therefore for every constant α with at most countable exceptions the operator $(I - \alpha\mathcal{B})(\omega)$ is invertible and in such case the equation (12) has the unique solution which must belong to the class **S**. The converse can be easily verified, we confirm the next result,

Theorem 2.2 *Under the assumptions (H), the SIE (5) has the unique* **S***-solution for every* α *with at most countable exceptions.*

3 Multi dimensional case

Let $Z(t,\omega)$ $((t,\omega) \in R^d \times \Omega)$ be such that the derivative, $\dot{Z}(t,\omega) = \dfrac{\partial^d}{\partial t_1 \cdots \partial t_d} Z(t,\omega)$ is well defined as a $L^2(\Omega)$-valued generalized random field on the Schwartz space $\mathcal{S}(R^d)$. We suppose the Z to have nice property such that the application,

$$\mathbf{S} \ni \varphi(x) \Longrightarrow \dot{Z}(\phi) = (\dot{Z}, \varphi) \in L^2(\Omega),$$

becomes continuous with respect to the topology in $L^2(R^p)$. Thus the application can be extended over the $L^2(R^p)$. Now let $\{\varphi_n\}_{n=1}^{\infty}$ be a complete orthonormal basis in the real Hilbert space $L^2(J)$.

Definition 1 *The stochastic integral* $\int f(t,\omega) d_\varphi Z(t)$ *of a random field* $f(t,\omega)$ *with respect to the pair* $(Z, \{\varphi_n\})$ *is defined as being the limit in probability of the following random series,*

$$\sum_{n=1}^{\infty} (f, \varphi_n)(\varphi_n, \dot{Z}).$$

In this paragraph, we are going to study the basic properties of the SIE of Fredholm type (3) for the random fields,

$$X(t,\omega) = f(t,\omega) + \alpha \int_J L(t,s,\omega) X(s) ds + \beta \int_J K(t,s,\omega) X(s) d_\varphi Z(s),$$
$$(\alpha, \beta = \text{constants}),$$

and show some results mainly following the article [3].

For the SIE in one-dimensional parameter case we have solved the equation by applying the integration by parts method, which does not work for such equation of multi-dimensional parameters. Thus to solve the above SIE we need to introduce another technique, a kind of stochastic Fourier transformation $\mathcal{T}_\epsilon$. This can be done when we suppose a kind of smoothness of the stochastic kernels and functions involved in the equation.

3.1 Stochastic Fourier transformation

For the simplicity of discussions, we will fix once for all, another orthonormal basis, $\{\psi_n\}$ in an arbitrary way and we set the next assumption (A) which concerns a regularity of the random kernels, K, L.

Assumption (A); *There exists a positive sequence* $\{\epsilon_n\}$ *such that,*

(A,1) $\{\epsilon_n\epsilon_m\gamma_{m,n}\} \in l^2$ *(P-a.s), where* $\gamma_{m,n} = \int \psi_m(t)\psi_n(t)d_\varphi Z(t)$,

(A,2) $\{k'_{m,n}\}$, $\{l'_{m,n}\} \in l^2$ *(P-a.s.) where* $k'_{m,n} = k_{m,n}/\epsilon_m\epsilon_n$, $l'_{m,n} = l_{m,n}/\epsilon_m$, $k_{m,n} = (K, \psi_m \otimes \psi_n)$, $l_{m,n} = (L, \psi_m \otimes \psi_n)$.

We will call such sequence $\{\epsilon_n\}$ *the admissible weight.*

Notice that if $\{\epsilon_n\}$, $\{\eta_n\}$ are admissible weights then the sequences, $\{(\epsilon \wedge \eta)_n\}$, $\{(\epsilon \vee \eta)_n\}$, given by $(\epsilon \wedge \eta)_n = \min \epsilon_n, \eta_n\}$, $(\epsilon \vee \eta)_n = \max\{\epsilon_n, \eta_n\}$, are also admissible. [Brownian sheet] In the case that Z is the Brownian sheet and $\{\psi_n\}$ is such that all elements are uniformly bounded on J, then any positive l^2–sequence satisfies the condition (A,1).

Definition 2 (ϵ-smoothness) *We will say that a random field* $g(t,\omega)$ *admits a sequence* $\{\epsilon_n\}$ *as the weight (or shortly,* $\{\epsilon_n\}$*-smooth) if there exists an admissible weight* $\{\epsilon_n\}$, *such that ;*

(t,1) The integral $\hat{g}_n = \int g(t,\omega)\psi_n(t)d_\varphi Z(t)$ *exists for all* $n \in \mathbf{N}$ *and* $\epsilon_n\hat{g}_n\} \in l^2$ $(P-a.s.)$.

(t,2) The following limit converges in probability,

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} \{\epsilon_n(\hat{g}_n - \int g(t,\omega)\psi_n(t)dZ^\varphi_m(t))\}^2 = 0.$$

We will denote by $\mathbf{S}_2$ *the totality of all such random fields that are* $\{\epsilon\}$*-smooth for some admissible weight* $\{\epsilon_n\}$.

It is easy to check that if a $\mathbf{S}_2$-field $g(t,\omega)$ admits two sequences, $\{\epsilon_n\}$, $\{\eta_n\}$ as the weights, then it also admits the sequences $\{(\epsilon \wedge \eta)_n\}$, $\{(\epsilon \vee \eta)_n\}$ as the weights.

Remark 2 *In the case that* $Z =$ *the Brownian sheet and the all elements of* $\{\psi_n\}$ *are uniformly bounded, we see that* $\mathbf{S}_2 \supset L^2(J)$ *and that every admissible sequence can be the weight for any* $g(t,\omega) \in L^2(J)$.

Associated to the notion of $\mathbf{S}_2$-fields, we introduce the linear stochastic transformation, $\mathcal{T}_\epsilon$ acting on $\mathbf{S}_2$, in the following,

Definition 3 (Stochastic Fourier transformation) *For a* $g(t,\omega) \in \mathbf{S}_2$ *admitting a* $\{\epsilon_n\}$ *as the weight, we set,*

$$\begin{aligned}(\mathcal{T}_\epsilon g)(t) \;&= \sum_n \epsilon_n \hat{g}_n \psi_n(t),\\ \textit{where} \quad \hat{g}_n(\omega) &= \int_J g(t,\omega)\psi_n(t) d_\varphi Z(t).\end{aligned} \tag{13}$$

We should notice that the transformation $\mathcal{T}_\epsilon$ depends on the weight $\{\epsilon_n\}$ and that for any $\{\epsilon\}$-smooth g, we have $\mathcal{T}_\epsilon g \in L^2(J)$, $(P-a.s.)$.

3.2 Results

Theorem 3.1 *For any* $f(t,\omega) \in \mathbf{S}_2$ *the following integral equation*

$$X(t,\omega) = f(t,\omega) + \int_J K(t,s,\omega) X(s) d_\varphi Z(s), \tag{14}$$

has the unique $\mathbf{S}_2$*-solution provided that the next condition (C) holds,*

(C); the homogeneous equation, $X(t,\omega) = \int_J K(t,s,\omega)X(s)d_\varphi Z(s)$, *does not have nontrivial* $\mathbf{S}_2$*-solutions.*

(Proof) Let $\{\epsilon_n\}$ be an admissible weight for the random field $f(t,\omega)$.
First we are going to show that the condition (C) is sufficient to assure the existence of a $\mathbf{S}_2$-solution X, which is unique among those functions that admit the same weight $\{\epsilon_n\}$.

Let X be an $\{\epsilon\}$-smooth solution of (14). Then, since

$$K(t,s,\omega) = \sum_{m,n} k_{m,n}\psi_m(t)\psi_n(s),$$

we get the following relation (15) by virtue of the condition (t,2),

$$\begin{aligned}X(t) &= f(t) + \sum_{m,n} \epsilon_m \epsilon_n k'_{m,n} \psi_m(t) \hat{x}_n,\\ \text{where} \quad \hat{x}_n(\omega) &= \int_J X(t,\omega)\psi_n(t) d_\varphi Z(t).\end{aligned} \tag{15}$$

Multiplying by $\psi_l(t)$ and taking the stochastic integration over J on both sides of the equation (15), we obtain, under the assumption (A,2) the next relation,

$$\hat{x}_l = \hat{f}_l + \sum_{m,n} \gamma_{l,m} k_{m,n} \hat{x}_n, \quad (\forall l \in N) \tag{16}$$

or equivalently,

$$\epsilon_l \hat{x}_l = \epsilon_l \hat{f}_l + \sum_{m,n} \epsilon_l \epsilon_m \gamma_{l,m} k'_{m,n} \epsilon_n \hat{x}_n. \tag{17}$$

So if we set,

$$\bar{K}(t,s,\omega) = \sum_{l,n} \epsilon_l \{\sum_m \epsilon_m \gamma_{l,m} k'_{m,n}\} \psi_l(t) \psi_n(s),$$

then by virtue of the condition (t,1), we see that the kernel $\bar{K}(\cdot,\cdot,\omega)$ is of Hilbert-Schmidt type for almost all ω and that the field, $Y = (\mathrm{T}_\epsilon X)(t,\omega)$ satisfies the following random integral equation,

$$Y(t) = (\mathcal{T}_\epsilon f)(t) + \int_J \bar{K}(t,s,\omega) Y(s) ds. \tag{18}$$

Conversely if we set $\hat{x}_n = (Y, \psi_n)/\epsilon_n$ for an L^2- solution Y of (18), then we see that the $\{\hat{x}_n\}$ satisfies the equation (16) and so the field $X(t)$ defined through the relation (15) becomes an $\mathbf{S}_2$–solution of (14). As is easily seen, this correspondence between the $\{\epsilon\}$-smooth solution of (14) and the L^2–solution of (18) is one-to-one and onto. Thus the question of the existence and the uniqueness of the $\{\epsilon\}$-smooth solution is reduced to the same question about the L^2–solutions of (18). Hence, by a simple application of The Riesz-Schauder Theory, we confirm that the condition (C) is sufficient for the validity of the prescribed result.

Next, we are going to show that this solution X which has the $\{\epsilon_n\}$ as the weight is unique among all $\{\epsilon\}$- smooth fields. So let X' be another $\mathbf{S}_2$–solution of (14) having a different sequence $\{\eta\}$ as the weight. Then it satisfies a similar relation as (15) from which we see the field $f(t,\omega)$ is $\{\eta\}$-smooth. Since all $\mathbf{S}_2$-fields f, X and X' are $\{(\epsilon \wedge \eta)\}$-smooth, the field X' and X must coincide with each other as the unique $\mathbf{S}_2$-solution admitting the same sequence as the weight. □

Corollary 3.1 *If all elements of the c.o.n.s.* $\{\psi_n\}$ *are continuous and uniformly bounded over* J *and if almost all sample of the field* $f(t,\omega)$ *are continuous. Then the* $\mathbf{S}_2$*-solution of (14) is also almost surely sample-continuous.*

(Proof) Evident from the equality (15) and the fact that,

$$\sum_{m,n} |\epsilon_m k'_{m,n} \epsilon_n \hat{x}_n| < +\infty \quad \text{(P-a.s.)}$$

□

Now we are to give a result for the general case (3) in the next,

Proposition 3.1 *Let* $f(t,\omega) \in \mathbf{S}_2$ *then for almost all* $\alpha,\ \beta \in R^1$ *the equation (3) has a unique* $\mathbf{S}_2$*- solution.*

(Proof) We notice that the condition (A,2) implies ;

$$(LX)(t,\omega) = \int_J L(t,s,\omega)X(s)ds \in \mathbf{S}_2$$

for any random field X. Let $\{\epsilon_n\}$ be a weight for the $f(t,\omega)$. Then following the same discussion as in the proof of Theorem (3.1), we see that any $\mathbf{S}_2$-solution X admitting the $\{\epsilon_n\}$ as weight, if exists, satisfies the following equation,

$$Y(t,\omega) = \{\mathcal{T}_\epsilon(f+\alpha LX)\}(t,\omega) + \beta \int_J \bar{K}(t,s,\omega)Y(s)ds, \tag{19}$$

where $Y(t,\omega) = (\mathcal{T}_\epsilon X)(t,\omega)$.

Since the operator $\bar{K}(\omega)$, given by ;

$$(L^2(J) \ni)Y \longrightarrow (\bar{K}Y)(t,\omega) = \int_J \bar{K}(t,s,\omega)Y(s)ds\ (\in L^2(J)),$$

is compact for almost all ω, we know that for all (but with at most countable exception) of β the operator $(I-\beta\bar{K})$ is invertible and for such β we get, by solving (19) in Y, the following expression,

$$(\mathcal{T}_\epsilon X)(t) = f_1(t) + \alpha(L'_\beta X)(t) \tag{20}$$

where

$$f_1(t) = (I-\beta\bar{K})^{-1}(\mathcal{T}_\epsilon f)(t)$$

and

$$(L'_\beta X)(t) = \{(I-\beta\bar{K})^{-1}(LX)\}(t).$$

On the other hand we have the next relation which can be derived in a same way as in the derivation of the (15);

$$X(t) = f(t) + \alpha(LX)(t) + \beta(K_1)\mathcal{T}_\epsilon X)(t) \tag{21}$$

where

$$(K_1Y)(t) = \int_J K_1(t,s,\omega)Y(s)ds,$$
$$K_1(t,s,\omega) = \sum_{m,n}(k_{m,n}/\epsilon_n)\psi_m(t)\psi_n(s).$$

Substituting the relation (20) into (21), we find that the solution X, if exists, satisfies the next

$$X(t) = f_2(t) + \alpha(L''X)(t) \tag{22}$$

where

$$f_2(t) = f(t) + \beta(K_1f_1)(t), \quad \text{and}$$
$$(L''Y)(t) = \{(L+\beta K_1L'_\beta)Y\}(t)\ (Y \in L^2(J)).$$

The operator L'' being compact for almost all ω, the equation (10) has for almost all α a unique $\mathbf{S}_2$-solution. Moreover, it is immediate to see, following the same argument as in the proof of Theorem 3.1, that this solution does not depend on the choice of the weight $\{\epsilon_n\}$ for the $f(t,\omega)$. $\square$

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