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# Closed dissipative systems: the sealed variation (閉じた散逸系：封印された変分構造)

Takashi SUZUKI (鈴木貴・阪大基礎工)*

**Abstract**

We pick up several model (C) equations, provided with the (skew) Lagrangian, especially, the ones with semi-dual variation, to discuss the dynamical stability of stationary solutions in a unified way.

## 1 Introduction

Thermal phenomena are described by dissipative systems. They are classified into isolated, closed, and open systems, provided with the microscopic structure based on micro-cannonical, cannonical, and grand-cannonical statistical mechanics, respectively. According to the "triple seale of self-organization", first, several features of self-organization are sealed in the closed system, second, total set of stationary states controls the non-equilibrium, and finally, the stationary states themselves are sealed in the (skew) Lagrangian, provided with the structure of dual variation.

Ginzburg-Landau theory is a phenomenology, consistent to the non-equilibrium thermodynamics. It is based on a (quasi-)free energy, denote by $\mathcal{F}$, associated with the order parameter $\varphi$. Then, the non-equilibrium mean field equation is described by the chemical potential

$$\mu = \delta\mathcal{F}(\varphi),$$

and is classified into model (A), (B), and (C) equations [13, 14].

*Division of Mathematical Science, Department of System Innovation, Graduate School of Engineering Science, Osaka University

In more detail, if $\Omega \subset \mathbf{R}^n$ $(n = 2, 3)$ denotes a bounded domain with smooth boundary $\partial\Omega$, then $\varphi$ is a function of the position $x \in \Omega$ and the time $t > 0$ indicating the status of the material, and $\mathcal{F}$ is a quantity determined by this $\varphi$. Thus, $\mathcal{F} = \mathcal{F}(\varphi)$ is regarded as a functional of $\varphi = \varphi(x, t)$, and its variation, $\delta\mathcal{F}(\varphi)$ is defined by

$$\langle \psi, \delta\mathcal{F}(\varphi) \rangle = \frac{d}{ds}\mathcal{F}(\varphi + s\psi)\Big|_{s=0}.$$

If $\langle\ ,\ \rangle$ is identified with the $L^2$ inner product, then model (A) equation is formulated as a gradient system,

$$\varphi_t = -K\delta\mathcal{F}(\varphi) \qquad \text{in } \Omega \times (0, T),$$

where $K$ is a positive quantity, possibly assocaited with $\varphi$. Then, it holds that

$$\frac{d}{dt}\mathcal{F}(\varphi) = -\int_\Omega K\delta\mathcal{F}(\varphi)^2 \le 0.$$

Model (B) equation, on the other hand, is described by

$$\varphi_t = \nabla \cdot (K\nabla\delta\mathcal{F}(\varphi)) \qquad \text{in } \Omega \times (0, T)$$
$$K\frac{\partial}{\partial\nu}\delta\mathcal{F}(\varphi)\Big|_{\partial\Omega} = 0.$$

In this case, we obtain

$$\frac{d}{dt}\int_\Omega \varphi = \int_{\partial\Omega} K\frac{\partial}{\partial\nu}\delta\mathcal{F}(\varphi) = 0$$
$$\frac{d}{dt}\mathcal{F}(\varphi) = -\int_\Omega K\,|\nabla\delta\mathcal{F}(\varphi)|^2 \le 0.$$

The stationary state is defined by the zero "free energy consumption", and therefore,

$$\delta\mathcal{F}(\varphi) = 0$$

in the model (A) equation, while

$$\delta\mathcal{F}(\varphi) = 0 \quad \text{constrained by} \quad \int_\Omega \varphi = \lambda$$

in the stationary (B) equation, where $\lambda$ is a prescribed constant. More precisely, stationary state of the model (A) equation is defined by

$$\frac{d}{ds}\mathcal{F}(\varphi + s\psi)\Big|_{s=0} = 0 \qquad \text{for all } \psi.$$

The model (B) equation, on the other hand, is concerned with the closed system, and the stationary state is defined by

$$\frac{d}{ds}\mathcal{F}(\varphi + s\psi)\Big|_{s=0} = 0 \qquad \text{fo all } \psi \text{ with } \int_\Omega \psi = 0$$
$$\int_\Omega \varphi = \lambda.$$

Similarly, linearized stability of the stationary state $\varphi$ means

$$Q(\psi,\psi) \equiv \frac{1}{2}\frac{d^2}{ds^2}\mathcal{F}(\varphi + s\psi)\Big|_{s=0} > 0 \qquad \text{for all } \psi \neq 0$$

in model (A) equation, while

$$Q(\psi,\psi) > 0 \qquad \text{for all } \psi \neq 0 \text{ with } \int_\Omega \psi = 0$$

in model (B) equation.

**Example 1** *Ginzburg-Landau's free energy,*

$$\mathcal{F}(\varphi) = \int_\Omega \frac{\xi^2}{2}|\nabla\varphi|^2 + W(\varphi)$$

*induces the Allen-Cahn equation [1]*

$$\varphi_t = K(\xi^2\Delta\varphi - W'(\varphi)) \qquad \text{in } \Omega \times (0,T)$$

*in phase separation as the model (A) equation, where $\xi > 0$ is a constant associated with the intermolecular force, $\varphi = \varphi(x,t)$ is the order parameter, $K > 0$ is a constant, and*

$$W(\varphi) = \frac{\varphi^4}{4} - \frac{\varphi^2}{2}.$$

*This $W = W(\varphi)$ is a double-well potential, and hence $\varphi = \pm 1$ are its bi-stable critical point. On the other hand, $\frac{\xi^2}{2}|\nabla\varphi|^2$ is the penalty term of van der Waals, associated with the surface tension. Usually, $\mathcal{F} = \mathcal{F}(\varphi)$ is taken to all $\varphi \in H^1(\Omega)$, and then the natural boundary condition*

$$\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

*is furthermore imposed.*

*The stationary state is described by*

$$-\xi^2\Delta\varphi = \varphi - \varphi^3 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega,$$

*and its stability is equivalent to the positivity of the first eigenvalue of the self-adjoint operator in* $L^2(\Omega)$,

$$A = -\xi^2\Delta - 1 + 3\varphi^2,$$

*with the domain*

$$D(A) = \left\{\psi \in H^2(\Omega) \mid \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0\right\}.$$

*From the general theory [20], any non-constant stationary solution* $\varphi$ *is linearly unstable if* $\Omega$ *is convex.*

**Example 2** *The same free energy induces the Cahn-Hilliard equation [4]*

$$\varphi_t = -K\Delta(\xi^2\Delta\varphi - W'(\varphi)) \qquad \text{in } \Omega \times (0,T)$$
$$\left.\frac{\partial}{\partial\nu}(\xi^2\Delta\varphi - W'(\varphi))\right|_{\partial\Omega} = 0$$

*in phase separation as the model (B) equation. Similarly to the above case, usually we impose*

$$\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

*furthermore, using* $\mathcal{F}(\varphi)$ *for all* $\varphi \in H^1(\Omega)$. *This means*

$$\varphi_t = -K\Delta(\xi^2\Delta\varphi - W'(\varphi)) \qquad \text{in } \Omega \times (0,T)$$
$$\frac{\partial\Delta\varphi}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega \times (0,T).$$

*The stataionary state* $\varphi$ *is defined by*

$$-\xi^2\Delta\varphi = \varphi - \varphi^3 - \frac{1}{|\Omega|}\int_\Omega(\varphi - \varphi^3) \quad \text{in } \Omega$$
$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega \varphi = \lambda,$$

*and its linearized stability is the positivity of the first eigenvalue of the self-adjoint operator in* $L^2_0(\Omega)$

$$A = -\xi^2\Delta + 1 - 3\varphi^2$$

*with the domain*

$$D(A) = \left\{\psi \in (H^2 \cap L^2_0)(\Omega) \mid \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0\right\},$$

*where*

$$L^2_0(\Omega) = \left\{\psi \in L^2(\Omega) \mid \int_\Omega \psi = 0\right\}.$$

**Example 3** *Ohta-Kawasaki's free energy [24],*

$$\mathcal{F}(\varphi) = \int_\Omega \frac{\xi^2}{2}|\nabla\varphi|^2 + W(\varphi) + \frac{\sigma}{2}\left|(-\Delta_N)^{-1/2}(\varphi - \overline{\varphi})\right|^2$$

*induces the Nishiura-Ohnishi equation [23] concerning the micro-phase separation in diblock copolymers,*

$$\begin{aligned}
&\varphi_t = -\Delta\left(\xi^2\Delta\varphi - W'(\varphi)\right) - \sigma(\varphi - \overline{\varphi}) \qquad \text{in } \Omega \times (0,T)\\
&\left.\frac{\partial}{\partial\nu}\left\{\xi^2\Delta\varphi - W'(\varphi)\right\}\right|_{\partial\Omega} = 0
\end{aligned}$$

*as the model (B) equation, where* $\sigma > 0$ *is a parameter associated with the length of the polymer chain and*

$$\overline{\varphi} = \frac{1}{|\Omega|}\int_\Omega \varphi.$$

*Similarly, we impose*

$$\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

*using all* $\varphi \in H^1(\Omega)$ *to calculate* $\delta\mathcal{F}(\varphi)$*, and this implies*

$$\begin{aligned}
&\varphi_t = -\Delta\left(\xi^2\Delta\varphi - W'(\varphi)\right) - \sigma(\varphi - \overline{\varphi}) \qquad \text{in } \Omega \times (0,T)\\
&\frac{\partial\Delta\varphi}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega.
\end{aligned}$$

*The stationary state is described by*

$$-\xi^2\Delta\varphi = \varphi - \varphi^3 - \frac{1}{|\Omega|}\int_\Omega(\varphi-\varphi^3) + \sigma\int_\Omega G(\cdot,x')\varphi(x')dx' \quad \text{in } \Omega$$

$$\frac{\partial\varphi}{\partial\nu} = 0 \qquad \text{on } \partial\Omega, \qquad \int_\Omega \varphi = \lambda,$$

*where* $G = G(x,x')$ *denotes the Green's function to*

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \left.\frac{\partial v}{\partial\nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0. \tag{1}$$

*Then, the linearized stablity of this stationary state is defined by the positivity of the first eigenvalue of the self-adjoint operator* $A$ *in* $L^2_0(\Omega)$ *defined by*

$$A\psi = -\xi^2\Delta\psi - \psi + 3\varphi^2\psi + \sigma\int_\Omega G(\cdot,x')\psi(x')dx',$$

*with the domain*

$$D(A) = \left\{\psi \in (H^2 \cap L^2_0)(\Omega) \mid \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0\right\}.$$

**Example 4** *Helmholtz' free energy*

$$\mathcal{F}(u) = \alpha\int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u(x)u(x')dxdx'$$

*induces the mean field equation of many self- gravitating particles, where* $u = u(x,t)$ *denotes the particle density. If the absolute temperature* $\alpha$ *is equal to* 1, *and the potential* $G = G(x,x')$ *is the Green's function to (1), then we obtain the simplified system of chemotaxis [16] as the model (B) equation with* $K = u$*:*

$$u_t = \nabla\cdot(u\nabla\delta\mathcal{F}(u)), \quad \left.u\frac{\partial}{\partial\nu}\delta\mathcal{F}(u)\right|_{\partial\Omega} = 0,$$

*that is,*

$$u_t = \nabla\cdot(\nabla u - u\nabla v)$$

$$
\begin{aligned}
&-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T)\\
&\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)\\
&\int_\Omega v = 0 \qquad (0 < t < T).
\end{aligned}
$$

*The stationary state is reduced to*

$$
-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right), \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0,
$$

*and in two space dimension, the quantized blowup mechanism of this state implies that of the non-equilibrium [29]. We note that the second term of this free energy is essentially the same as that of Ohta-Kawaski' s free energy.*

If the temperature $\alpha$ varies, it is preferable to use the equation provided with the increase of entropy other than the decrease of free energy [28], and then, Penrose-Fife and coupled Cahn-Hilliard equations are obtained for the phase transition and the phase separation, respectively.

# 2 Semi-unfolding-minimality

The purpose of the present paper is to pick up a common variational structure in several model (C) equations, and is to provide a unified approach to their dynamics. First, several phenomenological equations are provided with the Lyapunov function, and this functional induces a semi-dual variational structure to the stationary state, especially to the field component. In many cases, this structure guarantees the dynamical stability of the linearly stable stationary state, because the particle component is trivial in the stationary state. If the system is closed concerning the particle component, then this stationary state is realized as a nonlinear eigenvalue problem with non-local term.

**Example 5** *The first model (C) equation is the Fix-Caginalp equation [10, 5, 3, 19] describing non-isothermal phase transition:*

$$
\begin{aligned}
&\tau\varphi_t = \xi^2\Delta\varphi + (\varphi - \varphi^3) + 2u\\
&u_t + \frac{\ell}{2}\varphi_t = \kappa\Delta u \qquad \text{in } \Omega \times (0,T),
\end{aligned}
$$

*where* $\tau = K^{-1} > 0$, $\ell > 0$, $\kappa > 0$, $\varphi = \varphi(x,t)$, *and* $u = u(x,t)$ *denote relaxization time, latent heat, conductivity, order parameter, and relative temperature, respectively. This is a coupling of the model (A) equation using the free energy*

$$\mathcal{F}_u(\varphi) = \int_\Omega \frac{\xi^2}{2}|\nabla\varphi|^2 + W(\varphi) - 2u\varphi$$

*and the enthalpy equation for two phase Stefan problem:*

$$\left(u + \frac{\ell}{2}\varphi\right)_t = \kappa\Delta u.$$

*This free energy describes that the equilibrium is* $\varphi = \pm 1$ *with* $u = 0$.

*Actually, in the classical formulation [25, 17], the enthalpy* $H = u + \frac{\ell}{2}\varphi$ *is the maximal graph defined by the relation*

$$\varphi = \begin{cases} 1 & (u > 0) \\ -1 & (u < 0). \end{cases}$$

*Then, this system of equation is obtained by reformulating* $\varphi$ *as an order parmeter, subject to the above free energy.*

*If this system is open, then it holds that*

$$\frac{\partial\varphi}{\partial\nu} = u = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{2}$$

*In this case, we have*

$$\tau\|\varphi_t\|_2^2 = -\frac{\xi^2}{2}\frac{d}{dt}\|\nabla\varphi\|_2^2 - \frac{d}{dt}\int_\Omega W(\varphi) + 2(u,\varphi_t)$$
$$\frac{1}{2}\|u\|_2^2 + \frac{\ell}{2}(\varphi_t, u) = -\frac{\kappa}{2}\|\nabla u\|_2^2$$

*and therefore,*

$$\frac{d}{dt}\left\{\frac{1}{2}\|u\|_2^2 + \frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2 + \frac{\ell}{4}\int_\Omega W(\varphi)\right\}$$
$$= -\frac{\tau\ell}{4}\|\varphi_t\|_2^2 - \frac{\kappa}{2}\|\nabla u\|_2^2 \le 0. \tag{3}$$

*Thus,*

$$\mathcal{L}(\varphi,u) = \frac{1}{2}\|u\|_2^2 + \frac{\ell\xi^2}{8}\|\nabla\varphi\|_2^2 + \frac{\ell}{4}\int_\Omega W(\varphi)$$

*acts as a Lyapunov function.*

*In the stationary state, we have*

$$u = \overline{u} \equiv 0$$

*from the enthalpy equation*

$$\left(u + \frac{\ell}{2}\varphi\right)_t = \kappa \Delta u, \qquad u|_{\partial\Omega} = 0,$$

*and therefore, $\varphi = \overline{\varphi}$ satisfies*

$$-\xi^2 \Delta \varphi = \varphi - \varphi^3 \quad \text{in } \Omega, \qquad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

*from the order parameter equation. The latter problem has the variational structure defined by Ginzburg-Landau's free energy,*

$$\mathcal{J}(\varphi) = \int_\Omega \frac{\xi^2}{2} |\nabla \varphi|^2 + W(\varphi), \qquad \varphi \in H^1(\Omega).$$

*Thus, it is equivalent to $\delta \mathcal{F}(\varphi) = 0$ for $\varphi \in H^1(\Omega)$. Then, we obtain the semi-unfolding-minimality,*

$$\mathcal{L}(\varphi, u) \geq \mathcal{L}(\varphi, \overline{u}) = \mathcal{J}(\varphi).$$

**Example 6** *If the Fix-Caginalp system is closed, then it holds that*

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T)$$

*for (2). Equality (3) is valid even in this case, and the above $\mathcal{L}(u, \varphi)$ is again a Lyapunov function. The total enthalpy, on the other hand, is preserved in this case, and it holds that*

$$\frac{d}{dt} \int_\Omega \left(u + \frac{\ell}{2}\varphi\right) = 0.$$

*From the enthalpy equation*

$$\left(u + \frac{\ell}{2}\varphi\right)_t = \kappa \Delta u, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0,$$

*the stationary state of $u$ is a constant. This unknown constant $u = \overline{u}$ is to be determined by this invariant quantity, denoted by $a$. Thus, we obtain*

$$-\xi^2\Delta\varphi = \varphi - \varphi^3 + 2\overline{u} \quad in\ \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad on\ \partial\Omega$$
$$\overline{u}\,|\Omega| + \frac{\ell}{2}\int_\Omega \varphi = a$$

*or equivalently,*

$$-\xi^2\Delta\varphi = \varphi - \varphi^3 + \frac{2}{|\Omega|}\left(a - \frac{\ell}{2}\int_\Omega \varphi\right) \qquad in\ \Omega$$
$$\frac{\partial\varphi}{\partial\nu} = 0 \qquad on\ \partial\Omega.$$

*Regarding this $a$ as an eigenvalue, we see that the stationary state of this closed system is realized as a nonlinear eigenvalue problem with non-local term.*

*This problem has a variational function*

$$\mathcal{J}_a(\varphi) = \frac{\xi^2}{2}\|\nabla\varphi\|_2^2 + \int_\Omega W(\varphi) + \frac{2}{\ell\,|\Omega|}\left\{a - \frac{\ell}{2}\int_\Omega \varphi\right\}^2$$

*defined for $\varphi \in H^1(\Omega)$. Then, the semi-unfolding-minimality is obtained as*

$$\mathcal{L}(\overline{u}, \varphi) = \frac{\ell}{4}\mathcal{J}_a(\varphi)$$
$$\mathcal{L}(u, \varphi) \geq \mathcal{L}(\overline{u}, \varphi) \qquad for \quad \int_\Omega \left(u + \frac{\ell}{2}\varphi\right) = a$$

*by*

$$\left(\frac{1}{|\Omega|}\int_\Omega u\right)^2 \leq \frac{1}{|\Omega|}\int_\Omega u^2.$$

Such a structure of semi-unfolding-minimality is observed in the Penrose-Fife system of phase transition [28], coupled Cahn-Hilliard equation of phase separation [28, 2], and the Ginzburg-Landau theory for shape memory alloys [7, 8, 26, 27]. Several fundamental equations of self-interacting fluids have the same structure [15, 18]. See [30].

# 3 (skew) Gradient systems

Several other systems are derived from (skew) Lagrangian, with the stationary states being hard to reduce single equations.

In the gradient system, the Lagrangian acts as a Lyapunov function. For example, in the model (A) - model (B) equation

$$u_t = -L_u, \qquad \tau v_t = -L_v$$

it holds that

$$\frac{d}{dt}L(u,v) = -\int_\Omega L_u(u,v)^2 + \tau^{-1}L_v(u,v)^2 \le 0.$$

Then, the stationary state $(\overline{u},\overline{v})$ is defined by

$$L_u(\overline{u},\overline{v}) = L_v(\overline{u},\overline{v}) = 0,$$

and its linearized stability is formally described by the positivity of

$$A = \begin{pmatrix} L_{uu}(\overline{u},\overline{v}) & L_{uv}(\overline{u},\overline{v}) \\ L_{vu}(\overline{u},\overline{v}) & L_{vv}(\overline{u},\overline{v}) \end{pmatrix}.$$

This is nothing but the Hessian of $L$, and thus, linearly stable stationary solution derived from this Lagrangian is dynamically stable.

In the skew-gradient sytem using the skew Lagrangian, e.g.

$$u_t = -L_u, \qquad \tau v_t = L_v,$$

the stationary state is similarly defined by

$$L_u(\overline{u},\overline{v}) = L_v(\overline{u},\overline{v}) = 0,$$

and the linearized equation is formally

$$\frac{d}{dt}\begin{pmatrix} u \\ \tau v \end{pmatrix} + A\begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

where

$$A = \begin{pmatrix} L_{uu}(\overline{u},\overline{v}) & L_{uv}(\overline{u},\overline{v}) \\ -L_{uv}(\overline{u},\overline{v}) & L_{vv}(\overline{u},\overline{v}) \end{pmatrix}.$$

Then, its linearized stablity means that any eigenvalues of $A$ is in the right-half space, or

$$Re\ (Aw, w) > 0 \qquad \text{for all } w = \begin{pmatrix} u \\ v \end{pmatrix} \neq 0.$$

This condition is equivalent to the positivities of both $L_u(\overline{u}, \overline{v})$ and $L_v(\overline{u}, \overline{v})$, and such a stationary state is dynamically stable [31, 32].

**Example 7** *Eguchi-Oki-Matsumura equation [6] on phase separation of alloys is the model (B) - model (C) equation, using the Lagrangian*

$$\mathcal{L}(u, v) = \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{\xi^2}{2} |\nabla v|^2 + f(u, v),$$

*where*

$$f(u, v) = \frac{a}{2}u^2 - \frac{b}{2}v^2 + \frac{b'}{4}v^4 + \frac{g}{2}u^2v^2.$$

*Here, $u$ and $v$ stand for the concentration of the main component and the order parameter, respectively. Thus, we obtain*

$$\begin{aligned} &\tau u_t = \nabla \cdot \nabla \mathcal{L}_u(u, v) \\ &v_t = -\mathcal{L}_v(u, v) \qquad \text{in } \Omega \times (0, T) \\ &\frac{\partial}{\partial \nu}\mathcal{L}_u(u, v) = 0 \qquad \text{on } \partial\Omega \times (0, T), \end{aligned}$$

*and hence*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u = 0 \\ &\frac{d}{dt}\mathcal{L}(u, v) = -\int_\Omega \tau^{-1} |\nabla \mathcal{L}(u, v)|^2 + v_t^2 \leq 0. \end{aligned}$$

*Using all $u \in H^1(\Omega)$ and $v \in H^1(\Omega)$ in calculating $L_u$ and $L_v$, we obtain*

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T)$$

*as a natural boudary condition. Then, the stationary state is described by*

$$\begin{aligned} &-\Delta u + au + guv^2 = \text{constant}, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega u = \lambda \\ &-\xi^2 \Delta v - bv + gu^2 v = 0, \qquad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0. \end{aligned}$$

**Example 8** *Gierer-Meinhardt equation of morphogenesis [11] is the model (A) - model (A) equation*

$$ra_t = -\mathcal{L}_a, \qquad q\tau h_t = \mathcal{L}_h$$

*derived from the skew Lagrangian*

$$\mathcal{L}(a,h) = \int_\Omega \frac{r\varepsilon}{2}|\nabla a|^2 - \frac{qD}{2}|\nabla h|^2 - H(a,h)$$

*with*

$$H(a,h) = -\frac{r}{2}a^2 + r\sigma a + a^{p+1}h^q + \frac{q}{2}h^2$$

*in the case of* $p+1=r$, $q+1=s$. *If all* $a \in H^1(\Omega)$ *and* $h \in H^1(\Omega)$ *are taken to calculate* $\mathcal{L}_a$ *and* $\mathcal{L}_h$, *then we obtain*

$$\begin{aligned} a_t &= \varepsilon^2\Delta a - a + \frac{a^p}{h^q} + \sigma \\ \tau h_t &= D\Delta h - h + \frac{a^r}{h^s} \qquad \text{in } \Omega\times(0,T) \\ \frac{\partial a}{\partial \nu} &= \frac{\partial h}{\partial \nu} = 0 \qquad \text{on } \partial\Omega\times(0,T). \end{aligned}$$

*Here, shadow system takes a role in the global dynamics other then stationary solutions.*

# 4 Toland and Kuhn-Tucker dualities

Several (skew) Lagrangian's are defined from the free energy using Toland and Kuhn-Tucker dualities. In this case, stationary states split into the particle and the field components, provided with the structure of dual variation.

**Example 9** *Full-system of chemotaxis is the model (B) - model (A) equation*

$$\begin{aligned} u_t &= \nabla\cdot(u\nabla\mathcal{L}_u(u,v)) \\ \tau v_t &= -\mathcal{L}_v(u,v) \qquad \text{in } \Omega\times(0,T) \\ u\frac{\partial}{\partial\nu}\mathcal{L}_u(u,v) &= 0 \qquad \text{on } \partial\Omega\times(0,T) \end{aligned}$$

*derived from the Lagrangian*

$$\mathcal{L}(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \langle v, u\rangle,$$

*defined for*

$$\begin{aligned} &u \geq 0, \qquad \|u\|_1 = \lambda \\ &v \in H^1(\Omega), \qquad \int_\Omega v = 0, \end{aligned}$$

*and hence it holds that*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u = 0 \\ &\frac{d}{dt}\mathcal{L}(u,v) = -\int_\Omega u\,|L_u|^2 + \tau v_t^2 \leq 0. \end{aligned}$$

*Here, $\tau > 0$ denotes the relaxization time, and this system is associated with a chemical process in the formation of the field by particles, more precisely,*

$$\begin{aligned} &u_t = \nabla\cdot(\nabla u - u\nabla v) \\ &\tau v_t = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega\times(0,T) \\ &\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega\times(0,T) \\ &\int_\Omega v = 0 \qquad (0 < t < T). \end{aligned}$$

*Stationary particle state is*

$$\delta\mathcal{F}(u) = 0, \quad u \geq 0, \quad \|u\|_1 = \lambda,$$

*i.e.,*

$$(-\Delta_N)^{-1}\left(u - \frac{1}{|\Omega|}\int_\Omega u\right) = \log u + \textit{constant}, \quad u \geq 0, \quad \|u\|_1 = \lambda.$$

*Stationary field state is, on the other hand,*

$$\delta\mathcal{J}_\lambda(v) = 0, \qquad v \in H^1(\Omega), \quad \int_\Omega v = 0$$

*for*

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda \log \lambda - \lambda,$$

*i.e.,*

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right) \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega v = 0.$$

*These problems are equivalent through*

$$v = (-\Delta_N)^{-1}\left(u - \frac{1}{|\Omega|}\int_\Omega u\right), \quad \int_\Omega v = 0$$
$$u = \lambda \frac{e^v}{\int_\Omega e^v},$$

*and we obtain the unfolding*

$$\mathcal{L}|_{u=\lambda\frac{e^v}{\int_\Omega e^v}} = \mathcal{J}_\lambda$$
$$\mathcal{L}|_{v=(-\Delta_N)^{-1}\left(u-\frac{1}{|\Omega|}\int_\Omega u\right),\ \int_\Omega v=0} = \mathcal{F}$$

*and minimality*

$$\mathcal{L}(u,v) \geq \max\left\{\mathcal{F}(u), \mathcal{J}_\lambda(v) \mid u \geq 0,\ \|u\|_1 = 0,\ \int_\Omega v = 0\right\}.$$

*This structure guarantees the equivalence of the spectral and the dynamical stabilities. See [29].*

The above mentioned structure is written in the context of convex analysis. In more detail, given a Banach space $X$ over $\mathbf{R}$, and proper, convex, and lower semi-continuous functionals $F, G : X \to (-\infty, +\infty]$, we define the Lagrange function by

$$L(x,p) = G(x) + F^*(p) - \langle x, p\rangle$$

for $(x,p) \in X \times X^*$, where $X^*$ denotes the dual space and $F^*$ is the Legendre transformation of $F$:

$$F^*(p) = \sup_{x\in X}\{\langle x, p\rangle - F(x)\}.$$

This Lagrangian is associated with the free energy

$$\mathcal{J}^*(p) = \begin{cases} F^*(p) - G^*(p) & (p \in D(F^*)) \\ +\infty & (\text{otherwise}) \end{cases}$$

and the anti-free energy

$$\mathcal{J}(x) = \begin{cases} G(x) - F(x) & (x \in D(G)) \\ +\infty & (\text{otherwise}) \end{cases}$$

through

$$J^*(p) = \inf_{x \in X} L(x,p), \qquad J(x) = \inf_{p \in X^*} L(x,p).$$

Then, we obtain the unfolding-minimality, of which details are not described here. Furthermore, $\overline{p}$ and $\overline{x}$ are linearly stable local minimizers of $J^*$ and $J$ if and only if $(\overline{x}, \overline{p})$ is a linearly stable critical point of $\mathcal{L}$, and the former conditions are equivalent each other:

$$\overline{p} \in \partial G(\overline{x}) \cap \partial F(\overline{x}) \quad \Leftrightarrow \quad \overline{x} \in \partial F^*(\overline{p}) \cap \partial G^*(\overline{p}).$$

Skew Lagrangian, on the other hand, is introduced by

$$L(x,p) = \begin{cases} F^*(p) - G(x) + \langle x, p \rangle & (p \in D(F^*),\ x \in X) \\ +\infty & (\text{otherwise}) \end{cases}$$

Then, letting

$$J(x) = G(x) + F(-x), \qquad J^*(p) = F^*(p) + G^*(p),$$

we obtain a similar structure as above. Furthermore, $\overline{x}$ and $\overline{p}$ are linearly stable minimizers if and only if $(\overline{x}, \overline{p})$ is a linearly stable saddle point of $L$. This structure is a special case of the Kuhn-Tucker duality, but these $J$ and $J^*$ are convex, and therefore, linearly stable minimizers, if exits, are the only critical points in this case. Thus, dynamics of such a system is rather simple.

In the systems with semi-duality to the skew Lagrangian, however, multiple stationary can exist.

**Example 10** *FitzHugh-Nagumo equation concering nerve impluse [9, 21] is the model (A) - model (A) equation*

$$u_t = -\mathcal{L}_u(u,v), \qquad \tau v_t = \mathcal{L}_v(u,v)$$

*using the skew Lagrangian*

$$\mathcal{L}(u,v)=\int_\Omega \frac{\xi^2}{2}\left(|\nabla u|^2+W(u)\right)-\frac{\sigma}{2}\|\nabla v\|_2^2+\sigma\int_\Omega uv$$

*defined for*

$$u\in H^1(\Omega),\qquad v\in H^1(\Omega),\quad \int_\Omega v=0,$$

*i.e.,*

$$\begin{aligned}
&u_t=\xi^2\Delta u+W'(u)-v\\
&\tau v_t=\sigma\Delta v+\sigma\left(u-\frac{1}{|\Omega|}\int_\Omega u\right)\qquad \text{in } \Omega\times(0,T)\\
&\frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0\qquad \text{on } \partial\Omega\times(0,T)\\
&\int_\Omega v=0\qquad (0<t<T).
\end{aligned}$$

*Its stationary state is*

$$\delta\mathcal{F}(u)=0 \tag{4}$$

*combined with*

$$v=(-\Delta_N)^{-1}(u-\overline{u}),\qquad \int_\Omega v=0, \tag{5}$$

*where*

$$\begin{aligned}
&\mathcal{F}(u)=\int_\Omega \frac{\xi^2}{2}|\nabla u|^2+W(u)+\frac{\sigma}{2}\left|(-\Delta_N)^{-1/2}(u-\overline{u})\right|^2\\
&\overline{u}=\frac{1}{|\Omega|}\int_\Omega u
\end{aligned}$$

*is Ohta-Kawasaki's free energy. Here, $u$ is regarded as the particle density, $v$ denotes the field associated with it, and (4) is equivalent to*

$$\begin{aligned}
&-\xi^2\Delta u=u-u^3+\sigma\int_\Omega G(\cdot,x')\varphi(x')dx'\quad \text{in } \Omega\\
&\frac{\partial u}{\partial \nu}=0\quad \text{on } \partial\Omega.
\end{aligned}$$

*Actually, the last term of this $\mathcal{F}$ is equal to*

$$\frac{\sigma}{2}\int\int_{\Omega\times\Omega}G(x,x')u(x)u(x')dxdx'=\frac{\sigma}{2}\int_\Omega vu=\frac{\sigma}{2}\|\nabla v\|_2^2$$

*using the Green's function $G = G(x, x')$ and the solution $v = v(x)$ to (1), and therefore, we obtain the semi-unfolding*

$$\mathcal{L}|_{v=(-\Delta_N)^{-1}(u-\overline{u}),\ \int_\Omega v=0} = \mathcal{F}. \tag{6}$$

*Since this system is skew gradient, the linearized stability of the stationary state $(u, v)$ is reduced to the positivities of $L_{uu}(u, v)$ and $L_{vv}(u, v)$ from Yanagida's criterion [31, 32] mentioned in §3. The latter positivity is obvious, because $L_{vv}(u, v)$ is nothing but $-\Delta$ with the domain*

$$\left\{\psi \in (H^2 \cap L^2_0)(\Omega) \mid \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0\right\}.$$

*Thus, the linearized stability is described by the positivity of the first eigenvalue of*

$$A_{FHN} = -\xi^2\Delta - 1 + 3u^2$$

*with the domain*

$$D(A_{FHN}) = \left\{\psi \in H^2(\Omega) \mid \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0\right\}.$$

*The first eigenvalue of this operator is denoted by $\mu_{FHN}(u)$.*

*If model (A) equation for Ohta-Kawasaki's free energy is adopted, the stationary state $u$ is described in the same, but its linearlized stability is indicated by the positivity of the first eigenvalue of $A_{OK}$ defined by*

$$A_{OK}\psi = -\xi^2\Delta\psi - \psi + 3\psi^2 + \sigma\int_\Omega G(\cdot, x')\psi(x')dx'$$

*with the domain*

$$D(A_{OK}) = \left\{\psi \in H^2(\Omega) \mid \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0\right\}.$$

*If this first eigenvalue is denoted by $\mu_{OK}(u)$, then we obtain*

$$\mu_{OK}(u) > \mu_{FHN}(u).$$

*This relation, combined with the semi-unfolding (6), indicates that instability around a stationary state $(u, v)$ of the FitzHugh-Nagumo equation satisfying*

$$\mu_{OK}(u) > 0 > \mu_{HFN}(u)$$

*occurs to $v$ at the begining.*

## Conclusion

We have examined several dissipative systems derived from free energy, and observed one aspect of self-organization ("self-assembly"), realized as a triple seal of model (C) equations; closedness, nonlinear spectral mechanics, and dual variation. Details are the following.

1. Closed system
   (a) Model (B) equation describes the closed system.
   (b) Stationary state of this equation is the Euler-Lagrange equation of a variational problem with constraint.
2. Nonlinear spectral mechanics
   (a) Stationary state of the closed system is re-formulated as a nonlinear eigenvalue problem with non-local term.
   (b) Then, the total set of stationary solutions controls its non-equilibirium of dynamics.
3. Model (C) equation
   (a) Several model (C) equations are provided with the Lyapunov function, associated with the semi-unfolding- minimality.
   (b) Several other model (C) equations are described by (skew) gradient systems, provided with (skew) Lagrangian.
   (c) However, some other model (C) equations are provided with both properties and more (dual variation).
4. Dual variation
   (a) In the above mentioned model (C) equations, stationary states are equivalently formulated variations, in terms of the field and particles.
   (b) These structures are packaged into the (skew) Lagrangian, and then, the linearized stationary solution is dynamically stable.

**Note:** In mathematical theory of dynamical systems, *dissipative system* is defined by the presence of the attractor, and a typical exam-

ple of such systems is the gradient system with compact semi-orbits [12]. The *closed system* in the classical (equilibrium) thermodynamics, on the other hand, indicates the lack of the transport of materials between the outer system, whereby the transport of the temperature or that of the energy is permitted. More precisely, thermodynamical systems are classified into the isolated, the closed, and the open, as is described at the begining of this paper, and openness here means the transport of the material media between the outer system. Our title is a precise combination of these two notions of dissipativeness and closedness in different areas.

In the theory of non-equilibrium thermodynamics, however, *dissipativeness* indicates the dissipation of energy (or entropy) to the outer system, which is sometimes identified with the *openness* [22], and in this sense, this terminology of openness seems to be inconsistent between equilibrium and non-equilibrium thermodynamics.

The above mentioned definition of dissipativeness in the theory of dynamical systems, on the other hand, may not be a precise description of the phenomena observed by [22]. Actually, recent paradigm reveals two aspects of self-organization, *far equilibrium* ("*top-down self-organization*") and *self-assembly* ("*bottom-up self-organization*"), emphasizing the role of their *hierarchical developments.*

*Closed dissipative systems*, mathematically introduced in this paper, are certainly associated with the formation of self-assembly, where the total set of stationary states casts the driving force. The author thanks Professor Tomohiko Yamaguchi for stimulative discussions on the non-equilibrium thermodynamics.

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