数理解析研究所講究録 1464 巻 2006 年 40-57

Global Existence and Blow-up and other properties to Degenerate Keller-Segel Systems

Keller-Segel 系の時間大域解の存在・解の爆発・Barenblatt 解への漸近問題について

津田塾大学・情報数理科学科　杉山由恵 (Yoshie SUGIYAMA)

Department of Mathematics and Computer Science,
Tsuda University

1 Introduction

We consider the degenerate Keller-Segel system of Nagai type:

$$\text{(KS)} \qquad \begin{cases} u_t = \nabla \cdot \Big(\nabla u^m - \chi u \nabla v\Big), & x \in \mathbb{R}^N,\ t > 0, \\ 0 = \Delta v - \gamma v + \alpha u, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $m > 1,\ \alpha, \chi > 0,\ \gamma \geq 0$ and $N \geq 1$. This equation is often called as the Keller-Segel model describing the motion of the chemotaxis molds.

In this paper, we introduce our results concerning the properties of a weak solution for the degenerate Keller-Segel system (KS), which were obtained in [27], [41], [44], and [45] . The proofs for the global existence and finite time blow-up of solution for (KS) are given.

First of all, we give the definition of the weak solution (u, v) for (KS).

Definition *For $m > 1$, non-negative functions (u, v) defined in $[0, T) \times \mathbb{R}^N$ are said to be a weak solution of* (KS) *for $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$, $u_0^m \in H^1(\mathbb{R}^N)$ if*

i) $u \in L^\infty(0, T; L^2(\mathbb{R}^N))$, $u^m \in L^2(0, T; H^1(\mathbb{R}^N))$,

ii) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$,

iii) (u, v) *satisfies the equations in the sense of distribution: i.e.*

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} (\nabla u^m \cdot \nabla \varphi - \chi u \nabla v \cdot \nabla \varphi - u \cdot \varphi_t)\ dxdt &= \int_{\mathbb{R}^N} u_0(x) \cdot \varphi(x, 0)\ dx, \\ -\Delta v(x,t) + \gamma v(x,t) - \alpha u(x,t) &= 0 \qquad \text{a.a. } x \in \mathbb{R}^N,\ t \in (0, T), \end{aligned}$$

for any function $\varphi \in C^1(\overline{Q_T})$ which vanishes on $t = T$, where $Q_T = \mathbb{R}^N \times (0, T)$.

The following proposition gives the existence of a time "local" weak solution to (KS) and the uniform bound of the solution when $u_0 \in L^\infty(\mathbb{R}^N)$. The proof is based on the L^∞-energy method which is employed in [35]. The proof was given in [44].

Proposition 1.1 ([44]) **[time local existence of weak solution and its L^∞ uniform bound]** *Let $m > 1$, $\alpha, \chi > 0$, $\gamma \geq 0$. Then* (KS) *has a non-negative weak solution (u, v) on $(0, T_0)$ with $T_0 = \alpha^{-1}\Big(\|u_0\|_{L^\infty(\mathbb{R}^N)} + 2\Big)^{-2}$. Moreover, $u(t)$ satisfies the following a priori estimate*

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \;\leq\; \|u_0\|_{L^\infty(\mathbb{R}^N)} + 2 \qquad \text{for all } t \in [0, T_0]. \tag{1.1}$$

If the maximal existence time $T_{\max}$ of (u, v) is finite, then we have

$$\lim_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \;=\; \infty.$$

In the following theorem, we consider the case of $m > 2 - \frac{2}{N}$. The following theorem gives the existence of a time "global" weak solution to (KS) and the uniform bound of the solution when $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$. Recently, another degenerate case is treated by Laurencot and Wrzosek [23]. The time global L^∞ bound was also obtained in Kowalczyk [12] for the quasilinear Keller-Segel system of non-degenerate type and the existence of a solution was not considered.

Theorem 1.2 ([41]) **[time global existence of weak solution of $m > 2 - \frac{2}{N}$ case and its L^∞ uniform bound]** *Let $m > 2 - \frac{2}{N}$ and $\alpha, \chi > 0$, $\gamma \geq 0$. Then* (KS) *has a global weak solution (u, v). Moreover it satisfies a uniform estimate, i.e., that there exists $K_1 = K_1(\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, m, N)$ such that*

$$\sup_{t>0} \Big(\|u(t)\|_{L^r(\mathbb{R}^N)} + \|v(t)\|_{L^r(\mathbb{R}^N)}\Big) \leq K_1 \qquad \textit{for all } r \in [1, \infty].$$

In addition, in both cases (i) *and* (ii), *there exists a positive constant $K_2 = K_2(\|u_0\|_{L^1(\mathbb{R}^N)}$, $\|u_0\|_{L^2(\mathbb{R}^N)}, \|u_0\|_{L^m(\mathbb{R}^N)}$,*

$$\sup_{t>0} \|v(t)\|_{H^2(\mathbb{R}^N)} \;\leq\; K_2. \tag{1.2}$$

In the following theorem, we consider the case of $1 < m \leq 2 - \frac{2}{N}$ and the decay property of a weak solution (u, v) for (KS) with small initial data is given. (see [40] and [41]). On the other hand, the finite time blow-up of u for (KS) with large data is also given. We remark that the finite time blow-up was first formally obtained by [4] for Neumann problem, and then a rigorous complete proof using Bessel potential for (KS) was given. (see [44] for more detail.)

Theorem 1.3 ([27]),([41]),([44]) **[decay for small data and blow-up for large data of $1 < m \leq 2 - \frac{2}{N}$ case]** *Let $N \geq 3$, $1 < m \leq 2 - \frac{2}{N}$ and $\alpha, \chi > 0$, $\gamma \geq 0$ and suppose that the initial data u_0 is non-negative everywhere.*
(i) *We assume that the initial data is sufficiently small, i.e., for any fixed number $\ell \geq \frac{N(2-m)}{2}$ (≥ 1),*

$$\|u_0\|_{L^\ell(\mathbb{R}^N)} << 1. \tag{1.3}$$

then (KS) *has a global weak solution (u, v) and the weak solution satisfies*

$$\sup_{t>0}(1+t)^d \cdot (\|u(t)\|_{L^r(\mathbb{R}^N)} + \|v(t)\|_{L^r(\mathbb{R}^N)}) < \infty \qquad \textit{for } r \in \Big[\frac{N(2-m)}{2}, \infty\Big). \tag{1.4}$$

where

$$d = \frac{N}{\sigma}\left(1 - \frac{1}{r}\right), \quad \sigma = N(m-1)+2.$$

Moreover, the weak solution satisfies

$$t^{\frac{N}{\sigma+\delta}} \left|u(x,t) - G(x,t; \|u_0\|_{L^1(\mathbb{R}^N)})\right| \;\to\; 0 \quad \text{as } t \to \infty \tag{1.5}$$

uniformly with respect to x in the set $|x| \le Rt^{\frac{1}{\sigma}}$, where δ and R are any fixed positive constant and

$$\begin{aligned} M &:= \int_{\mathbb{R}^N} \left(A - \frac{m-1}{2m\sigma} \cdot |x|^2\right)_+^{\frac{1}{m-1}} dx, \\ G(x,t;M) &:= t^{-\frac{N}{\sigma}}\left(A - \frac{m-1}{2m\sigma} \cdot \frac{|x|^2}{t^{\frac{2}{\sigma}}}\right)_+^{\frac{1}{m-1}}. \end{aligned} \tag{1.6}$$

(ii) *We assume that $\gamma = 1$ and the initial data $u_0 \in L^1 \cap L^m(\mathbb{R}^N)$ with $u_0|x|^2 \in L^1(\mathbb{R}^N)$ satisfies the following condition:*

$$\frac{2}{(m-1)\chi}\int_{\mathbb{R}^N} u_0^m \, dx \;<\; \int_{\mathbb{R}^N} u_0 \cdot v_0 \, dx, \tag{H1}$$

where $v_0 = G * u_0$ with the Bessel kernel G. Then the weak solution does not exists globally in time, i.e., that there exists $T_{\max} < \infty$ such that for some initial data u_0 the weak solution blows up in a finite time $T_{\max}$ in the following sense:

$$\operatorname{limsup}_{t\to T_{\max}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \;=\; \infty.$$

In the following theorem, we consider the case of $1 < m \le 2 - \frac{2}{N}$ and construct an initial function which assures the global existence for $\frac{\int u_0 \cdot v_0(x)\,dx}{\int u_0^m(x)\,dx}$ small data and blow-up for large $\|u_0\|_{L^{\frac{N(2-m)}{2}}}$ data.

Theorem 1.4 ([45]) **[global existence for $\frac{\int u_0 \cdot v_0(x)\,dx}{\int u_0^m(x)\,dx}$ small data and blow-up for large $\|u_0\|_{L^{\frac{N(2-m)}{2}}}$ data of $1 < m \le 2 - \frac{2}{N}$ case]** *Let $N \ge 3$, $1 < m \le 2 - \frac{2}{N}$ and $\alpha, \chi > 0$, $\gamma \ge 0$.*

(i) *We take the initial data u_0 by $A\big(1 - \frac{|x|^N}{b^N}\big)_+$ with positive constants A and b. We also assume that*

$$\frac{\int u_0 \cdot v_0(x)\,dx}{\int u_0^m(x)\,dx} << 1, \tag{1.7}$$

where $v_0 = G * u_0$ with the Bessel potential G. Then, the problem (KS) *has a global weak solution (u,v).*

(ii) *We take the initial data u_0 by $A\big(1 - \frac{|x|^N}{b^N}\big)_+^{\frac{2}{N(2-m)}}$ with $A, b > 0$. If $\int u_0^{\frac{N(2-m)}{2}}\,dx$ is sufficiently large such that*

$$\|u_0\|^{2-m}_{L^{\frac{N(2-m)}{2}}(\mathbb{R}^N)} \;\ge\; C_N \cdot e^{2b\sqrt{\gamma}} \tag{1.8}$$

for some $C_N = C_N(\alpha, \chi, m, N)$, Then, a weak solution (u,v) of (KS) *blows up in a finite time.*

By combining Theorem 1.3 (ii) with Theorem 1.4 (i), it is seen that the size of $\frac{\int u_0 \cdot v_0(x)\, dx}{\int u_0^m(x)\, dx}$ divides the situation of the solution (u, v) into the global existence and the finite time blow-up. Simultaneously, by combining Theorem 1.3 (i) with Theorem 1.4 (ii), the size $\int_{\mathbb{R}^N} u_0^{\frac{N(2-m)}{2}}\, dx$ together with the geometrical restriction can divide the situation too.

We now consider the Fujita's exponent case: $m = 2 - \frac{2}{N}$ and obtain the upper bound (resp. the lower bound) on the size of the $L^1 (= L^{\frac{N(2-m)}{2}})$-norm which assures the global existence (resp. the finite time blow-up), which reads:

Theorem 1.5 ([45]) [**the L^1 upper and lower bound for time global existence and blow-up; the critical case of $m = 2 - \frac{2}{N}$**] *Let $N \geq 3$, $m = 2 - \frac{2}{N}$ and $\alpha, \chi > 0$, $\gamma \geq 0$.*

(i) *We suppose that*

$$\|u_0\|_{L^1(\mathbb{R}^N)} \leq \left(\frac{2N^2\pi}{\alpha\chi}\right)^{\frac{N}{2}} \cdot \left[\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right]. \tag{1.9}$$

Then, the problem (KS) *has a global weak solution (u, v) and*

$$\sup_{t>0} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C(N)$$

(ii) *We assume the same assumption as* Theorem 1.4 (ii) *and suppose that*

$$\|u_0\|_{L^1(\mathbb{R}^N)} > \left(\frac{2^{2(N-1)} \cdot N^{2-\frac{2}{N}} \cdot \pi^{\frac{1}{2}}}{\alpha\chi}\right)^{\frac{N}{2}} \cdot \frac{1}{\Gamma(\frac{N}{2})} \cdot \left[\frac{\Gamma(\frac{N}{2}) \cdot \Gamma(\frac{N-1}{2})}{\Gamma(N-1)}\right]^{\frac{N}{2}}. \tag{1.10}$$

Then, in the case of $\gamma = 0$, a weak solution (u, v) of (KS) *blows up in a finite time.*

Moreover, in the case of $\gamma > 0$, we suppose that (1.10) *is satisfied and $\gamma << A^{2-m}$ or $b^2 \cdot \gamma << 1$. Then, a weak solution (u, v) of* (KS) *blows up in a finite time.*

Remark 1 When we take $m = 1$ and $N = 2$, formally, we obtain

$$\|u_0\|_{L^1(\mathbb{R}^2)} \leq \left(\frac{2 \cdot 2^2\pi}{\alpha\chi}\right) \cdot \left[\frac{\Gamma(1)}{\Gamma(2)}\right] = \frac{8\pi}{\alpha\chi} \tag{1.11}$$

and

$$\|u_0\|_{L^1(\mathbb{R}^2)} > \left(\frac{2^2 \cdot 2 \cdot \pi^{\frac{1}{2}}}{\alpha\chi}\right) \cdot \frac{1}{\Gamma(1)} \cdot \left[\frac{\Gamma(1) \cdot \Gamma(\frac{1}{2})}{\Gamma(1)}\right] = \frac{8\pi}{\alpha\chi}. \tag{1.12}$$

We will use the simplified notations:

1) $\|\cdot\|_{L^r} = \|\cdot\|_{L^r(\mathbb{R}^N)}, (1 \leq r \leq \infty)$, $\int \cdot\, dx := \int_{\mathbb{R}^N} \cdot\, dx$.
2) $Q_T := (0, T) \times \mathbb{R}^N$,

3) When the weak derivatives ∇u, D^2u and u_t are in $L^p(Q_T)$ for some $p \geq 1$, we say that $u \in W_p^{2,1}(Q_T)$, *i.e.*,

$$W_p^{2,1}(Q_T) \;:=\; \Big\{u \in L^p(0,T;W^{2,p}(\mathbb{R}^N)) \cap W^{1,p}(0,T;L^p(\mathbb{R}^N));$$
$$\|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|D^2u\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)} < \infty\Big\}.$$

2 Preliminary Lemmas

The following representation is one from elliptic theory. (see E.M.Stein [39, Ch V Sec 6.5].)

Let $N \geq 3,\ 1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^N)$ and consider the following problem:

$$\text{(E)} \qquad -\Delta z + z = f \quad \text{for } x \in \mathbb{R}^N.$$

Then the function $z(x) \in L^p(\mathbb{R}^N)$ given by

$$\text{(2.1)} \qquad z(x) \;=\; \int_{\mathbb{R}^N} G(x-y)\cdot f(y)\,dy$$

is the *strong solution* of (E) in $\mathbb{R}^N$, *i.e.*, that (E) is satisfied almost everywhere, where $G(x)$ is the Bessel potential which can be express as

$$\text{(2.2)} \qquad G(x) \;=\; \gamma_N e^{-|x|}\int_0^\infty e^{-|x|s}\cdot\Big(s+\frac{s^2}{2}\Big)^{\frac{N-3}{2}}\,ds$$

with the constant γ_N given by

$$\gamma_N^{-1} \;=\; 2(2\pi)^{\frac{N-1}{2}}\cdot\Gamma\Big(\frac{N-1}{2}\Big).$$

For $G(x)$, we obtain the following lemma.

Lemma 2.1 *It holds that for $x, y \in R^N$ $(x \neq y)$,*

$$\text{(2.3)} \qquad x\cdot\nabla G(x) \;\leq\; -(N-2)\cdot G(x) \;\leq\; 0,$$

Proof of Lemma 2.1)

We differentiate (2.2) with respect to x, then for $x \neq 0$ it holds that

$$\text{(2.4)} \qquad \nabla G(x) \;=\; -\gamma_N\frac{x}{|x|}\cdot e^{-|x|}\int_0^\infty e^{-|x|s}(1+s)\cdot\Big(s+\frac{s^2}{2}\Big)^{\frac{N-3}{2}}\,ds.$$

By (2.4),

$$x\cdot\nabla G(x) \;=\; -\gamma_N|x|\cdot e^{-|x|}\int_0^\infty e^{-|x|s}(1+s)\cdot\Big(s+\frac{s^2}{2}\Big)^{\frac{N-3}{2}}\,ds.$$

For $N \geq 3$, the integration by parts yields that

$$
\begin{aligned}
x \cdot \nabla G(x) &= \gamma_N \cdot e^{-|x|} \int_0^\infty \frac{de^{-|x|s}}{ds} \cdot (1+s) \cdot \left(s + \frac{s^2}{2}\right)^{\frac{N-3}{2}} ds \\
&= -\gamma_N \cdot e^{-|x|} \int_0^\infty e^{-|x|s} \cdot \frac{d}{ds}\left[(1+s) \cdot \left(s + \frac{s^2}{2}\right)^{\frac{N-3}{2}}\right] ds. \qquad (2.5)
\end{aligned}
$$

It is seen that

$$
\begin{aligned}
\frac{d}{ds}\left[(1+s) \cdot \left(s + \frac{s^2}{2}\right)^{\frac{N-3}{2}}\right] &= \left(s + \frac{s^2}{2}\right)^{\frac{N-5}{2}} \cdot \left(s + \frac{s^2}{2} + (N-3) \cdot \frac{(1+s)^2}{2}\right) \\
&\geq (N-2)\left(s + \frac{s^2}{2}\right)^{\frac{N-3}{2}}. \qquad (2.6)
\end{aligned}
$$

Substituting (2.6) into (2.5),

$$
x \cdot \nabla G(x) \leq -(N-2) \cdot \gamma_N \cdot e^{-|x|} \int_0^\infty e^{-|x|s} \cdot \left(s + \frac{s^2}{2}\right)^{\frac{N-3}{2}} ds = -(N-2) \cdot G(x).
$$

Thus the proof of Lemma 2.1 is completed. Q.E.D.

The following lemma is shown by Hölder's inequality.

Lemma 2.2 (the moment inequality) *Let $p \geq 1$ and $|x|^p f \in L^1(\mathbb{R}^N)$. Then,*

$$
\int_{R^N} |f(x)| \cdot |x| \, dx \leq \left(\int_{\mathbb{R}^N} |f(x)| \, dx\right)^{\frac{p-1}{p}} \cdot \left(\int_{\mathbb{R}^N} |f(x)| \cdot |x|^p \, dx\right)^{\frac{1}{p}},
$$

The following lemma, due to M. Nakao, gives us a version of Gagliardo-Nirenberg inequality. (see Nakao[33, Lemma 3].)

Lemma 2.3 (Nakao[33]) *Let $m \geq 1$, $u \in L^{q_1}(\mathbb{R}^N)$ with $q_1 \geq 1$ and $u^{\frac{r+m-1}{2}} \in H^1(\mathbb{R}^N)$ with $r > 0$.*

If $q_2 \geq \frac{r+m-1}{2}$ and

$$
\begin{cases}
1 \leq q_1 \leq q_2 \leq \infty & \text{when } N = 1, \\
1 \leq q_1 \leq q_2 < \infty & \text{when } N = 2, \\
1 \leq q_1 \leq q_2 \leq \frac{(r+m-1)N}{N-2} & \text{when } N \geq 3,
\end{cases}
$$

then there exists a positive constant C_s depending only on q_1, q_2, r, N such that

$$
\|u\|_{L^{q_2}} \leq C_s^{\frac{2}{r+m-1}} \|u\|_{L^{q_1}}^{1-\theta} \cdot \|\nabla u^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\theta}{r+m-1}}, \qquad (2.7)
$$

where

$$
\theta = \frac{r+m-1}{2} \cdot \left(\frac{1}{q_1} - \frac{1}{q_2}\right) \cdot \frac{1}{\frac{1}{N} - \frac{1}{2} + \frac{r+m-1}{2q_1}}
$$

and C_s has at most a polynomial growth in q_1 and q_2.

3 Approximated Problem

In order to justify the formal arguments, we introduce the following approximated equation of (KS):

$$
(\mathrm{KS})_\varepsilon \begin{cases} u_{\varepsilon t}(x,t) &= \nabla\cdot\Big(\nabla(u_\varepsilon+\varepsilon)^m - \chi u_\varepsilon\cdot\nabla v_\varepsilon\Big), \quad (x,t)\in\mathbb{R}^N\times(0,T), \quad \cdots(1),\\ 0 &= \Delta v_\varepsilon - \gamma v_\varepsilon + \alpha u_\varepsilon, \quad (x,t)\in\mathbb{R}^N\times(0,T), \quad \cdots(2),\\ u_\varepsilon(x,0) &= u_{0\varepsilon}(x), \quad x\in\mathbb{R}^N, \end{cases}
$$

where ε is a positive parameter and $(u_{0\varepsilon},v_{0\varepsilon})$ is an approximation for the initial data (u_0,v_0) such that

(A.1) $0\le u_{0\varepsilon}\in W^{2,p}(\mathbb{R}^N)$, for all $p\in[1,\infty]$, for all $\varepsilon\in(0,1]$,

(A.2) $\|u_{0\varepsilon}\|_{L^p}\le\|u_0\|_{L^p}$, for all $p\in[1,\infty]$, for all $\varepsilon\in(0,1]$,

(A.3) $\|\nabla u_{0\varepsilon}\|_{L^2}\le\|\nabla u_0\|_{L^2}$, for all $\varepsilon\in(0,1]$,

(A.4) $u_{0\varepsilon}\to u_0$, strongly in $L^p(\mathbb{R}^N)$ as $\varepsilon\to0$, for some $p>\max\{2,N\}$.

We call $(u_\varepsilon,v_\varepsilon)$ a strong solution of $(\mathrm{KS})_\varepsilon$ if it belongs to $W_p^{2,1}\times W_p^{2,1}(Q_T)$ for some $p\ge1$ and the equations (1),(2) in $(\mathrm{KS})_\varepsilon$ are satisfied almost everywhere.

The following convergence is given in [41]: For any fixed positive number there exists a subsequence $\{u_{\varepsilon_n}\}$ such that

(3.1)	$u_{\varepsilon_n}\rightharpoonup u$	weakly	in $L^2((0,T);L^2(\mathbb{R}^N))$,
(3.2)	$u_{\varepsilon_n}^m\to u^m$	strongly	in $C((0,T);L^2_{loc}(\mathbb{R}^N))$,
(3.3)	$\nabla u_{\varepsilon_n}^m\rightharpoonup\nabla u^m$	weakly	in $L^2((0,T);L^2(\mathbb{R}^N))$,
(3.4)	$v_{\varepsilon_n}\to v$	strongly	in $C((0,T);L^2_{loc}(\mathbb{R}^N))$,
(3.5)	$\nabla v_{\varepsilon_n}\rightharpoonup\chi=\nabla v$	weakly	in $L^2(0,T;L^2(\mathbb{R}^N))$.

(see (4.11), (4.14) and (4.15) in section 4 in [41] .)

4 Proof of Proposition 1.1

As for the proof of Proposition 1.1, we refer to [44].

5 Proof of Theorem 1.2

We multiply (1) in $(\mathrm{KS})_\varepsilon$ by u_ε^{r-1} and integrate over $\mathbb{R}^N$.

$$
\frac{1}{r}\cdot\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r
$$

$$
= -m(r-1)\int u_\varepsilon^{m-1}u_\varepsilon^{r-2}|\nabla u_\varepsilon|^2\,dx + (r-1)\chi\int u_\varepsilon\nabla v_\varepsilon\cdot u_\varepsilon^{r-2}\nabla u_\varepsilon\,dx \tag{5.1}
$$

$$
= -\frac{4m(r-1)}{(r+m-1)^2}\int|\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx - \frac{r-1}{r}\cdot\chi\int u_\varepsilon^r\cdot\Delta v_\varepsilon\,dx. \tag{5.2}
$$

Substituting (2) of $(\mathrm{KS})_\varepsilon$: $\Delta v_\varepsilon = \gamma v_\varepsilon - \alpha u_\varepsilon$ into (5.1) and noting that u_ε and v_ε are non-negative,

$$(5.3) \quad \frac{d}{dt}\|u_\varepsilon\|_{L^r}^r \;\le\; -\frac{4m(r-1)r}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx + \;\alpha\chi\cdot(r-1)\int u_\varepsilon^{r+1}\,dx.$$

From Lemma 2.3

$$(5.4) \qquad \|u_\varepsilon\|_{L^{r+1}} \;\le\; C_s^{\frac{2}{r+m-1}}\|u_\varepsilon\|_{L^1}^{1-\theta_1}\cdot\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\theta_1}{r+m-1}},$$

where

$$\theta_1 \;=\; \frac{r+m-1}{2}\cdot\left(1-\frac{1}{r+1}\right)\cdot\frac{1}{\dfrac{1}{N}-\dfrac{1}{2}+\dfrac{r+m-1}{2}}$$

for

$$(5.5) \qquad \begin{cases} r\in\left[\max(1,m-3),\infty\right] & \textit{if } N=1,\ m>1,\\ r\in\left[\max(1,m-3),\infty\right) & \textit{if } N=2,\ m>1.\\ r\in\left[\max(1,m-3,\frac{N(2-m)}{2}-1),\ \infty\right) & \textit{if } N\ge 3,\ m>1.\end{cases}$$

It is easy to verify that $\dfrac{2\theta_1\cdot(r+1)}{r+m-1} < 2$ if $m>2-\frac{2}{N}$. Therefore, by Young's inequality,

$$(5.6) \qquad \alpha\chi\cdot\|u_\varepsilon\|_{L^{r+1}}^{r+1} \;\le\; C_{m,r}+\frac{2mr}{(r+m-1)^2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2$$
$$\text{if } r \text{ satisfies (5.5) and } m>2-\frac{2}{N},$$

where $C_{m,r}$ is a positive number depending only on $m,\alpha,\chi,r,N,\|u_{0\varepsilon}\|_{L^1}$ and has at most a polynomial growth in r. This number $C_{m,r}$ will have different values in different places.

Again, from Lemma 2.3,

$$(5.7) \qquad \|u_\varepsilon\|_{L^r}^r \;\le\; \left(C_s^{\frac{2}{r+m-1}}\|u_\varepsilon\|_{L^1}^{1-\theta_2}\cdot\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\theta_2}{r+m-1}}\right)^r \quad \text{for } r\ge m-1,$$

where

$$\theta_2 \;=\; \frac{r+m-1}{2}\cdot\left(1-\frac{1}{r}\right)\cdot\frac{1}{\dfrac{1}{N}-\dfrac{1}{2}+\dfrac{r+m-1}{2}}$$

and C_s has at most a polynomial growth in r.

Since $\dfrac{2\theta_2\cdot r}{r+m-1} < 2$ by $m>1-\frac{2}{N}$, and $r\ge 1$, Young's inequality and (5.7) yield

$$(5.8) \qquad \|u_\varepsilon\|_{L^r}^r \;\le\; \frac{2m(r-1)}{(r+m-1)^2}\int|\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx \;+\; C_{m,r}.$$

By combining (5.15) and (5.8) with (5.3),

$$\begin{aligned} \frac{d}{dt}\|u_\varepsilon\|_{L^r}^r &\le -\frac{2m(r-1)}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx + C_{m,r} \\ &\le -\|u_\varepsilon\|_{L^r}^r + C_{m,r}. \end{aligned} \tag{5.9}$$

Hence, for r in (5.5),

$$\sup_{t>0}\|u_\varepsilon(t)\|_{L^r} \le \|u_0\|_{L^r} + C_{m,r} =: R_r. \tag{5.10}$$

From (2) in $(\mathrm{KS})_\varepsilon$, for any $p \in [1,\infty)$, there exists a constant $C_p = C_p(\alpha,\gamma,p)$

$$\begin{aligned} \sup_{t>0}\|\nabla v_\varepsilon(t)\|_{L^p(\mathbf{R}^N)} &\le C_p \sup_{t>0}\|u_\varepsilon\|_{L^p(\mathbf{R}^N)} \le C_p \cdot R_p, \\ \sup_{t>0}\|\Delta v_\varepsilon(t)\|_{L^p(\mathbf{R}^N)} &\le \alpha \sup_{t>0}\|v_\varepsilon(t)\|_{L^p} + \gamma \sup_{t>0}\|u_\varepsilon(t)\|_{L^p(\mathbf{R}^N)} \\ &\le (\alpha+\gamma)\sup_{t>0}\|u_\varepsilon(t)\|_{L^p(\mathbf{R}^N)} \le (\alpha+\gamma)R_p. \end{aligned}$$

Hence, Gagliardo-Nirenberg inequality yields that

$$\begin{aligned} \sup_{t>0}\|\nabla v_\varepsilon(t)\|_{L^\infty(\mathbf{R}^N)} &\le C_N \cdot \sup_{t>0}\|\nabla v_\varepsilon\|_{L^2(\mathbf{R}^N)}^{\frac{2}{N(N+1)+2}} \cdot \sup_{t>0}\|\Delta v_\varepsilon\|_{L^{N+1}(\mathbf{R}^N)}^{\frac{N(N+1)}{N(N+1)+2}} \\ &\le C_N\Big(R_2^{\frac{2}{N(N+1)+2}} + R_{N+1}^{\frac{N(N+1)}{N(N+1)+2}}\Big) =: M_{\nabla v} < \infty, \end{aligned} \tag{5.11}$$

where $C_N = C_N(N)$.

We are now going to obtain the time global $L^\infty(\mathbb{R}^N)$-bound for u_ε by using (5.10) and (5.11).

From (5.1) and Young' inequality,

$$\begin{aligned} &\frac{1}{r}\cdot\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r \\ &= -\frac{4m(r-1)}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx + (r-1)\chi\int u_\varepsilon \nabla v_\varepsilon \cdot u_\varepsilon^{r-2}\nabla u_\varepsilon\,dx \\ &\le -\frac{2m(r-1)}{(r+m-1)^2}\int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2\,dx + \frac{r-1}{m}\cdot\chi\cdot M_{\nabla v}^2\int u_\varepsilon^{r+1-m}\,dx. \end{aligned} \tag{5.12}$$

By Lemma 2.3,

$$\|u_\varepsilon\|_{L^{r+1-m}} \le C_s^{\frac{2}{r+m-1}}\|u_\varepsilon\|_{L^{\frac{r}{2}}}^{1-\theta_3}\cdot\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\theta_3}{r+m-1}}, \tag{5.13}$$

where

$$\theta_3 = \frac{r+m-1}{2}\cdot\Big(\frac{2}{r}-\frac{1}{r+1-m}\Big)\cdot\frac{1}{\frac{1}{N}-\frac{1}{2}+\frac{r+m-1}{r}}$$

for

$$\text{(5.14)} \qquad \begin{cases} r \in \left[\max\{m, 3(m-1), 2\}, \infty\right] & \textit{if } N = 1,\ m > 1, \\ r \in \left[\max\{m, 3(m-1), 2\}, \infty\right) & \textit{if } N \geq 2,\ m > 1. \end{cases}$$

It is easy to verify that $\dfrac{2\theta_3 \cdot (r+1-m)}{r+m-1} < 2$ and $(r+1-m)(1-\theta_3) \cdot \frac{r+m-1}{r+m-1-\theta_3(r+1-m)} \leq r$ by $m > 1$. Therefore, Young's inequality yields that

$$\text{(5.15)} \quad \frac{r-1}{m} \cdot M^2_{\nabla v} \|u_\varepsilon\|^{r+1-m}_{L^{r+1-m}} \leq C_{m,r} + C_{m,r}\|u_\varepsilon\|^r_{L^{\frac{r}{2}}} + \frac{2m(r-1)}{(r+m-1)^2} \|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|^2_{L^2}$$

if r satisfies (5.14) and $m > 1$.

Substituting (5.15) into (5.12),

$$\text{(5.16)} \quad \frac{d}{dt}\|u_\varepsilon\|^r_{L^r} \leq -\frac{2mr(r-1)}{(r+m-1)^2} \int |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2 \, dx + C_{m,r}\|u_\varepsilon\|^r_{L^{\frac{r}{2}}} + C_{m,r}.$$

Moreover, substituting (5.8) into (5.16),

$$\text{(5.17)} \qquad \frac{d}{dt}\|u_\varepsilon\|^r_{L^r} + \|u_\varepsilon\|^r_{L^r} \leq C_{m,r}\|u_\varepsilon\|^r_{L^{\frac{r}{2}}} + C_{m,r}.$$

By using (5.17), the Moser's iteration technique yields the $L^\infty(\mathbb{R}^N)$-bound for u_ε globally in time which is independent of ε. (see Alikakos [1].)

In consequence, the $L^\infty(\mathbb{R}^N)$-bound for u_ε globally in time is obtained. By (3.1)–(3.5) and the convergence argument which is used in [41], we complete the proof of Theorem 1.2. Q.E.D.

6 Proof of Theorem 1.3-(ii)

As for the proof of Theorem 1.3-(i), we refer to [27] and [41].

The finite time blow-up was first formally obtained by [4] for Neumann problem. They consider the second equation as $0 = \Delta v + u$ and gave a proof by using Riesz potential. Then, we [?] gave a rigorous complete proof for the Cauchy problem (KS) (with the absorption term in the second equation) using the Bessel potential. Those results were obtained independently each other. In this paper, we give a proof of the blow-up of solution for (KS) using the Bessel potential.

We show the crucial inequality for the weak solution of (KS) in the following proposition:

Proposition 6.1 (the L^m apriori estimate) *Let $m > 1$, $\alpha, \chi > 0$, $\gamma \geq 0$, and $(u_\varepsilon, v_\varepsilon)$ be a strong solution of $(\mathrm{KS})_\varepsilon$ in $W^{2,1}_p \times W^{2,1}_p(Q_T)$ and suppose that the non-negative functions $u_0 \in L^1 \cap L^m(\mathbb{R}^N)$. Then the strong solution $(u_\varepsilon, v_\varepsilon)$ of $(\mathrm{KS})_\varepsilon$ satisfies*

$$\begin{aligned} &\frac{1}{m-1}\int (u_\varepsilon(t)+\varepsilon)^m \, dx - \frac{\chi}{2}\int u_\varepsilon(t) \cdot v_\varepsilon(t) \, dx \\ \text{(6.1)} \qquad &\leq \frac{1}{m-1}\int (u_{0\varepsilon}+\varepsilon)^m \, dx - \frac{\chi}{2}\int u_{0\varepsilon} \cdot v_{0\varepsilon} \, dx \quad \text{for } t \in (0,T). \end{aligned}$$

Proof of Proposition 6.1)

To give the rigorous proof, we should multiply (1) in $(\mathrm{KS})_\varepsilon$ by $\Big(\frac{m(u_\varepsilon+\varepsilon)^{m-1}}{m-1} - \chi v_\varepsilon\Big)$ and integrate over $\mathbb{R}^N$. However, for the sake of simplicity, we multiply (1) in $(\mathrm{KS})_\varepsilon$ by $\Big(\frac{m u_\varepsilon^{m-1}}{m-1} - \chi v_\varepsilon\Big)$ and integrate over $\mathbb{R}^N$. Then we get

$$\begin{aligned} \int u_{\varepsilon t}\Big(\frac{m u_\varepsilon^{m-1}}{m-1} - \chi v_\varepsilon\Big)\,dx &= -\int \Big(\nabla u_\varepsilon^m - u_\varepsilon\cdot\chi\nabla v_\varepsilon\Big)\cdot\nabla\Big(\frac{m u_\varepsilon^{m-1}}{m-1} - \chi v_\varepsilon\Big)\,dx \\ &= -\int u\cdot\Big|\nabla\Big(\frac{m}{m-1}u_\varepsilon^{m-1} - \chi v_\varepsilon\Big)\Big|^2\,dx \le 0. \end{aligned} \tag{6.2}$$

We now follow the argument in [32].

$$\text{The left-hand side of (6.2)} = \frac{d}{dt}\Big(\frac{1}{m-1}\int u_\varepsilon^m\,dx - \chi\int u_\varepsilon\cdot v_\varepsilon\,dx\Big) + J, \tag{6.3}$$

where

$$J := \chi\int u_\varepsilon\cdot v_{\varepsilon t}\,dx.$$

Substituting (2) of $(\mathrm{KS})_\varepsilon$: $u_\varepsilon = \frac{1}{\alpha}\Big(-\Delta v_\varepsilon + \gamma v_\varepsilon\Big)$ into J, we have

$$J = \frac{\chi}{2\alpha}\cdot\frac{d}{dt}\int(|\nabla v_\varepsilon|^2 + \gamma v_\varepsilon^2)\,dx.$$

Moreover, by (2) of $(\mathrm{KS})_\varepsilon$,

$$\alpha\int u_\varepsilon\cdot v_\varepsilon\,dx = \int(|\nabla v_\varepsilon|^2 + \gamma v_\varepsilon^2)\,dx.$$

Thus, we observe that

$$J = \frac{\chi}{2}\cdot\frac{d}{dt}\int u_\varepsilon\cdot v_\varepsilon\,dx. \tag{6.4}$$

By substituting (6.4) into (6.3), we obtain

$$\text{the left-hand side of (6.2)} = \frac{d}{dt}\Big(\frac{1}{m-1}\int u_\varepsilon^m\,dx - \frac{\chi}{2}\int u_\varepsilon\cdot v_\varepsilon\,dx\Big). \tag{6.5}$$

We denote $W(t)$ by

$$W(t) := \frac{1}{m-1}\int u_\varepsilon^m\,dx - \frac{\chi}{2}\int u_\varepsilon\cdot v_\varepsilon\,dx. \tag{6.6}$$

Then from (6.2), (6.5) and (6.6),

$$\frac{d}{dt}W(t)\,dx = -\int u_\varepsilon\cdot\Big|\nabla\Big(\frac{m}{m-1}u_\varepsilon^{m-1} - \chi v_\varepsilon\Big)\Big|^2\,dx \le 0. \tag{6.7}$$

By integrating (6.7) with respect to the time variable from 0 to t,

$$W(t) + \int_0^t \int_{\mathbf{R}^N} u_\varepsilon \Big| \nabla \Big(\frac{m}{m-1} u_\varepsilon^{m-1} - \chi v_\varepsilon \Big) \Big|^2 \, dxdt \quad \leq \quad W(0).$$

Thus we establish the following *a priori* estimate for $W(t)$.

$$\begin{aligned} W(t) \ &\leq \ W(0) \\ &= \ \frac{1}{m-1} \|u_{0\varepsilon}\|_{L^m}^m - \frac{\chi}{2} \int u_{0\varepsilon} \cdot v_{0\varepsilon} \, dx. \end{aligned} \tag{6.8}$$

From (6.8), we find the following estimate:

$$\frac{1}{m-1} \int u_\varepsilon^m \, dx - \frac{\chi}{2} \int u_\varepsilon \cdot v_\varepsilon(t) \, dx \ \leq \ \frac{1}{m-1} \int u_{0\varepsilon}^m \, dx - \frac{\chi}{2} \int u_{0\varepsilon} \cdot v_{0\varepsilon} \, dx. \tag{6.9}$$

From the similar argument by multiplying (1) in $(\mathrm{KS})_\varepsilon$ by $\Big(\frac{m(u_\varepsilon + \varepsilon)^{m-1}}{m-1} - \chi v_\varepsilon \Big)$, we obtain

$$\begin{aligned} &\frac{1}{m-1} \int (u_\varepsilon(t) + \varepsilon)^m \, dx - \frac{\chi}{2} \int u_\varepsilon(t) \cdot v_\varepsilon(t) \, dx \\ &\leq \ \frac{1}{m-1} \int (u_{0\varepsilon} + \varepsilon)^m \, dx - \frac{\chi}{2} \int u_{0\varepsilon} \cdot v_{0\varepsilon} \, dx \quad \text{for } t \in (0,T). \end{aligned}$$

Thus we complete the proof of Proposition 6.1. Q.E.D.

The following lemma is a key tool which is essentially due to theorem 1.3, which reads:

Lemma 6.2 *Let $N \geq 3$, $1 < m < 2 - \frac{2}{N}$, $\alpha, \chi > 0$ $\gamma = 1$, and (u,v) be the weak solution of* (KS) *corresponding to the initial data u_0 and suppose that u_0 is non-negative everywhere. Assume that $\int u_0(x)|x|^2 \, dx < +\infty$. and (u,v) be the weak solution of* (KS) *corresponding to the initial data u_0 and suppose that u_0 is non-negative everywhere. Then,*

$$\int u(x,t)|x|^2 \, dx < +\infty \quad \text{for} \quad 0 < t < T_{\max} : \text{the maximal existence time of solution,} \tag{6.10}$$

and it hold that

$$\begin{aligned} &\int u(t) \cdot |x|^2 \, dx - \int u_0 \cdot |x|^2 \, dx \\ &\leq \ 2Nt \cdot \int_{\mathbf{R}^N} \Big(u_0^m - \frac{(m-1)\chi}{2} \cdot u_0 \cdot v_0 \Big) \, dx \qquad \text{for } t \in (0,T). \end{aligned} \tag{6.11}$$

Proof of Lemma 6.2) As for the proof of (6.10), we refer to [44]. We are now going to prove (6.11).

By virtue of the integration by parts and the converge in (3.1)–(3.5), it holds that

$$\begin{aligned} &\int u(t) \cdot |x|^2 \, dx - \int u_0 \cdot |x|^2 \, dx \\ &\leq \ 2 \int_0^t \int N u^m(s) \ + \ \chi u(s) \nabla v(s) \cdot x \, dxds \qquad \text{for } t \in (0,T). \end{aligned} \tag{6.12}$$

(see [44] in detail.)

Using any fix number $\delta > 0$,

$$
\begin{aligned}
&\int u(t) \cdot |x|^2 \, dx - \int u_0 \cdot |x|^2 \, dx \\
&\le 2 \int_0^t \int \Big(N u^m(s) - \delta \cdot u(s)v(s) \Big) + \Big(\chi u(s) \nabla v(s) \cdot x + \delta \cdot u(s)v(s) \Big) \, dx ds \\
&\qquad \text{for } t \in (0, T).
\end{aligned}
\tag{6.13}
$$

From Proposition 6.1, we observe that

$$
\int_{\mathbf{R}^N} \Big(u^m(s) - \frac{(m-1)\chi}{2} \cdot u(s)v(s) \Big) \, dx \le \int_{\mathbf{R}^N} \Big(u_0^m - \frac{(m-1)\chi}{2} \cdot u_0 v_0 \Big) \, dx. \tag{6.14}
$$

Using (6.14) and taking by $\delta = \frac{N(m-1)\chi}{2}$,

$$
\begin{aligned}
&\int N u^m(s) - \delta \cdot u(s)v(s) \, dx \\
&= N \int \Big(u^m(s) - \frac{\delta}{N} \cdot u(s)v(s) \Big) \, dx \\
&= N \int \Big(u^m(s) - \frac{(m-1)\chi}{2} \cdot u(s)v(s) \Big) \, dx \\
&\le N \int_{\mathbf{R}^N} \Big(u_0^m - \frac{(m-1)\chi}{2} \cdot u_0 v_0 \Big) \, dx.
\end{aligned}
\tag{6.15}
$$

We are now going to estimate the second term on the right-hand side of (6.12). We use the following representation of v:

$$
v(x,t) = \alpha \int G(x-y) u(y,t) dy. \tag{6.16}
$$

Using (6.16) and $\delta = \frac{N(m-1)\chi}{2}$,

$$
\begin{aligned}
&\chi \int u \cdot \nabla v \cdot x \, dx + N(m-1)\chi \int uv \, dx \\
&= \alpha\chi \int \int u(x,s)u(y,s) \Big(x \cdot \nabla G(x-y) + \frac{N(m-1)}{2} \cdot G(x-y) \Big) \, dx dy \\
&= \frac{\alpha\chi}{2} \int \int u(x,s)u(y,s) \cdot \Big((x-y) \cdot \nabla G(x-y) + N(m-1) \cdot G(x-y) \Big) \, dx dy.
\end{aligned}
$$

By Lemma 2.1 and the assumption $m \le 2 - \frac{2}{N}$ in Theorem 1.3,

$$
\begin{aligned}
&\chi \int u \cdot \nabla v \cdot x \, dx + N(m-1)\chi \int uv \, dx \\
&\le -\frac{\alpha\chi}{2} \Big((N-2) - N(m-1) \Big) \int \int u(x,s)u(y,s) \cdot G(x-y) \, dx dy \\
&\le 0.
\end{aligned}
\tag{6.17}
$$

Combining (6.17) with (6.13),

$$\int u(t)\cdot|x|^2\,dx-\int u_0\cdot|x|^2\,dx$$

$$(6.18)\qquad \le\ 2N\cdot t\int_{\mathbf{R}^N}\left(u_0^m-\frac{(m-1)\chi}{2}u_0\cdot v_0\right)dx\qquad \text{for } t\in(0,T).$$

Thus we complete the proof of Lemma 6.2. Q.E.D.

Proof of Theorem 1.3-(ii))

We are going to prove Theorem 1.3 by contradiction.

Suppose $T_{\max}=\infty$, that is, the weak solution of (KS) is solvable globally in time. Then, from Lemma 6.2, it follows that

$$(6.19)\ M(t)\ \le\ M(0)+2N\cdot t\int_{\mathbf{R}^N}\left(u_0^m-\frac{(m-1)\chi}{2}u_0\cdot v_0\right)dx=:\ H(t)\quad \text{for } t>0.$$

By virtue of (H1) in Theorem 1.3 *i.e.*, that $k:=-\int_{\mathbf{R}^N}\left(u_0^m-\frac{(m-1)\chi}{2}u_0\cdot v_0\right)dx>0$,

$$(6.20)\qquad H'(t)\ =\ -2N\cdot k\ <\ 0\quad \text{for } t>0.$$

Hence, the equation $H(t)=0$ has a solution $T_*=-\frac{M(0)}{k}$ and $M(t)=0$ at $t=T_*$. This contracts $M(t)>0$ for $t\in(0,\infty)$. Thus we conclude that $T_{\max}<\infty$. On the other hand, by Proposition 1.1, we can extend the maximal existence time of the weak solution for (KS) as long as $\|u(t)\|_{L^\infty}$ is bounded. Hence, we observe that the weak solution of (KS) blows up in a finite time. Thus we complete the proof of (ii) in Theorem 1.3. As for (i) in Theorem 1.3, we refer [40] and [41]. Q.E.D.

7 Proof of Theorem 1.4 and 1.5

As for the proof of Theorem 1.4 and 1.5, we refer to [45].

8 Keller-Segel model with a power factor in drift term

We rewrite the first equation of (KS) by substituting the second equation: $\Delta v=v-u$ (with $\alpha=\gamma=1$) as follows:

$$(\mathrm{E})\qquad u_t\ =\ \Delta u^m-\nabla u\cdot\nabla v-u\Delta v\ =\ \Delta u^m-\nabla u\cdot\nabla v-uv+u^2.$$

Since this equation (E) has three terms: $u_t,\Delta u^m$ and u^2, the first equation in (KS) is analogous to the following equation with $q=2$.

$$(\mathrm{PS})\qquad \begin{cases} u_t=\Delta u^m+u^q & x\in\mathbb{R}^N,\ t>0,\\ u(x,0)=u_0(x), & x\in\mathbb{R}^N.\end{cases}$$

It is well known that the critical exponent $q = m + \frac{2}{N}$ divides the situation of the global existence and non-existence of the solution to the above equation (PS). This exponent is called as the Fujita exponent [10]. Indeed, when $q > m + \frac{2}{N}$, it can be globally solvable for small initial data. When $q < m + \frac{2}{N}$ and $q = m + \frac{2}{N}$, it was proved that (all) non-negative solutions of (PS) blow up in a finite time without any restriction on the size of the initial data. (see for example [11], [13], [21] and [26]).

As for the case of $q \geq 2$, we obtained the following theorem in [27] and [43].

Theorem 8.1 (time global existence of $\tau = 0$ case) *Let $\tau = 0$, $q \geq 2$ and suppose that u_0 is non-negative. Then*
(i) *when $m > q - \frac{2}{N}$,* (KS) *has a global weak solution (u, v).*
(ii) *When $1 < m \leq q - \frac{2}{N}$, we also assume that the initial data is sufficiently small, i.e., $\|u_0\|_{L^{\frac{N(q-m)}{2}}(\mathbf{R}^N)} << 1$, then* (KS) *has a global weak solution (u, v).*

Moreover, it satisfies a uniform estimate, i.e., that in both cases (i) *and* (ii)*, there exists $K_1 = K_1(\|u_0\|_{L^r(\mathbf{R}^N)}, m, q, N)$ such that*

$$\sup_{t>0} \left(\|u(t)\|_{L^r(\mathbf{R}^N)} + \|v(t)\|_{L^r(\mathbf{R}^N)} \right) \leq K_1 \qquad \text{for all } r \in [1, \infty]. \tag{8.1}$$

In addition, in both cases (i) *and* (ii)*, there exists a positive constant $K_2 = K_2(\|u_0\|_{L^2(\mathbf{R}^N)}, m, q, N)$ such that*

$$\sup_{t>0} \|v(t)\|_{H^2(\mathbf{R}^N)} \leq K_2. \tag{8.2}$$

We assume that the initial data is sufficiently small, i.e., for any fixed number $\ell \geq \frac{N(q-m)}{2}$ (≥ 1),

$$\|u_0\|_{L^\ell(\mathbf{R}^N)} << 1. \tag{8.3}$$

then (KS) *has a global weak solution (u, v) and the weak solution satisfies*

$$\sup_{t>0} (1+t)^d \cdot \left(\|u(t)\|_{L^r(\mathbf{R}^N)} + \|v(t)\|_{L^r(\mathbf{R}^N)}\right) < \infty \qquad \text{for } r \in \left[\frac{N(q-m)}{2}, \infty\right). \tag{8.4}$$

where

$$d = \frac{N}{\sigma}\left(1 - \frac{1}{r}\right), \quad \sigma = N(m-1) + 2.$$

Moreover, the weak solution satisfies

$$t^{\frac{N}{\sigma + \delta}} \left| u(x,t) - G(x, t; \|u_0\|_{L^1(\mathbf{R}^N)}) \right| \to 0 \quad \text{as } t \to \infty \tag{8.5}$$

uniformly with respect to x in the set $|x| \leq R t^{\frac{1}{\sigma}}$, where δ and R are any fixed positive constant and

$$\begin{aligned} M &:= \int_{\mathbf{R}^N} \left(A - \frac{m-1}{2m\sigma} \cdot |x|^2\right)_+^{\frac{1}{m-1}} dx, \\ G(x, t; M) &:= t^{-\frac{N}{\sigma}} \left(A - \frac{m-1}{2m\sigma} \cdot \frac{|x|^2}{t^{\frac{2}{\sigma}}}\right)_+^{\frac{1}{m-1}}. \end{aligned} \tag{8.6}$$

Thus, we observe that the critical exponent $m = q - \frac{2}{N}$ of (KS) is equal to the Fujita's exponent for (PS) with $q = 2$. Consequently, we can see that the critical exponent between the existence and non-existence of the solutions for (KS) and (PS) is same as each other.

Acknowledgments: This article was written while the author stayed at Max Planck Institute for Mathematics in Leipzig. The author would like to express her sincere gratitude to Professors S.Luckhaus and A.Stevens and other all members at MPI for their cordial hospitality; to Professors T.Nagai and T.Ogawa who provided her both encouragement and helpful advice; to Professor P.Biler who provided her the information about their work; to Professor M.A.Herrero who provided his fruitful comments. The author also wishes to express her sincerest gratitude to Professors S.Luckhaus and J.L.Velazquez for their personal communications and helpful comments.

References

[1] N.D.ALIKAKOS, L^p bounds of solutions of reaction-diffusion equations, *Comm. Partial Differential Equations*, **4** (1979), 827-868.

[2] J.M.BALL, Remarks on blow up and nonexistence theorems for nonlinear evolutionary equations, *Quart. J. Math. Oxford Ser.*, **28** (1977), 473-486.

[3] G.I.BARENBLATT, On some unsteady motions of a fluid and a gas in a porous medium, *Prikl. Mat. Mekh.* **16** (1952), 67–78.

[4] P.BILER, T.NADZIEJA, AND R.STANCZY, Nonisothermal systems of self-attracting Fermi-Dirac particles, *Banach Center Pulb.*, **66** (2004), no. 2, 61-78.

[5] P.BILER, Existence and nonexistence of solutions for a model of gravitational interaction of particles, *III. Colloq. Math.*, **68** (1995), no. 2, 229–239

[6] P.BILER, T.NADZIEJA, Existence and nonexistence of solutions for a model of gravitational interaction of particles, *I. Colloq. Math.*, **66** (1994), no. 2, 319–334.

[7] H.BREZIS, Analyse fonctionnelle, Theorie et applications, Masson, (1983).

[8] S.CHILDRESS AND J.K.PERCUS, Nonlinear aspects of chemotaxis , *Math. Biosci.*, **56** (1981), 217–237.

[9] J.I.DIAZ, T.NAGAI, AND J.M.RAKOTOSON, Symmetrization techniques on unbounded domains: Application to a chemotaxis system on $\mathbb{R}^N$, *J.Diffierential Equations*, **145** (1998), 156–183.

[10] H.FUJITA, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. I 13*, (1966), 109–124.

[11] V.A.GALAKTIONOV, S.P.KURDYUMOV, A.P.MIKHAILOV AND A.A.SAMARSKIIN, On unbounded solutions of the Cauchy problem for a parabolic equation $u_t = \nabla \cdot (u^{\sigma \nabla u}) + u^\beta$, *Sov. Phys., Dokl.*, **25** (1980), 458–459.

[12] R.KOWALCZYK, Preventing Blow-up in a Chemotaxis Model, *J. Math. Anal. and Appl.*, (2005).

[13] V.A.GALAKTIONOV, Blow-up for quasilinear heat equations with critical Fujita's exponents, *Proc. Roy. Soc. Edinburgh Sect. A*, **124** (1994), 517–525.

[14] D.GILBARG AND N.S.TRUDINGER, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin-New York, (1983).

[15] M.A.HERRERO, J.L.VELAZQUEZ, A blow-up mechanism for a chemotaxis model, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **24** (1997), no. 4, 633–683.

[16] M.A.HERRERO, J.L.VELAZQUEZ, Singularity patterns in a chemotaxis model, *Math. Ann.*, **306** (1996), no. 3, 583–623.

[17] M.A.HERRERO, J.L.VELAZQUEZ, Chemotactic collapse for the Keller-Segel model, *J. Math. Biol.*, **35** (1996), no. 2, 177–194.

[18] D.HORSTMANN, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences.I., *I.Jahresber.Deutsch.Math.-Verein.*, **105** (2003), no.3, 103–165.

[19] D.HORSTMANN, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences, *II.Jahresber.Deutsch.Math.-Verein.*, **106** (2004), no. 2, 51–69.

[20] W.JÄGER AND S.LUCKHAUS, On explosions of solutions to a system of partial differential equations modelling chemotaxis, *Trans. Amer. Math. Soc.*, **329** (1992), 819-824.

[21] T.KAWANAGO, Existence and behavior of solutions for $u_t = \Delta(u^m) + u^\ell$, *Advances in Mathematical Sciences and Applications*, **7**, No.1, (1997), 367-400.

[22] M.KUROKIBA AND T.OGAWA, Finite time blow-up of the solution for the nonlinear parabolic equation of the drift diffusion type, *Diff. Integral Equations*, **16**, (2003), 427-452.

[23] P.LAURENCOT AND D.WRZOSEK, A chemotaxis model with threshold density and degenerate diffusion, *preprint.*

[24] H.A.LEVINE AND P.E.SACKS, Some existence and nonexistence theorems for solutions of degenerate parabolic equations, *J. Diff. Eqns*, **52**, No.1, (1984), 135-161.

[25] G.L.LIEBERMAN, Second Order Parabolic Differential Equations, World Scientific publishing Co. Pte. Ltd. (1996).

[26] K.MOCHIZUKI AND R.SUZUKI, Critical exponent and critical blow-up for quasilinear parabolic equations, *Israel J. Math*, **98**, (1997), 141-156.

[27] S.LUCKHAUS AND Y. SUGIYAMA, Large time behavior of solutions in subcritical cases to degenerate Keller-Segel systems, *to submit.*

[28] T.NAGAI, Blow-up of radially symmetric solutions to a chemotaxis system, *Adv. Math. Sci. Appl.*, **5**, (1995), 581-601.

[29] T.NAGAI, Behavior of solutions to a parabolic system modeling chemotaxis, *J. Korean Math. Soc.*, **37**, (2000), 721-733.

[30] T.NAGAI, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, *J. Inequal. Appl.*, **6**, (2001), 37-55.

[31] T.NAGAI,T.SENBA, AND T.SUZUKI, Concentration behavior of blow-up solutions for a simplified system of chemotaxis, *RIMS Koukyuroku* , **1181** (2001), 140–176.

[32] T.NAGAI, T. SENBA AND K. YOSHIDA, Application of the Moser-Trudinger inequality to a parabolic system of chemotaxis, *Funkc. Ekvacioj*, **40**, (1997), 411-433.

[33] M.NAKAO, Global solutions for some nonlinear parabolic equations with nonmonotonic perturbations, *Nonlinear Analysis, Theory, Method & Applications*, **10**, No.3, (1986), 299-314.

[34] O.A.OLEINIK AND S.N.KRUZKOV, Quasi-linear second-order parabolic equations with many independent variables, *Russian Math. Surveys*, **16**, (1961), 105-146.

[35] M.ÔTANI AND Y.SUGIYAMA, Lipschitz continuous solutions of some doubly nonlinear parabolic equations, *Discrete and Continuous Dynamical Systems*, **8**, (2002), no.3, 647-670.

[36] A.PAZY, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, (1983).

[37] A.A.SAMARSKII, V.A.GALAKTIONOV, S.P.KURDYUMOV, A.P. MIKHAILOV, Blow-up in quasilinear parabolic equations, Translated from the 1987 Russian.

[38] T.SENBA, T.SUZUKI, Local and norm behavior of blowup solutions to a parabolic system of chemotaxis, *J. Korean Math.Soc.*, **37**, No.6, (2000), 929–941

[39] E.M.STEIN, Singular integrals and differentiability properties of functions , Princeton Mathematical Series, No.30 Princeton University Press, Princeton, N.J. (1970).

[40] Y. SUGIYAMA, Global existence and decay properties of solutions for some degenerate quasi-linear parabolic systems modelling chemotaxis , *Proceedings of the Fourth World Congress of Nonlinear Analysts WCNA-2004, in press.*

[41] Y. SUGIYAMA, Time Global Existence and Asymptotic Behavior of Solutions to Degenerate Quasi-linear Parabolic Systems for Chemotaxis-Growth Models, *to submit.*

[42] Y. SUGIYAMA, Global existence of solutions of the Keller-Segel model with a nonlinear chemotactical sensitivity function , *RIMS Koukyuroku* , *to appear.*

[43] Y. SUGIYAMA AND H. KUNII, Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term , *to submit.*

[44] Y. SUGIYAMA, Global existence in the super critical cases and finite time blow-up in the subcritical cases to degenerate Keller-Segel systems , *to submit.*

[45] Y. SUGIYAMA, Two types of conditions for global existence and blow-up to a degenerate Keller-Segel system, *to submit.*