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Additive structure

Masanori Sawa

Graduate School of Information Science, Nagoya University,
Furo-cho, Chikusa-ku, Nagoya, 464-8601, Japan.
e-mail: discrete@jim.math.cm.is.nagoya-u.ac.jp

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1 Definitions

A BIB design is an ordered pair $(V, \mathcal{B})$ with v points ($|V| = v$) and b blocks of size k, each point appearing in exactly r blocks, each pair of points appearing in exactly λ blocks, which is widely denoted by $\mathrm{B}(v, b, r, k, \lambda)$, or $\mathrm{B}(v, k, \lambda)$ for short [10]. The value r is called replication number and λ coincidence number. A BIB design originates in the design of experiments for statistical analysis, but now it is of combinatorial interest as well. Let $\boldsymbol{N} = (n_{ij})$ be a $v \times b$ incidence matrix of a BIB design, where $n_{ij} = 1$ or 0 for all i $(= 1, 2, ..., v)$ and j $(= 1, 2, ..., b)$, according as the ith point occurs in the jth block or otherwise. Hence an incidence matrix $\boldsymbol{N}$ satisfies the following conditions: (i) $n_{ij} = 0$ or 1 for all i, j, (ii) $\sum_{j=1}^{b} n_{ij} = r$ for all i, (iii) $\sum_{i=1}^{v} n_{ij} = k$ for all j, (iv) $\sum_{j=1}^{b} n_{ij} n_{i'j} = \lambda$ for all i, i' $(i \neq i') = 1, 2, ..., v$.

Let $2 \leq \ell \leq s$. A set of ℓ $\mathrm{B}(v, k, \lambda)$ design, say $\{(V, \mathcal{B}_i) : i = 1, \ldots, \ell\}$ where $\mathcal{B}_i = \{B_j^{(i)} : j = 1, \ldots, b\}$, is said *pairwise additive* if there is a numbering of blocks in each $\mathcal{B}_i$ such that

(A) for any pair $\{h, h'\} \subset V$, $(V, \mathcal{B}_{(h,h')})$ is a $\mathrm{B}(v = sk, k^* = 2k, \lambda^* = 2r(2k-1)/(sk-1))$ where $\mathcal{B}_{(h,h')} = \{B_j^{(h)} \cup B_j^{(h')} : j = 1, \ldots, b\}$.

When $\ell = s$, such s BIB designs are said to have *additive structure*. In this case, it holds that for any j, $\bigcup_{i=1}^{s} B_j^{(i)} = V$ (1.1). The notion of additive structure has been introduced by Matsubara et al. [11]. The same authors [20] analyzed the existence of such structure and proposed its mathematical applications. When $k = 3$ and $\lambda = 1$, instead of *additive structure*, the term *compatibly minimal partition* is used by Colbourn and Rosa [3]. It is easy to state the condition (A) in terms of incidence matrices. Let $\boldsymbol{N}_i$ be incidence matrices of ℓ pairwise additive BIB designs with parameters v, b, r, k, λ, then Condition (A) is rewritten as

$N_{i_1} + N_{i_2}$ is an incidence matrix of a $\mathrm{B}(v = sk, k^* = 2k, \lambda^* = 2r(2k-1)/(sk-1))$ for any distinct $i_1, i_2 \in \{1, 2, ..., \ell\}$,

which makes the proof of Proposition 3.1 easy. Since $n_{ij} = 0$ or 1 for all i, j, if $\ell = s$, then the relation (1.1) implies that $\sum_{i=1}^{s} N_i = J_{v\times b}$, where $J_{v\times b}$ is the $v \times b$ matrix all of whose elements are 1.

Suppose that pairwise additive $\mathrm{B}(v = sk, b, r, k, \lambda)$ exist. Then, for any $\{h, h'\} \subset V$, $(V, \mathcal{B}_{(h,h')})$ is a BIB design with parameters

$$v^* = v, b^* = b, r^* = 2r, k^* = 2k, \lambda^* = 2r(2k-1)/(sk-1).$$

Since λ^* must be a positive integer and $(k-1, 2k-1) = 1$, it holds that

$$2\lambda \equiv 0 \mod (k-1). \tag{1.2}$$

It follows from (1.2) pairwise additive symmetric BIB designs cannot exist for $s \geq 3$ and $k \geq 2$. Furthermore, by using (1.2), characterizations of parameters of BIB designs with pairwise additive structure can be made. Especially, we find that it is combinatorially meaningful to focus on the case that $k > \lambda$, noting the following facts. If k is an odd integer, then by (1.2), it holds that $\lambda \geq (k-1)/2$, and hence BIB designs with $s(2\lambda+1)$ points and blocks of size $k = 2\lambda + 1$ are minimal among BIB designs with pairwise additive structure. If k is an even integer, then similarily $\lambda \geq k-1$, and hence BIB designs with $s(\lambda+1)$ points and blocks of size $k = \lambda + 1$ are minimal. Furthermore, by the well known relation of BIB designs that $\lambda = (k-1)r/(sk-1)$, if $(sk-1, k-1) = 1$ and there exists a BIB design with v points and blocks of size k, then it is a minimal possible design for given v and k. Therefore, if $(sk-1, 2k-1) = 1$, $(V, \mathcal{B}_{(i,i')})$ generates a $\mathrm{B}(v, 2k, 2k-1)$ minimal in terms of coincidence numbers among BIB designs with $v = sk$ points and blocks of size $2k$. Thus, we will combinatorially focus on the case that $k = 2\lambda + 1$ and $k = \lambda + 1$. Pairwise additive BIB designs with $k = 2\lambda + 1$ or $k = \lambda + 1$ have the following parameters:

$$v = sk, b = s(sk-1), r = sk-1, k, \lambda = k-1, \tag{1.3}$$

$$v = sk, b = \frac{s(sk-1)}{2}, r = \frac{sk-1}{2}, k, \lambda = \frac{k-1}{2}. \tag{1.4}$$

We note that the 2-copy of a BIB design with $v = sk$ and $k = 2\lambda + 1$ yields a BIB design with $v^* = sk^*$ and $k^* = \lambda^* + 1$.

Two lists are given; one is a list of parameters s, v, b, r, k, λ for which additive BIB designs with $k > \lambda$ exist, and the other is a list of admissible parameters of BIB designs for which the existence of additive BIB designs is not known. In the latter, "Yes" shows the existence of the design, and ? means that the existence is unknown.

Table 1. Additive BIB designs with $s \geq 3$, $v \leq 100$, $2 \leq k, r \leq 20$ and $k > \lambda$.

No.	s	v	b	r	k	λ
1	3	6	15	5	2	1
2	3	9	12	4	3	1
3	3	9	24	8	3	2
4	3	12	33	11	4	3
5	3	18	51	17	6	5
6	3	21	60	20	7	6
7	3	27	39	13	9	4
8	4	8	28	7	2	1
9	4	12	44	11	3	2
10	4	16	60	15	4	3
11	5	15	70	14	3	2
12	5	25	60	12	5	2
13	8	16	120	15	2	1
14	9	27	117	13	3	1

Table 2. Unknown additive BIB designs with $s \geq 3$, $v \leq 100$, $2 \leq k, r \leq 20$ and $k > \lambda$.

No.	s	v	b	r	k	λ	Existence
1	3	15	42	14	5	4	Yes
2	3	21	30	10	7	3	Yes
3	3	33	48	16	11	5	Yes
4	3	39	57	19	13	6	?
5	4	20	76	19	5	4	Yes
6	5	10	45	9	2	1	Yes
7	5	15	35	7	3	1	Yes
8	5	20	95	19	4	3	Yes
9	5	35	85	17	7	3	Yes
10	6	12	66	11	2	1	Yes
11	6	18	102	17	3	2	Yes
12	7	14	91	13	2	1	Yes
13	7	21	70	10	3	1	Yes
14	7	21	140	20	3	2	Yes
15	7	35	119	17	5	2	Yes
16	9	18	153	17	2	1	Yes
17	10	20	190	19	2	1	Yes
18	11	33	176	16	3	1	Yes
19	13	39	247	19	3	1	Yes

Some characterizations can be made for $k \leq \lambda$. For example, in case of $k = \lambda$, pairwise additive BIB designs are either one of (i) 2-copies of complete designs or (ii) 3-fold triple systems:

$$\begin{array}{llllr} v = 2s, & b = 2s(2s-1), & r = 2(2s-1), & k = \lambda = 2, & (3.2) \\ v = 3(2\ell+1), & b = 3(2\ell+1)(3\ell+1), & r = 3(3\ell+1), & k = \lambda = 3, & (3.3) \end{array}$$

where $\ell \geq 1$. We omit characterizations for the case because pairwise additive designs with $k < \lambda$ have large parameters.

2 Links with perpendicular arrays

A perpendicular array, denoted by $\mathrm{PA}_d(g, s)$, is a matrix with g rows and $d\binom{s}{2}$ columns such that every pair of an s-set appears in exactly d columns among every two rows (see [1], [18]), where $g \geq 1$. d is called index. When $d = 1$, we suppress the index in the notation and write $\mathrm{PA}(g, s)$. Some necessary conditions for perpendicular arrays can be obtained as follows.

Theorem 2.1 [8]. Suppose that $0 \leq t' \leq t$ and $\binom{k}{t} \geq \binom{k}{t'}$, then, a $\mathrm{PA}_d(t, k, s)$ is also a $\mathrm{PA}_{d'}(t', k, s)$, where

$$d' = d\binom{s-t'}{t-t'}/\binom{t}{t'}.$$

In case of $t = 2$ and $t' = 1$, if $g \geq 3$ and there exists a $\mathrm{PA}(g, s)$, then every element appears in each row of the PA equally, and hence s must be an odd integer. We can easily construct a perpendicular array with $g = 2$ and the above property, so that combining these facts with the definition of perpendicular arrays, we have the following.

Theorem 2.2 [20]. g pairwise additive $\mathrm{B}(v = s, b = ds(s-1)/2, r = d(s-1)/2, k = 1, d = 0)$ is equivalent to a perpendicular array $\mathrm{PA}_d(g, s)$ for some $d \geq 1$.

Especially, we note that when $g = s$, the notion of additive BIB designs with s points and $k = 1$ is equivalent to that of a $\mathrm{PA}_d(s, s)$. It is well known that there exists a $\mathrm{PA}(s, s)$ for an odd prime power s [15]. For $d \geq 2$, some results on the existence of a $\mathrm{PA}_d(s, s)$ are known. We note that there are some connections between t-designs and perpendicular arrays; for example, if there exists a t-$(v, t+1, t)$ design, then there exists a $\mathrm{PA}_{d_1}(t, t+1, v)$ where $d_1 = d/(d, t+1)$ [9]. Many informations on perpendicular arrays are available in literature [1], [4].

3 Links with nested designs

Preece [16] introduced the concept of a nested BIB design for the design of experiment in statistics. Many papers on this topics have been published. A nested B(v; b_1, b_2; k_1, k_2) is a triple $(V, \mathcal{B}_1, \mathcal{B}_2)$ with v points ($|V| := v$) and two systems of blocks ($|\mathcal{B}_i| := b_i$), $i = 1, 2$, such that (i) the first system is nested within the second, i.e., each block in $\mathcal{B}_2$ is partitioned into l subblocks of size k_1 and the resulting subblocks form $\mathcal{B}_1$, say, $b_1 = lb_2$ and $k_2 = lk_1$, (ii) $(V, \mathcal{B}_1)$ is a BIB design with v points and b_1 blocks of k_1 points each, (iii) $(V, \mathcal{B}_2)$ is a BIB design with v points and b_2 blocks of k_2 points each. Similarly, Morgan et al. have reviewed and extended the concept of nested BIB designs [13]. A multiply nested BIB design [13] is an $(m+1)$-tuple $(V, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m)$ with v points and m systems of blocks ($|\mathcal{B}_i| = b_i$), $i = 1, \ldots, m$, such that (i) the jth system is nested within the ith system, $i > j$, (ii) for each i ($1 \le i \le m$), $(V, \mathcal{B}_i)$ is a BIB design with v points, b_i blocks of k_i points each. Such a design is denoted by MNB(v; b_1, b_2, ..., b_m ; k_1, k_2, ..., k_m). The spectrum of nested BIB designs is available within the scope of $v \le 16$ and $r \le 30$ [13]. Sawa et al. [20] have proposed a new method of constructing nested (resolvable) BIB designs.

Proposition 3.1 [20]. For $1 \le \ell \le s$, let $\{(V, \mathcal{B}_i) : i = 1, \ldots, \ell\}$ be a set of ℓ pairwise additive B($v = sk, k, \lambda$), where blocks in $\mathcal{B}_i = \{B_j^{(i)} : j = 1, \ldots, b\}$ are ordered suitably. Let

$$\mathcal{B}_R = \{\bigcup_{\ell \in R} B_j^{(\ell)} : j = 1, \ldots, b\}$$

for $R \subset \{1, \ldots, \ell\}$, then $(V, \mathcal{B}_R)$ yields a B(v, gk, λ^*), where $g = |R| \le \ell$.

Proof. Let $\boldsymbol{N}_i$, $i = 1, \ldots, \ell$, be incidence matrices of $(V, \mathcal{B}_i)$ with pairwise additive structure, then $(\boldsymbol{N}_i + \boldsymbol{N}_j)(\boldsymbol{N}_i + \boldsymbol{N}_j)^T = \lambda^* \boldsymbol{I} + (r^* - \lambda^*)\boldsymbol{J}$. Hence, it holds that

$$(\sum_{i \in R} \boldsymbol{N}_i)(\sum_{i \in R} \boldsymbol{N}_i)^T = \sum_{\substack{i,j \in R \\ i \ne j}} (\boldsymbol{N}_i + \boldsymbol{N}_j)(\boldsymbol{N}_i + \boldsymbol{N}_j)^T - (g-2) \sum_{i \in R} \boldsymbol{N}_i \boldsymbol{N}_i^T,$$

which completes the proof. $\square$

By using Proposition 3.1, if there exist pairwise additive BIB designs with v points and blocks of size k each, then a sum of any g incidence matrices generates a BIB design with v points and blocks of size gk each.

Multiply nested BIB designs with $v = sk$ points can be also constructed by use of Proposition 3.1.

Theorem 3.2 [20]. Let $m = \lfloor \log_2 \ell \rfloor + 1$, where $\lfloor x \rfloor$ means the greatest integer y such that $y \le x$. If there exist ℓ pairwise additive B($v = sk, k, \lambda$), then an MNB(v; $b_1 = 2^{m-1}b_m$, $b_2 = 2^{m-2}b_m$, $\ldots$, b_m ; k_1, $k_2 = 2k_1$, $\ldots$, $k_m = 2^{m-1}k_1$) can be constructed. In particular, when $\ell = s = 2^{m-1}$, a resolvable multiply nested BIB design can be obtained.

4 Links with combinatorial geometries

Morgan [14] has constructed series of BIB designs by taking union of blocks of symmetric BIB designs sutably. This method of constructing designs has been referred as 'the union method" by Rahilly [17]. The union method is considered under the situation that each block of the resulting design is composed of blocks each being not necessarily disjoint. In the sense, the notion of additive designs are included in the union method, thinking about Proposition 3.1.

Rahilly [17] also has focused on a parallelism, and took the union method. Suppose that there exists a resolvable $\mathrm{B}(v = sk, k, \lambda)$, $(V, \mathcal{B})$, with $2 \le \ell \le s-1$. Rahilly has constructed BIB designs by taking the union of any ℓ blocks in each parallel class. Hence, we have the following result.

Theorem 4.1 [17]. If there exists a resolvable $\mathrm{B}(v = sk, b, r, k, \lambda)$, then there exists a $\mathrm{B}(v = sk, \ell k, \lambda\binom{t-1}{\ell-1} + (r-\lambda)\binom{t-2}{\ell-2})$ for $2 \le \ell \le s-1$.

Using a parallelism in different way from Rahilly's, we can get the following new result.

Theorem 4.2 [20]. If there exists a resolvable $\mathrm{B}(sk, b, r, k, \lambda)$ and a $\mathrm{PA}(g, s)$, then there exists a $\mathrm{B}(sk,\ s(s-1)r/2,\ (s-1)\ell r/2,\ \ell k,\ \lambda\ell(\ell k-1)/(k-1))$ for any $1 \le \ell \le g \le s$ and $g \ge 2$.

As series of BIB designs with block size being a prime power, it is well known that the set of t-flats in $\mathrm{AG}(n, s)$ forms a BIB design with $v = s^n$, $k = s^t$ and $\lambda = \begin{bmatrix} n-1 \\ t-1 \end{bmatrix}_q$. By applying Theorems 4.1 and 4.2 each to the fact that the well-known necessary conditions for a resolvable $\mathrm{B}(v = s^n, k = s^m, \lambda)$ with $n > m$ are sufficient for any prime power s [23], series of BIB designs with $v = s^n$ points and blocks of size ℓs^m can be obtained. We observe that when $\ell = 2$, two resulting designs of these two theorems each are the same, but when $\ell \ge 3$, the resulting BIB design of Theorems 4.1 has much larger coincidence number than that Theorem 4.2. For illustrative purpose, two series of BIB designs, each of which is obtained by applying Theorems 4.1 and 4.2 to the Woelfel's result with $m = n-1$ and $\lambda = (s^n-1)/(s-1)$, are given.

Theorem 4.3 [17]. There exists a $\mathrm{B}(v = s^n, k = \ell s^{n-1}, \lambda = \frac{s^n-1}{s-1}\binom{s-1}{\ell-1} + \frac{s^n-s}{s-1}\binom{s-2}{\ell-2})$.

Theorem 4.4 [20]. There exists a $\mathrm{B}(v = s^n, k = \ell s^{n-1}, \lambda = \ell(\ell s^{n-1}-1)/2)$.

We note that when $\ell = 2$, the resulting BIB designs have the minimal coincidence numbers for given $v = s^n$ and $k = s^{n-1}$. More generally, Jimbo and Sawa have constructed [19] new series of BIB designs by $\mathrm{AG}(n, s)$

Theorem 4.5 [19]. There exists (i) a $\mathrm{B}(v = s^n, k = ls, \lambda = (ls - 1)l/2)$ for any $2 \le l \le s$ and s being any odd prime power (ii) a $\mathrm{B}(v = 3^n, k = 3l, \lambda = (3l - 1)l/2)$ for any $2 \le l \le 3^{n-1}$.

For $\ell = 2$, the minimalities of λ for given v and k can be analyzed.

Mahmoodian and Shirdarreh [12] showed that Morgan's BIB designs in [14] are simple. Rahilly also investigated the simpleness of his designs [17]. Jimbo and Sawa have shown that the BIB designs in Theorem 4.5 are simple for $2 \le l \le (p+1)/2$.

Theorem 4.6 [19]. Let $s = p^m$ be an odd prime power. There exists a simple $\mathrm{B}(v = s^n, k = ls, \lambda = (ls - 1)l/2)$ for any $2 \le l \le (p+1)/2$.

In the last of this section, we introduce some results on the union method of constructing designs with higher regular incidence structure.

Theorem 4.7 [6]. If $(2m+1, 3) = 1$, then there exists a simple $\mathrm{S}_3(3, 4, 2(2m+1))$.

There are many results of constructing simple t-designs by using a parallelism of a resolvable design. For example, see [21], [22].

5 Additive structure

Arguments in this section is all characteristic of the case that $\ell = s$. A new method of constructing BIB designs is provided in Proposition 5.1. For given v and k, minimalities of the resulting BIB designs, together with the property of resolvablity, have been analyzed [20]. We will introduce one of such constructions without proofs.

A difference matrix, denoted by $\mathrm{D}(g, \lambda; s) := (d_{mn})$, based on a group $(G, *)$ of order s, is a $g \times \lambda s$ matrix satisfying the condition that for any x in G, there exist exactly λ columns in which x is represented by $d_{mn} * (d_{m'n})^{-1}$ among the mth and m'th row of the matrix. λ is called index. Necessarily, the number of columns of $\mathrm{D}(g, \lambda; s)$ is λs. For $\mathrm{D}(g, \lambda; s)$, a row with all entries x for some x in G is possibly included, and there exist at most one such a row [5]. Some existence results of difference matrices should be referred to literatures [2], [5]. Here we use a difference matrix with more conditions that (i) among two rows without a row of all x, if α and β appears in the mth and m'th row precisely ℓ times, then β and α appears in the mth and m'th row precisely ℓ times (ii) a row with all entries x for some x in G is possibly included. Such a difference matrix is called *symmetric*, and denoted by $\mathrm{SD}(g, \lambda; s)$. When $g = s$, we suppress the index λ in the notation.

By use of $\mathrm{SD}(s, \lambda)$, we have the following theorem.

Theorem 5.1 [20]. Let c and d be integers with $2\lambda c \equiv 0 \mod d(k-1)$. If there exist additive $\mathrm{B}(v = sk, b, r, k, \lambda)$, an $\mathrm{SD}(s, c)$ based on a group $(G, *)$ and a $\mathrm{PA}_d(s, s)$, then there exist additive $\mathrm{B}(v^* = s^2k, b^* = cs[(s+1)r - s\lambda]$, $r^* = c[(s+1)r - s\lambda]$, $k^* = sk$, $\lambda^* = cr)$.

By the assumption of additive BIB designs, the condition that $2\lambda c \equiv 0 \mod d(k-1)$ is always satisfied for the case of d dividing c.

Corollary 5.2 [20]. Let s be an odd integer. If there exist additive $\mathrm{B}(v = sk, b, r, k, \lambda)$, an $s \times s^2$ OA and a $\mathrm{PA}(s, s)$, then there exist additive $\mathrm{B}(v^* = s^2k, b^* = s^2[(s+1)r - s\lambda]$, $r^* = s[(s+1)r - s\lambda]$, $k^* = sk$, $\lambda^* = sr)$.

Proof. Obviously, an $s \times s^2$ OA is regarded as an $\mathrm{SD}(s, s)$. □

The resulting BIB designs given in Corollary 5.2 have the large coincidence numbers, thinking about the minimalities of pairwise additive BIB designs. In order to get additive BIB designs with small coincidence numbers, the existence of symmetric difference matrices with small indices is essentially required. It can be shown [20] from the definition of SD that if there exists an $\mathrm{SD}(s, \lambda)$ and s is an even integer, then λ is also an even integer. Concerning the arguments of the existence of symmetric difference matrices, Sawa et al. [20] showed that when s is a prime power, there exists an $\mathrm{SD}(s, 2)$, and when s is an odd prime, there exists an $\mathrm{SD}(s, 1)$. For other value $\lambda \leq 2$, we cannot find whether there exists an $\mathrm{SD}(s, \lambda)$. Since a $\mathrm{D}(s, 1)$ generates an $s \times s^2$ OA, it may exist only for a prime power.

Theorem 5.3 [20]. If there exist additive $\mathrm{B}(v = sk, b, r, k, \lambda)$, then there exist additive BIB designs with parameters

(i) $v^* = s^2k, b^* = 2s[(s+1)r - s\lambda], r^* = 2[(s+1)r - s\lambda], k^* = sk, \lambda^* = 2r$

for a prime power s, and

(ii) $v^* = s^2k, b^* = s[(s+1)r - s\lambda], r^* = (s+1)r - s\lambda, k^* = sk, \lambda^* = r$

for an odd prime s.

Proof. Apply Theorem 5.1 and the facts mentioned above. □

In Theorem 5.3 (ii), we start from additive BIB designs with v points and $k = \lambda + 1$ or $k = 2\lambda + 1$ respectively, and then additive BIB designs with $v^* = sv$ points and $k^* = \lambda^* + 1$ or $k^* = 2\lambda^* + 1$ respectively, can be constructed recursively.

6 Related unsolved problem

Firstly, there are many admissible parameters left, for which it is still unknown that additive BIB designs exist or not.

Problem 1 Do BIB designs in Table 2 have additive structure?

Secondly, in order that we can get minimal additive BIB designs by using Theorem 5.3, an SD$(s, 1)$ is required. Unfortunately, Sawa et al. [20] cannot find whether an SD$(s, 1)$ exists or not for an odd prime power $s = p^m$, where $m \geq 2$.

Problem 2 Does there exist an SD$(s, 1)$ for an odd prime power $s = p^m$ and $m \geq 2$?

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