BROWN-PETERSON COHOMOLOGY OF $BPU(p)$

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Let $p$ be a fixed odd prime and denote by $BP^{*}(X)$ (resp. $P(m)^{*}(X)$) the Brown-Peterson cohomology of a space $X$ with the coefficient ring $BP^{*} = \mathbb{Z}_{(p)}[v_{1}, v_{2}, \cdots]$ (resp. $P(m)^{*} = \mathbb{Z}/p[v_{m}, v_{m+1}, \cdots]$) where $\deg v_{k} = -2p^{k} + 2$. We denote by $PU(n)$ the projective unitary group which is the quotient of the unitary group $U(n)$ by its center $S^{1}$. Recall that the cohomologies of $PU(p)$ and exceptional Lie groups $F_{4}, E_{6}, E_{7}, E_{8}$ have odd torsion elements. In this paper, we compute the Brown-Peterson cohomologies of classifying spaces $BG$ of these Lie groups $G$ as $BP^{*}$-modules using the Adams spectral sequence. Let us write $H^{*}(X; \mathbb{Z}/p)$ by simply $H^{*}(X)$ and let $A$ be the mod $p$ Steenrod algebra.

Our main result is as follows:

**Theorem 0.1.** Let $(G, p)$ be one of cases $(G = PU(p), p)$ for an arbitrary odd prime $p$ and $G = F_{4}, E_{6}, E_{7}$ for $p = 3$, and $G = E_{8}$ for $p = 5$. Then the $E_{2}$-terms of the Adams spectral sequences abutting to $BP^{*}(BG)$ and $P(m)^{*}(BG)$ for $m \geq 1$

$$\mathrm{Ext}_{A}^{s,t}(H^{*}(BP), H^{*}(BG)), \quad \mathrm{Ext}_{A}^{s,t}(H^{*}(P(m)), H^{*}(BG))$$

have no odd degree elements.

As an immediate consequence is as follows:

**Corollary 0.2.** For $(G, p)$ in Theorem 1.1, the Adams spectral sequence abutting to $BP^{*}(BG)$ and $P(m)^{*}(BG)$ in the previous theorem collapse at the $E_{2}$-level. In particular $BP^{odd}(BG) = P(m)^{odd}(BG) = 0$.

Recall $K(m)^{*}(X) \cong K(m)^{*} \otimes_{P(m)^{*}} P(m)^{*}(X)$ is the Morava $K$-theory. From above theorem and corollary, we see $K(m)^{odd}(BPU(p)) = 0$. Then we have the following corollary ([Ko-Ya],[Ra-Wi-Ya])

**Corollary 0.3.** For $(G, p)$ in Theorem 1.1, the following holds:

1. $BP^{*}(BG)$ is $BP^{*}$-flat for $BP^{*}(BP)$-modules, i.e.,
   $BP^{*}(BG \times X) \cong BP^{*}(BG) \otimes_{BP^{*}} BP^{*}(X)$ for all finite complexes $X$

2. $K(n)^{*}(BG) \cong K(n)^{*} \otimes_{BP^{*}} BP^{*}(BG)$.

3. $P(n)^{*}(BG) \cong P(n)^{*} \otimes_{BP^{*}} BP^{*}(BG)$.

We give the $BP^{*}$-module structure of $BP^{*}(BPU(p))$ more explicitly, in this talk.

**Theorem 0.4.** There is a $BP^{*}$-algebra isomorphism

$$0 \rightarrow BP^{*} \hat{\otimes} M \rightarrow grBP^{*}(BPU(p)) \rightarrow BP^{*} \hat{\otimes} IN/(f_{0}, f_{1}) \rightarrow 0$$

where

1. $M \cong \mathbb{Z}_{(p)}[x_{4}, x_{6}, \cdots, x_{2p}]$ as $\mathbb{Z}_{(p)}$-modules (but not $\mathbb{Z}_{(p)}$-algebras).

2. $IN \cong \mathbb{Z}_{(p)}[x_{2p+2}, x_{2p+2}(p-1)] \{x_{2p+2}\}$; the principal ideal of $\mathbb{Z}[x_{2p+2}, x_{2p+2}(p-1)]$

generated by $x_{2p+2}$. 


3. relations \( f_0, f_1 \) are given with modulo \( (p, v_1, v_2, \ldots)^2 \)

\[
f_0 \equiv v_0 - v_2 x_{2p+1}^2 + \cdots, \quad f_1 \equiv v_1 - v_2 x_{2p(p-1)} + \cdots.
\]

**Remark 0.5.** In the above theorem, suffix i of \( x_i \) means its degree. \( BP^* (BPU(p)) \) does not contain the subalgebra \( BP^* \otimes \mathbb{Z}(p)[x_4, \ldots, x_{2p}] \), but contains a subalgebra which is isomorphic as \( BP^* \)-modules to the above \( BP^* \)-subalgebra.

For an algebraic group \( G \) over \( \mathbb{C} \), Totaro defines its Chow ring \([Tc]\) and conjectures that \( BP^*(BG) \otimes_{BP^*} \mathbb{Z}(p) \cong CH^*(BG)(p) \). Recall that \( PGL(p, \mathbb{C}) \) is the algebraic group over \( \mathbb{C} \) corresponding the Lie group \( PU(p) \).

**Theorem 0.6.** There is the isomorphism

\[
BP^*(BPU(p)) \otimes_{BP^*} \mathbb{Z}(p) \cong CH^*(BGL(p, \mathbb{C}))(p).
\]

Hence there is the additive isomorphism

\[
CH^*(BGL(p, \mathbb{C}))(p) \cong \mathbb{Z}(p)[x_4, x_6, \ldots, x_{2p}] \oplus \mathbb{F}_p[x_{2p+2}, x_{2p(p-1)}\{x_{2p+2}\}.
\]

**Remark.** Recently Vistoli \([V]\) also determined the additive structure of the Chow ring and integral cohomology of \( BGL(p, \mathbb{F}_p) \) by using stratified methods of Vessoz. Moreover he shows that for \( G = PGL(p, \mathbb{C}) \)

\[
H^*(G; \mathbb{Z}) \rightarrow H^*(BT; \mathbb{Z})^{W_G(T)}
\]

is epic.

Let \( MGL^*, *(X) \) be the motivic cobordism ring defined by V.Voevodsky \([Vo]\) and \( MGL^{2*, *}(X) = \oplus_i MGL^{2i, i}(X) \).

**Corollary 0.7.** \( MGL^{2*, *}(BGL(p, \mathbb{C}))(p) \cong MU^*(BPU(p))(p) \).

We prove Theorem 1.1 using the Adams spectral sequence converging to the Brown-Peterson cohomology. The \( E_1 \)-term of the spectral sequence could be given by

\[
\mathbb{F}_p[v_0, v_1, \ldots] \otimes H^*(X) \quad \text{with} \quad d_1 x = \sum_{k=0}^{\infty} v_k Q_k x
\]

where \( Q_k \)'s are Milnor's operations. By the change-of-rings isomorphism, the \( E_2 \)-term is

\[
\text{Ext}_A(H^*(BP), H^*(X)) \cong \text{Ext}_E(\mathbb{F}_p, H^*(X))
\]

where \( E = \Lambda(Q_0, Q_1, \cdots) \). The \( E_\infty \)-term is given by \( gr BP^*(X) \).

To state the cohomology \( H^*(BPU(p)) \), we recall the Dickson algebra. Let \( A_n \) be an elementary abelian \( p \)-group of rank \( n \), and

\[
H^*(B A_n) \cong \mathbb{F}_p[t_1, \ldots, t_n] \otimes \Lambda(dt_1, \ldots, dt_n) \quad \text{with} \quad \beta(dt_i) = t_i.
\]

The Dickson algebra is

\[
D_n = \mathbb{F}_p[t_1, \ldots, t_n]^{GL(n, \mathbb{F}_p)} \cong \mathbb{F}_p[c_{n,0}, \ldots, c_{n,n-1}]
\]

with \( |c_{n,i}| = 2(p^n - p^i) \). The invariant ring under \( SL(n, \mathbb{F}_p) \) is also given

\[
SD_n = \mathbb{F}_p[t_1, \ldots, t_n]^{SL(n, \mathbb{F}_p)} \cong D_n\{c_{n,0}, \ldots, e_n^{p-2}\} \quad \text{with} \quad e_n^{p-1} = c_{n,0}.
\]

We also recall the Muǐ's \([Mu]\) result by using \( Q_i \) by \([Ka-Mi]\)

\[
gr H^*(BA)^{SL(n, \mathbb{F}_p)} \cong SD_n(e_n) \oplus SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}\{u_n\}
\]

where \( u_n = dt_1 \cdots dt_n \) and \( e_n = Q_0 \cdots Q_{n-1} u_n \).
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Theorem 0.8. There is the short exact sequence

$$0 \to M/p \to H^*(BPU(p)) \to N \to 0$$

where $M/p$ is the trivial $E$-module given in Theorem 1.4 and

$$N = SD_2 \otimes \Lambda(Q_0, Q_1\{u_2\} \cong \mathbb{F}[x_{2p+2}, x_{2(p^2-p)}] \otimes \Lambda(Q_0, Q_1\{u_2\})$$

identifying $x_{2p+2} = e_2$ and $x_{2(p^2-p)} = c_{2,1}$.

This theorem is proved by using the following facts. The group $G = PU(p)$ has just two conjugacy classes of maximal elementary abelian $p$-subgroups, one of which is toral and the other is non-toral $A$ of $\text{rank}_p = 2$. The cohomology $H^*(BG)$ is detected by this two subgroups. The restriction image to the non-toral subgroup is $i_A^*(H^*PU(p)) \cong H^*(BA^{SL(2,F_p)})$. Similar (but not same) facts also hold for the exceptional Lie groups in Theorem 1.1.

Algebraic main result in this talk is as follows:

Theorem 0.9. For $m \geq 0$, define $f_0, \ldots, f_{n-1}$ in $P(m)^* \hat{\otimes} SD_n$ by

$$d_1 u_n = \sum_{k \geq m} v_k Q_k(u_n) = f_0 Q_0 u_n + \cdots + f_{n-1} Q_{n-1} u_n.$$

Then the sequence $f_0, \ldots, f_{n-1}$ is a regular sequence in $P(m)^* \hat{\otimes} SD_n$.

With the notation in this theorem, we prove that the complex

$$C = (P(m)^* \hat{\otimes} SD_n \otimes \Lambda(Q_0, Q_1, \cdots, Q_{n-1}\{u_n\}), d_1)$$

with the differential $d_1 u_n = \sum_{i=0}^{n-1} f_i Q_i u_n$ is a K"{o}nnul complex. This means that

$$H_i(C, d_1) = \begin{cases} P(m)^* \hat{\otimes} SD_n\{e_n\}/(f_0, \cdots, f_{n-1}) & \text{for } i = 0 \\ 0 & \text{for } i \geq 1. \end{cases}$$

Thus Theorem 1.1 follows from the above theorem.

Remark about the convergence of the Adams spectral sequence. By Theorem 15.6 in Boardman’s paper [Bo2], since $H^*(BP)$ is of finite type, the above Adams spectral sequence is conditionally convergent. Moreover, since we prove the above Adams spectral sequence collapses at the $E_2$-level, by the remark after Theorem 7.1 in [Bo1], the above Adams spectral sequence is strongly convergent, so that we know the Brown-Peterson cohomology up to group extension.

REFERENCES


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