Note on blocks of $p$-solvable groups with same Brauer category

熊本大学理学部　渡辺アツミ（Atumi Watanabe）
Department of Mathematics. Faculty of Science
Kumamoto University

1

Let $p$ be a prime and let $\mathcal{O}$ be a complete discrete valuation ring with an algebraically closed residue field $k$ of characteristic $p$. Let $G$ be finite group and $b$ be a block of $G$ with maximal $(G, b)$-subpair $(P, e_P)$ where $b$ is a block idempotent of $\mathcal{O}G$. For any subgroup $Q$ of $P$, let $(Q, e_Q)$ be a unique $(G, b)$-subpair contained in $(P, e_P)$. Following Kessar, Linckelmann and Robinson [4], we denote by $\mathcal{F}_{(P, e_P)}(G, b)$ the category whose objects are subgroups of $P$ and for $Q, R \leq P$, whose set of morphisms from $Q$ to $R$ are the set of group homomorphisms $\varphi : Q \to R$ such that there exists $x \in G$ such that $x(Q, e_Q) \subseteq (R, e_R)$ and $\varphi(u) = xux^{-1}$ for all $u \in Q$. We call $\mathcal{F}_{(P, e_P)}(G, b)$ the Brauer category of $b$. Let $\mathcal{B}_G(b)$ be the Brauer category of $b$ in the sense of Thévenaz [10], § 47. The categories $\mathcal{F}_{(P, e_P)}(G, b)$ and $\mathcal{B}_G(b)$ are equivalent. Let $R$ be a normal subgroup of $P$ such that $N_G(P) \subseteq N_G(R)$ and $c$ be the Brauer correspondent of $b$ in $N_G(R)$, that is, $c$ is a unique block of $N_G(R)$ such that $\text{Br}_P(c) = \text{Br}_P(b)$ where $\text{Br}_P$ is the Brauer homomorphism from $(\mathcal{O}G)^P$ onto $kC_G(P)$. Set $N = N_G(R)$. The notations $R, c$ and $N$ are fixed. Thus $b = c^G$ and $(P, e_P)$ is a maximal $(N, c)$-subpair. The arguments in the proof of Theorem in Kessar-Linckelmann [5] imply the following.

Theorem 1 Assume that $G$ is $p$-solvable. With the above notations, suppose that $\mathcal{F}_{(P, e_P)}(G, b) = \mathcal{F}_{(P, e_P)}(N, c)$. Then there is an indecomposable $\mathcal{O}Gb-\mathcal{O}Nc$-bimodule $M$ which satisfies the following.

(i) $M$ and its $\mathcal{O}$-dual $M^*$ induce a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Nc$.

(ii) As an $\mathcal{O}(G \times N)$-module $M$ has a vertex $\Delta P$ and an endo-permutation $\mathcal{O}(\Delta P)$-module as a source where $\Delta P = \{ (u, u) \mid u \in P \}$.

Let $H^*_c(P, e_P)(G, b)$ be the cohomology ring of $b$ in the sense of Linckelmann[6], [7], that is, $H^*_c(P, e_P)(G, b)$ is the subring of $H^*(P, k)$ consisting of $\zeta \in H^*(P, k)$ satisfying $\text{res}_Q \zeta = ^g\text{res}_Q \zeta$ for all $Q \leq P$ and, for all $g \in N_G(Q, e_Q)$. We prove the following.

Theorem 2 Assume that $G$ is $p$-solvable. With the above notations, if $H^*_c(P, e_P)(G, b) = H^*_c(P, e_P)(N, c)$, then $\mathcal{F}_{(P, e_P)}(G, b) = \mathcal{F}_{(P, e_P)}(N, c)$. 
2

We prove Theorem 1 using the following.

**Lemma 1** (Harris-Linckelmann [3], Lemma 4.2) Assume that $G$ is $p$-solvable. For any $p$-subgroup $Q$ of $G$, we have $O_p'(N_G(Q)) = O_p'(G) \cap N_G(Q) = O_p'(G) \cap C_G(Q) = O_p'(C_G(Q))$.

**Proposition 1** (Harris-Linckelmann [2], Proposition 3.1 (iii)) Let $G$ be a $p$-solvable group and $b$ be a block of $G$ such that $b$ covers a $G$-invariant block of $O_p'(G)$. Then $b$ is of principal type. That is, for any $p$-subgroup $Q$ of $G$, $Br_Q(b)$ is a block of $kC_G(Q)$.

**Proposition 2** (Fong[1]; Puig[9]) Let $G$ be a $p$-solvable group and $b$ be a block of $G$ with defect group $P$. Then the following holds.

(i) There is a subgroup $H$ of $G$ and an $H$-invariant block $e$ of $O_p'(H)$ such that $O_p'(G)P \subseteq H$ and $OGB \cong \text{Ind}^G_H(\mathcal{O}He)$ as interior $G$-algebras.

(ii) $P$ is a Sylow $p$-subgroup of $H$ and $P$ is a defect group of $e$ as a block of $H$. Moreover let $(P,e')$ be a maximal $(H,c)$-subpair and let $e_P = \text{Tr}^G_P O_{G,H}(e')$. Then $(P,e_P)$ is a maximal $(G,b)$-subpair.

Note that in the above proposition $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_{(P,e_P)}(H,e)$ since $OGB \cong \text{Ind}^G_H(\mathcal{O}He)$ as interior $G$-algebras.

**Proposition 3** ([5]. Proposition 6) With the notations in the above proposition, let $R$ be a subgroup of $P$ such that $N_G(P) \subseteq N_G(R)$. Denote by $c$ the Brauer correspondent of $b$ in $N_G(R)$, and by $f$ the Brauer correspondent of $e$ in $N_H(R)$. Then $f$ is an $N_H(R)$-invariant block of $O_p'(N_H(R))$ and $ON_G(R)c \cong \text{Ind}^{N_G(R)}_{N_H(R)}(ON_H(R)f)$ as interior $N_G(R)$-algebras.

The following is shown in the proof of Theorem in [5].

**Theorem 3** (Kessar-Linckelmann) Let $G$ be a $p$-solvable group and $b$ be a block of $G$ with defect group $P$. Let $R$ be a subgroup of $P$ such that $N_G(P) \subseteq N_G(R)$ and let $e$ be the Brauer correspondent of $b$ in $N$ where we set $N = N_G(R)$. If $b$ covers a $G$-invariant block of $O_p'(G)$ and if $G = O_p'(G)N$, then there is an indecomposable $OGB \cdot ONc$-bimodule $M$ which satisfies the following.

(i) $M$ and its $O$-dual $M^*$ induce a Morita equivalence between $OGB$ and $ONc$.

(ii) As an $O(G \times N)$-module $M$ has a vertex $\Delta P$ and an endo-permutation $O(\Delta P)$-module as a source.

**Proof of Theorem 1.** We prove by induction on $|G|$. Let $H$, $e$, $e'_p$ and $e_p$ be as in Proposition 2, and let $f$ be as in Proposition 3. We may assume that $e_p$'s in Theorem 1 and Proposition 2 are equal by replacing $H$, $e$, $e'_p$, and $f$, by $H^x$, $e^x$, $(e'_p)^x$ and $f^x$ respectively for some $x \in N_G(P)$ if necessary. By Proposition 2,

By Proposition 3, \((P, e''_P)\) is a maximal \((N_H(R), f)\)-subpair and

\[
\]

So by the assumption we have \(F_{(P, e''_P)}(H, e) = F_{(P, e''_P)}(N_H(R), f)\). Since \(OGB \cong \text{Ind}_{H}^{G}(OH_e)\) as interior \(G\)-algebras, the \(OGB-OHe\)-bimodule \(bOGe = OGe\) and the \(OHe(OGB\)- bimodule \(eOGe \) induce a Morita equivalence between \(OGB\) and \(OHe\). Similarly the \(ONc-ON_H(R)f\)-bimodule \(ONf\) and the \(ON_H(R)f-ONc\)-bimodule \(fON\) induce a Morita equivalence between \(ONc\) and \(ON_H(R)f\). Suppose that \(H < G\). By the induction hypothesis for \(H\) and \(e\), there is an indecomposable \(OHe-\)

\(ON_H(R)f\)- bimodule \(M_0\) such that \(M_0\) and \(M_0^e\) induce a Morita equivalence between \(OHe\) and \(ON_H(R)f\). and that \(M_0\) as an \(O(H \times N_H(R))\)-module has a vertex \(\Delta P\) and an endo-permutation \(O(\Delta P)\)-module as a source. Set \(M = bOGB \otimes_{OHe} M_0 \otimes_{ON_H(R)f} ONc \cong M_0^{G \times N}\). Then \(M\) satisfies (i) and (ii) in Theorem 1. Therefore we may assume that \(H = G\). Then \(b = e\).

Let \(Y = O_{e''_P}(G)\). Then \(b\) is a \(G\)-invariant block of \(Y\) because \(Y/O_{e''_P}(G)\) is a \(p\)-group. Furthermore we have \(Y = O_{e''_P}(G)(Y \cap P)\). Set \(Q = P \cap Y\). Then \(Q\) is a defect group of \(b\) as a block of \(Y\). Now since \(G\) is constrained, \(C_Y(Q) = C_G(Q)\). Therefore we see that \((Q, e_Q)\) is a maximal \((Y, b)\)-subpair. By the Frattini argument and the assumption that \(F_{(P, e''_P)}(G, b) = F_{(P, e''_P)}(N, c)\).

\[
G = N_G(Q, e_Q)Y \subseteq N_Y(Q)C_Y(Q)Y \subseteq NY \subseteq NO_{e''_P}(G).
\]

So we have \(G = NO_{e''_P}(G)\). This and Theorem 3 complete the proof.

Proof of Theorem 2. We prove by induction on \(|G|\). Let \(H\), \(e\), \(e'\), and \(e_P\) be as in Proposition 2, and let \(f\) be as in Proposition 3. We may assume that \(e_P\) is in Theorem 2 and Proposition 2 are equal as in the proof of Theorem 1. Since \(F_{(P, e''_P)}(G, b) = F_{(P, e''_P)}(H, e)\) and \(F_{(P, e''_P)}(N, c) = F_{(P, e''_P)}(N_H(R), f)\) we have

\[
\]

\[
\]

From the assumption, we have \(H^*_e(P, e''_P)(H, e) = H^*_e(P, e''_P)(N_H(R), f)\). Suppose that \(H < G\). Then by the induction hypothesis, \(F_{(P, e''_P)}(H, e) = F_{(P, e''_P)}(N_H(R), f)\), and hence \(F_{(P, e''_P)}(G, b) = F_{(P, e''_P)}(N, c)\). Therefore we may assume that \(H = G\). Then \(b\) covers a \(G\)-invariant block of \(O_{e''_P}(G)\) and \(P\) is a Sylow \(p\)-subgroup of \(G\). Note that the element \(b \in O_{e''_P}(G)\).

From Proposition 1, \(b\) is of principal type. On the other hand, by Lemma 1, \(B_{R}(b)\) is an \(N\)-invariant block idempotent of \(kO_{e''_P}(N)\) and \(c\) is a lifting of \(B_{R}(b)\) to \(ON\). So by Proposition 1, \(c\) is also of principal type. So we may assume that \(b\) is a principal block. Therefore by a theorem of Mislin [8], we obtain \(F_{(P, e''_P)}(G, b) = F_{(P, e''_P)}(N, c)\). This completes the proof.
References


