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Towards Main Conjectures for Modular Forms

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This short note is a summary of talks given by the author at the RIMS workshop on automorphic forms and automorphic L-functions held in January 2005. The talks were essentially reports on two projects whose aims are to identify certain p-adic L-functions with the characteristic polynomials of Selmer groups of certain "big" Galois representations. That is, to prove various main conjectures in Iwasawa and thereby establish "special value formulae" for various classical L-functions (such as the Hasse-Weil L-functions of elliptic curves). One of these projects is being carried out jointly with Eric Urban and the other with Michael Harris and Jian-Shu Li.

The p-adic L-functions of interest herein interpolate special values of L-functions of modular forms twisted by Hecke characters. The "big" Galois representations piece together all the relevant twists of the p-adic Galois representations associated to these modular forms.

Throughout, p denotes a fixed odd prime. We denote by K any imaginary quadratic field in which p splits. We let $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ and $G_K = \mathrm{Gal}(\bar{\mathbf{Q}}/K)$. For each prime v of K we fix an embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{K}_v$. This determines decomposition and inertia groups D_v and I_v. We also fix embeddings $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$. We let Σ be any finite set of primes distinct from p.

1. The L-functions

Let $f = \sum_{m=1}^{\infty} a_m q^m$ be a normalized holomorphic cuspidal eigenform of weight $k \geq 2$, level N, and character χ. For a Hecke character ψ of K of infinity-type z^{-n}, $n \geq 0$, we let $L^{\Sigma}(f,\psi,s)$ be the L-function of f twisted by ψ:

$$L^{\Sigma}(f,\psi,s) = \sum_{(\mathfrak{m},\Sigma)=1} a_m \psi(\mathfrak{m}) N(\mathfrak{m})^{-s}, \quad (m) = \mathfrak{m} \cap \mathbf{Z},$$

with $\mathfrak{m}$ running over integral ideals of K, $(\mathfrak{m},\Sigma) = 1$ meaning that $\mathfrak{m}$ is coprime to the primes in Σ, and always taking $m > 0$. If g_ψ is the CM-form of weight $n+1$ associated to ψ, then $L^{\Sigma}(f,\psi,s)$ is the Rankin-Selberg convolution $L^{\Sigma}(f \times g_\psi, s)$. We distinguish two cases:

Case 1. In the first case we take $n = 0$ (i.e., ψ is finite). Furthermore, we assume that

$$p|N \quad \text{and} \quad |a_p|_p = 1. \qquad \text{(ord)}$$

That is, f is a p-normalized p-ordinary eigenform.

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Case 2. In this case we take $k = 2$ and $n = 3$ and assume that $p \nmid N$.

We now describe the p-adic L-function in each of these cases. Let $\mathcal{O}$ be the ring of integers of a finite extension F of $\mathbf{Q}_p$ containing all the a_m's and let λ be a uniformizer of $\mathcal{O}$. Let K_∞/K be the rank-two $\mathbf{Z}_p$-extension of K and let $H = \mathrm{Gal}(K_\infty/K)$. Then H decomposes as $H = H^+ \oplus H^-$, with any complex conjugation acting on $H^\pm$ as ± 1. Let $\Lambda = \mathcal{O}[\![H]\!]$ and let $\Psi : G_K \to \Lambda^\times$ be the character obtained by composing the projection to H with the canonical inclusion of H in $\Lambda^\times$. Let $\gamma^\pm$ be a topological generator of $H^\pm$ ($H^\pm \cong \mathbf{Z}_p$). There is an isomorphism $\Lambda \to \mathcal{O}[\![X_+, X_-]\!]$, $\gamma^\pm \mapsto X_\pm$. Given a pair of pth-power roots of unity $\underline{\zeta} = (\zeta_+, \zeta_-)$, let $\psi_{\underline{\zeta}} : G_K \to \mathcal{O}_{\underline{\zeta}}^\times$ be the finite order character that sends $\gamma^\pm$ to $\zeta_\pm$. Similarly, let $\phi_{\underline{\zeta}} : \Lambda \twoheadrightarrow \mathcal{O}_{\underline{\zeta}}$ be the $\mathcal{O}$-algebra homorphism sending $\gamma^\pm$ to $\zeta_\pm$ (so $\psi_{\underline{\zeta}} = \phi_{\underline{\zeta}} \circ \Phi$). Here $\mathcal{O}_{\underline{\zeta}}$ is the ring of integers of the finite extension $F_{\underline{\zeta}}$ of F obtained by adjoining both ζ_+ and ζ_-. Let $\mathfrak{p}_{\underline{\zeta}}$ be the kernel of $\phi_{\underline{\zeta}}$. Let $\Lambda^* = \mathrm{Hom}_{cts}(\Lambda, \mathbf{Q}_p/\mathbf{Z}_p)$ be the Pontryagin dual of Λ. Let $\Lambda_\pm = \mathcal{O}[\![H/H^\mp]\!]$, let $\Phi_\pm : \Lambda \twoheadrightarrow \Lambda_\pm$ be the canonical surjection, and let $\mathfrak{p}_\pm$ be the kernel of $\Phi_\pm$. (Note that $\Lambda_\pm \cong \mathcal{O}[\![X_\pm]\!]$.)

Case 1. There exists an element $\mathcal{L}_f^\Sigma \in \Lambda$ such that for any $\phi_{\underline{\zeta}}$

$$\phi_{\underline{\zeta}}(\mathcal{L}_f^\Sigma) = a(f, \underline{\zeta}) L^\Sigma(f, \psi_{\underline{\zeta}}, k-1)/\Omega_f,$$

where $a(f, \underline{\zeta})$ is essentially a Gauss sum and Ω_f is a period for f (cf. [GS]). Here we identify the finite Galois character $\psi_{\underline{\zeta}}$ with a finite Hecke character for K via the usual global reciprocity map for K.

Case 2. Let ψ be any fixed Hecke character of K of infinity-type z^{-3}. We will assume that $\mathcal{O}$ is large enough that it contains the values $\psi(\mathrm{Frob}_\mathfrak{p})$ for almost all prime ideals $\mathfrak{p}$ of K. There exists an element $\mathcal{L}_{f,\psi}^\Sigma \in \Lambda$ such that for any $\phi_{\underline{\zeta}}$

$$\phi_{\underline{\zeta}}(\mathcal{L}_{f,\psi}^\Sigma) = b(f, \underline{\zeta}) L^\Sigma(f, \psi\psi_{\underline{\zeta}}, 3)/\Omega_\psi,$$

where $b(f, \underline{\zeta})$ is essentially a Gauss sum and Ω_ψ is a CM-period determined by ψ. Again, we identify $\psi_{\underline{\zeta}}$ with a finite Hecke character via class field theory. In this case the existence of $\mathcal{L}_{f,\psi}^\Sigma$ was established by many authors, but we indicate a different proof below. We let $\mathcal{L}_{f,\psi,0}^\Sigma = \Phi(\mathcal{L}_{f,\psi}^\Sigma) \in \Lambda_-$.

2. The Selmer groups

Let V be a two-dimensional F-space and let $\rho_f : G_\mathbf{Q} \to \mathrm{GL}(V)$ be the usual Galois representation associated to f. So ρ_f is unramified at primes $\ell \nmid Np$, and for such primes $\mathrm{trace}\rho_f(\mathrm{Frob}_\ell) = a_\ell$. Let $T \subseteq V$ be any $G_\mathbf{Q}$-stable $\mathcal{O}$-lattice. To simplify matters we will assume that

$$G_K \text{ acts irreducibly on } T/\lambda T. \tag{irr}$$

Following Greenberg [G1,G2], we define certain Selmer groups in both cases.

Case 1. The condition (ord) together with the splitting of p in K ensures that for $v|p$

$$\rho_f|_{D_v} \cong \begin{pmatrix} \chi_{1,v}\epsilon^{k-1} & * \\ & \chi_{2,v} \end{pmatrix}, \quad \chi_{2,v}|_{I_v} = 1,$$

with ϵ the p-adic cyclotomic character. Let $V_v^+ \subseteq V$ be the F-line on which D_v acts via $\chi_{1,v}\epsilon^{k-1}$. Let $T_v^+ = T \cap V_v^+$. Let $W = T \otimes_{\mathcal{O}} \Lambda^*$ and let G_K act on W via $\rho_f \otimes \Psi\epsilon^{2-k}$. Then W is a discrete G_K-module. Let $W_v^+ = T_v^+ \otimes_{\mathcal{O}} \Lambda^*$. This is a D_v-stable submodule of W.

The Selmer group we consider in this case is defined to be:

$$\mathrm{Sel}^{\Sigma}(W) = \ker\left\{ H^1(G_K, W) \to \prod_{v|p} H^1(I_v, W/W^+) \times \prod_{w|\ell, \ell\notin\Sigma} H^1(I_w, W) \right\}, \quad \text{(Sel)}$$

where the arrow is the product of the obvious restriction maps. This is a discrete Λ-module and we let $S^{\Sigma}(W)$ be its Pontryagin dual, which is a finitely-generated Λ-module.

We note that if Σ contains all divisors of N other than p then $\mathrm{Sel}^{\Sigma}(W)[\mathfrak{p}_{\underline{\zeta}}]$ is just the Selmer group $\mathrm{Sel}^{\Sigma}(f, \underline{\zeta})$, where the latter is defined as in (Sel) but with W and W_v^+ replaced by $T \otimes_{\mathcal{O}} F_{\underline{\zeta}}/\mathcal{O}_{\underline{\zeta}}$ and $T_v^+ \otimes_{\mathcal{O}} F_{\underline{\zeta}}/\mathcal{O}_{\underline{\zeta}}$, respectively, and the G_K-action is via $\rho_f \otimes \psi_{\underline{\zeta}}\epsilon^{2-k}$. Then $S^{\Sigma}(W)/\mathfrak{p}_{\underline{\zeta}}S^{\Sigma}(W)$ is naturally isomorphic to the Pontryagin dual of $\mathrm{Sel}^{\Sigma}(f, \underline{\zeta})$,

Case 2. Let $\psi_p : G_K \to \mathcal{O}^\times$ be the character such that $\psi_p(\mathrm{Frob}_{\mathfrak{p}}) = \psi(\mathfrak{p})$ for any prime ideal $\mathfrak{p}$ of K that does not divide p or the conductor of ψ. Then ψ_p is Hodge-Tate at each prime $v|p$ with Hodge-Tate weights being 3 at one of these primes and 0 at the other. We let v_1 and v_2 be the two primes dividing p, ordered so that the weight at v_1 is 3. Then the representation $\rho_f \otimes \psi_p\epsilon^{-2}$ of G_K on V is Hodge-Tate at each v_i. The weights at v_1 are $2, 1$ and the weights at v_2 are $-1, -2$.

Let $V_{v_1}^+ = V$ and $V_{v_2}^+ = 0$. Let W be as in Case 1 but with the G_K-action now being by $\rho_f \otimes \Psi\psi_p\epsilon^{-2}$. Defining T_v^+ as in Case 1 but with our present definitions of the V_v^+'s, we also define W_v^+ as in Case 1. We then let $\mathrm{Sel}^{\Sigma}(W, \psi)$ be defined by the right-hand side of (Sel) and let $S^{\Sigma}(W, \psi)$ be the Pontryagin dual of $\mathrm{Sel}^{\Sigma}(W, \psi)$. We define $\mathrm{Sel}^{\Sigma}(f, \psi, \underline{\zeta})$ to be the obvious analog of $\mathrm{Sel}^{\Sigma}(f, \underline{\zeta})$.

We define a variant of this Selmer group, $Sel^{\Sigma}(W_-, \psi)$, with W, W_v replaced by $W_-, W_{-,v}$, where the latter are defined by replacing Λ^* with Λ_-^*. Clearly, if Σ contains the primes that divide N other than p then $Sel^{\Sigma}(W_-, \psi) = \mathrm{Sel}^{\Sigma}(W, \psi)[\mathfrak{p}_-]$. Let $S^{\Sigma}(W_-, \psi)$ be the Pontryagin dual of $\mathrm{Sel}^{\Sigma}(W_-, \psi)$.

3. The connections

The connection between the p-adic L-functions and the Selmer groups in each of the cases is summarized by a "main conjecture." These are just special cases of the conjectures in [G1,G2]

Main Conjecture. (Case 1) *In the situation of Case* 1,

(i) $S^\Sigma(W)$ *is a torsion* Λ*-module.*

(ii) *For any height-one prime* P *of* Λ,

$$\text{length}_{\Lambda_P}(S^\Sigma(W)_P) = \text{ord}_P(\mathcal{L}_f^\Sigma).$$

The main conjectures in Case 2 take exactly the same form:

Main Conjecture. (Case 2) *In the situation of Case* 2,

(i) $S^\Sigma(W,\psi)$ *is a torsion* Λ*-module.*

(ii) *For any height-one prime* P *of* Λ,

$$\text{length}_{\Lambda_P}(S^\Sigma(W,\psi)_P) = \text{ord}_P(\mathcal{L}_{f,\psi}^\Sigma).$$

Main Conjecture. (Case 2)$_-$ *In the situation of Case* 2,

(i) $S^\Sigma(W_-,\psi)$ *is a torsion* Λ_-*-module.*

(ii) *For any height-one prime* P *of* Λ_-,

$$\text{length}_{\Lambda_{-,P}}(S^\Sigma(W_-,\psi)_P) = \text{ord}_P(\mathcal{L}_{f,\psi,-}^\Sigma).$$

Remarks. (1) Perhaps the most significant consequence of part (ii) of these conjectures is that if Σ contains all the primes dividing N other than p then

$$\#\text{Sel}^\Sigma(f,\underline{\zeta}) = \#\mathcal{O}_{\underline{\zeta}}/(a(f,\underline{\zeta})L^\Sigma(f,\psi_{\underline{\zeta}},k-1)/\Omega_f), \quad \text{(Case 1)}$$

$$\#\text{Sel}^\Sigma(f,\psi,\underline{\zeta}) = \#\mathcal{O}_{\underline{\zeta}}/(b(f,\underline{\zeta})L^\Sigma(f,\psi\psi_{\underline{\zeta}},3)/\Omega_\psi), \quad \text{(Case 2)}.$$

If we only know the Main Conjecture (Case 2)$_-$ then the last equality can only be deduced when $\zeta_+ = 1$. All this follows from an easy argument employing Fitting ideals.

(2) Clearly the Main Conjecture (Case 2)$_-$ follows from the Main Conjecture (Case 2), at least if Σ contains all primes dividing N other than p.

Sections 4 and 5 of this paper discuss efforts to establish these conjectures. In Case 1, significant progress was made by K. Kato in [K], where he proves (among many other important results) that $S^\Sigma(W)/\mathfrak{p}_+S^\Sigma(W)$ is a torsion Λ_+-module whose characteristic ideal contains $\Phi_+(\mathcal{L}_f^\Sigma)$. Part (i) of the Main Conjecture (Case 1) follows from this. In joint work with E. Urban, we prove that the equality in part (ii) of the conjecture can at least be replaced by $\geq$ in many cases. When combined with Kato's results this proves the Main Conjecture in these cases. For example:

Theorem. (Kato, Skinner-Urban) *Assuming the situation of Case 1, if $\chi\epsilon^{k-2}$ is trivial modulo λ, if there exists a prime $\ell || N$ different from p such that $T/\lambda T$ is ramified at ℓ, and if the action of $G_{\mathbf{Q}}$ on $T/\lambda T$ is irreducible, then the Main Conjecture is true.*

Remark. (3) The joint work with Urban makes use of four-dimensional p-adic Galois representations attached to certain automorphic representations of a quasi-split unitary group in four variables. The establishment of these representations is an on-going project, so the careful reader may wish to view this theorem as conditional.

The following is a nice corollary of this theorem.

Corollary. *Let E be a semistable elliptic curve of $\mathbf{Q}$. Suppose E has good ordinary reduction at p and $G_{\mathbf{Q}}$ acts irreducibly on $E[p]$. If Σ contains all the primes of bad reduction and $L(E,1) \neq 0$, then $|L^{\Sigma}(E,1)/\Omega_E|_p$ has the value predicted by the Birch-Swinnerton-Dyer conjecture.*

Recall that for any E over $\mathbf{Q}$, there is a normalized weight 2 eigenform f of trivial character such that $L(E,s) = L(f,s)$. If E has ordinary reduction at p then f satisfies (ord). If $G_{\mathbf{Q}}$ acts trivially on $E[p]$, then (irr) holds and the periods of E and f are the same up to p-adic multiples (cf. [GV]). As a consequence of a standard "level-lowering" argument, the assumption that E has semistable reduction ensures that a prime ℓ as in the hypotheses of the Theorem exists. Thus all the hypotheses of the Theorem are satisfied for this choice of f. The Corollary then follows easily from the Theorem and Remark (1).

4. First Project

This is joint work with Eric Urban. We work in the setting of Case 1, proving in many instances that $\mathrm{length}_P(S^{\Sigma}(W)_P) \geq \mathrm{ord}_P(\mathcal{L}_f^{\Sigma})$, notation being as in part (ii) of the Main Conjecture (Case 1). Our strategy for doing this follows that used by Wiles in his proof of the (cyclotomic) Main Conjecture for totally real fields [W]. This strategy is as follows.

Step 1. To each $\underline{\zeta}$ we associate an Eisenstein series $E(f,\underline{\zeta})$ on the Hermitian upper-half space of degree 2 (relative to the imaginary quadratic field K; cf [F]). This is a holomorphic modular form of weight k. This Eisenstein series has a q-expansion indexed by semi-definite Hermitian matrices in some lattice $L \subset M_2(K)$: $E(f,\underline{\zeta}) = \sum_{T\in L, T\geq 0} a_T(\underline{\zeta})q^T$. The Eisenstein series is normalized so that each $a_T(\underline{\zeta}) \in \mathcal{O}_{\underline{\zeta}}$ and if $\det(T) = 0$, then $a_T(\underline{\zeta})$ is divisible by $a(f,\underline{\zeta})L^{\Sigma}(f,\psi_{\underline{\zeta}},k-1)/\Omega_f$.

Step 2. We show that there exists a formal q-expansion $\mathcal{E} = \sum_{T\in L, T\geq 0} A_T q^T$, $A_T \in \Lambda$ such that $E(f,\underline{\zeta}) = \sum \phi_{\underline{\zeta}}(A_n)q^T$.

Step 3. We show that some $A_T \in \Lambda^\times$, $\det(T) > 0$.

Step 4. For $P \subset \Lambda$ a height-one prime and $r = \mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathcal{F},K})$, using the theory of ordinary p-adic modular forms (as developed by Hida) we show that there is a cuspform $\mathcal{G}_P \in \Lambda[\![q^T]\!]$ such that if $\mathcal{G}_P = \sum_{T>0} B_T q^T$ then

$$B_T \equiv A_T \bmod P^r.$$

(That $\mathcal{G}_P$ is a cuspform means that each $\sum \phi_{\underline{\zeta}}(H_T)q^T$ is a cuspform on the Hermitian upper half-space of degree two.)

Step 5. We use the Galois representations associated to eigenforms on the Hermitian upper half-space together with the congruences from Step 4 to construct subgroups in the Selmer group $\mathrm{Sel}^{\Sigma}(W)$.

The spirit of this step can be sketched as follows. Let P be as in Step 4 with $r \geq 1$. Then Step 4 implies that there exists a finite, integrally closed extension Λ' of Λ, a prime $P' \subset \Lambda'$ extending P, and a cuspidal Λ'-eigenform $\mathcal{G}$ such that the hecke eigenvalues of $\mathcal{G}$ are congruent modulo P' to those of $\mathcal{E}$. To simplify notation, we will assume $\Lambda' = \Lambda$ and $P' = P$. Let $k_P = \Lambda_P/P\Lambda_P$. Then, assuming the existence of the (conjectured) four-dimensional G_K-representations associated to the specializations $\phi_{\underline{\zeta}}(\mathcal{G})$ and the generic irreducibility of these representations, one can deduce the existence of a representation

$$\rho_P : G_K \to \mathrm{GL}_4(k_P), \quad \rho_P = \begin{pmatrix} \chi\Psi^c\epsilon & *' & *'' \\ 0 & \rho_f & * \\ 0 & 0 & \Psi^{-1}\epsilon^{k-2} \end{pmatrix}$$

that is unramified away from the primes above p and the places in Σ and is such that $\rho_P|_{D_p}$ is split but the quotient representation

$$\rho'_P = \begin{pmatrix} \rho_{\mathcal{F}} & * \\ 0 & \Psi^{-1}\epsilon^{k-2} \end{pmatrix}$$

is not. Galois cohomology classes of degree 1 (i.e. in H^1) classify such extensions, so it should not be hard to believe that from the existence of ρ'_P it is possible to deduce that $\mathrm{length}_P(S^{\Sigma}(W)_P) \geq 1$.

Remarks. (4) Steps 1 and 2 can be carried out using the pull-backs of Siegel Eisenstein series on the Hermitian upper half-space of degree 3.

(5) Step 3 requires an explicit computation. The $a_T(\underline{\zeta})$'s turn out to be essentially special values of Rankin-Selberg convolution L-functions. And under the hypothesis of the existence of a prime ℓ as in the statement of the theorem and the irreduciblity of the action of $G_{\mathbf{Q}}$ on $T/\lambda T$, we show that a result of Vatsal [V] implies that some $a_T(\underline{\zeta})$, $\det(T) > 0$, some $\underline{\zeta}$, is a unit in $\mathcal{O}_{\underline{\zeta}}$.

(6) Step 4 is fairly straightforward, as is Step 5, provided we know the Galois representations exist.

(7) In practice, for technical reasons we have to introduce another variable, allowing the weight to vary. This is possible since (ord) implies that f belongs to a Hida family.

5. Second Project

This is joint with M. Harris and J.-S. Li. As much as possible we try to work in a setting where the role of f in Case 2 is played by a cuspidal automorphic representation π on some unitary group $U(r,s)$, but we stick to the setting of Case 2 in our discussion. The ultimate goal of this project is to prove a generalization of the Main Conjecture (Case $2)_-$.

We assume that f is in the image of the Jacquet-Langlands correspondence from a definite unitary group to GL_2. That is, there is a two-dimensional K-space V and a definite Hermitian pairing $<-,->: V \times V \to K$ such that if $G = U(V)$ is the algebraic group over $\mathbf{Q}$ determined by this pairing, then there is an automorphic form $\varphi : G(\mathbf{A}) \to \mathbf{C}$ having the same Hecke eigenvalues as f. Without loss of generality, it can be assumed that φ has values in $\mathcal{O}$.

The first part of the project is to construct the p-adic L-functions $\mathcal{L}^{\Sigma}_{f,\psi}$ and $\mathcal{L}^{\Sigma}_{f,\psi,0}$. This is essentially completed in full generality (i.e., working on $U(r,s)$). Our method generalizes that of Katz in [Ka], which should be viewed as the $U(1)$ version. In the setting of Case 2, we begin the construction by introducing the Hermitian space $W = V \oplus V$ and the pairing $<(x,x'),(y,y')>_W = <x,y> - <x',y'>$. Given ψ and $\underline{\zeta}$ we constuct an explicit Seigel-Eisenstein series $E(\psi,\underline{\zeta})$ on $U(W)$ such that, roughly speaking,

$$< \varphi \otimes \varphi, E(\psi,\underline{\zeta}) >_{G\times G} = b(f,\underline{\zeta}) L^{\Sigma}(f,\psi\psi_{\underline{\zeta}},3) < \varphi,\varphi >_G, \qquad (*)$$

where the pairings are the Petersson pairings on $G \times G$ and G, respectively, and $E(\psi,\underline{\zeta})$ is restricted to a form on $G \times G$ via the obvious embedding $U(V)\times U(V) \hookrightarrow U(W)$. The formula (*) is a specific instance of the "doubling method" of [GPSR]. The Eisenstein series $E(\psi,\underline{\zeta})$ is normalized to have Fourier coefficients in $\mathcal{O}_{\underline{\zeta}}$, so its restrictions to $G \times G$ is a form taking values in $\mathcal{O}_{\underline{\zeta}} \cdot$ CM-period. By (*) and the definition of the pairings, the RHS of (*) also takes values in $\mathcal{O}_{\underline{\zeta}} \cdot (\text{CM-period})^2$. The p-adic L-function is obtained by showing that the $E(\psi,\underline{\zeta})$'s are values of a p-adic measure on H (done by interpolating their fourier coefficients): using this Eisenstein measure it is easy to see that the RHS of (*) is the value of a measure on H, and a short argument involving congruence ideals then proves the same of $b(f,\underline{\zeta})L^{\Sigma}(f,\psi\psi_{\underline{\zeta}},3)/\Omega_{\underline{\zeta}}$.

The second goal of the project is to relate the p-adic L-functions to the Selmer groups is a manner similar to the statement of the Main Conjecture (Case $2)_-$.

The strategy we hope to follow is very similar to that in the first project, only we do not use Eisenstein series. Instead we introduce another definite Hermitian space V' of dimension three over K and let $G' = U(V')$. We then consider an explicit theta lift $\theta_\psi(\varphi, \underline{\zeta})$ of the form φ. This exists only when $\zeta^+ = 1$ and is a $\mathcal{O}_{\underline{\zeta}}$-valued form on $G'(\mathbf{A})$. The inner-product formula of Rallis [R] implies, roughly, that

$$< \theta_\psi(\varphi, \underline{\zeta}), \theta_\psi(\varphi, \underline{\zeta}) >_{G'} = b(f, \underline{\zeta}) L^\Sigma(f, \psi\psi_{\underline{\zeta}}, 3)/\Omega_\psi < \varphi, \varphi >_G . \qquad (**)$$

Then analogously to Steps 2 and 3 of the strategy for Case 1 (see Section 4), we want to show that the $\theta_\psi(\varphi, \underline{\zeta})$ are values of a measure $\Theta_\psi(\varphi)$ on $H^- = H/H^+$, which can be viewed roughly as a Λ_--valued form on G', and show that some value of one of these theta-lifts is a p-adic unit. Then (**) leads to something like

$$< \Theta_\psi(\varphi), \Theta_\psi(\varphi) >_{G'} = \mathcal{L}^\Sigma_{f,\psi,0} < \varphi, \varphi >_G . \qquad (***)$$

The LHS of (***) can be interpreted as measuring congruences between $\Theta_\psi(\varphi)$ and other Λ_--valued forms on G'. The next step is to make sence of the two pieces appearing on the RHS. The inner-product $< \varphi, \varphi >_G$ should measure all the congruences between φ and other forms on G, and thus it should also measure all the congruences between $\Theta_\psi(\varphi)$ and other theta-lifts. The remaining term on the LHS of (***), the p-adic L-function, should then measure congruences between $\Theta_\psi(\varphi)$ and non-theta-lifts. These non-lifts should have irreducible three-dimensional Galois representations associated to them, and proceeding much as in Step 5 of Case 1 one should be able to relate the divisibility of the L-function to elements in the Selmer group.

This is very much still work in progress.

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