Certain classes of analytic functions concerned with uniformly starlike and convex functions

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Abstract
Applying the coefficient inequalities of functions $f(z)$ belonging to the subclasses $\mathcal{M}D(\alpha,\beta)$ and $\mathcal{N}D(\alpha,\beta)$ of certain analytic functions in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{M}D^*(\alpha,\beta)$ and $\mathcal{N}D^*(\alpha,\beta)$ are introduced. The object of the present paper is to derive some convolution properties of functions $f(z)$ in the classes $\mathcal{M}D^*(\alpha,\beta)$ and $\mathcal{N}D^*(\alpha,\beta)$.

1 Introduction
Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{ z \in \mathbb{C} | |z| < 1 \}$. Shams, Kulkarni and Jahangiri [4] have studied the subclass $\mathcal{S}D(\alpha,\beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

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for some $\alpha (\alpha \geq 0)$ and for some $\beta (0 \leq \beta < 1)$. The subclass $\mathcal{K}D(\alpha, \beta)$ is defined by $f(z) \in \mathcal{K}D(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{S}D(\alpha, \beta)$. In view of the classes $\mathcal{S}D(\alpha, \beta)$ and $\mathcal{K}D(\alpha, \beta)$, we introduce the subclass $\mathcal{M}D(\alpha, \beta)$ consisting of all functions $f(z) \in \mathcal{A}$ which satisfy

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$

for some $\alpha (\alpha \leq 0)$ and for some $\beta (\beta > 1)$. The class $\mathcal{N}D(\alpha, \beta)$ is also considered as the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy $zf'(z) \in \mathcal{M}D(\alpha, \beta)$. We discuss some properties of functions $f(z)$ belonging to the classes $\mathcal{M}D(\alpha, \beta)$ and $\mathcal{N}D(\alpha, \beta)$.

We note if $f(z) \in \mathcal{M}D(\alpha, \beta)$, then, for $\alpha < -1$, $zf'(z)/f(z)$ lies in the region $G \equiv G(\alpha, \beta) \equiv \{ w = u + iv : \text{Re} at < \alpha|w - 1| + \beta \}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

$$\left( u - \frac{\alpha^2 - \beta}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2 (\beta - 1)^2}{(\alpha^2 - 1)^2}$$

with vertices at the points

$$\left( \frac{\alpha^2 - \beta}{\alpha^2 - 1}, \frac{\beta - 1}{\alpha^2 - 1} \right), \left( \frac{\alpha^2 - \beta}{\alpha^2 - 1}, \frac{1 - \beta}{\alpha^2 - 1} \right), \left( \frac{\alpha + \beta}{\alpha + 1}, 0 \right), \left( \frac{\alpha - \beta}{\alpha - 1}, 0 \right).$$

Since $\frac{\alpha + \beta}{\alpha + 1} < 1 < \frac{\alpha - \beta}{\alpha - 1} < \beta$, we have $\mathcal{M}D(\alpha, \beta) \subset \mathcal{M}D(0, \beta) \equiv \mathcal{M}(\beta)$. For $\alpha = -1$, if $f(z) \in \mathcal{M}D(\alpha, \beta)$, then $zf'(z)/f(z)$ belongs to the region which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2 - \beta^2 + 1}{2(\beta - 1)}.$$ 

In the case of $f(z) \in \mathcal{N}D(\alpha, \beta)$, $zf''(z)/f(z)$ lies in the region which contains $w = 0$ and is bounded by the ellipse

$$\left( u + \frac{\beta - 1}{\alpha^2 - 1} \right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2 (\beta - 1)^2}{(\alpha^2 - 1)^2} \quad (\alpha < -1)$$

with vertices at the points

$$\left( \frac{1 - \beta}{\alpha^2 - 1}, \frac{\beta - 1}{\sqrt{\alpha^2 - 1}} \right), \left( \frac{1 - \beta}{\alpha^2 - 1}, \frac{\beta - 1}{\sqrt{\alpha^2 - 1}} \right), \left( \frac{1 - \beta}{\alpha - 1}, 0 \right), \left( \frac{\beta - 1}{\alpha + 1}, 0 \right).$$

Since $\frac{\beta - 1}{\alpha + 1} < 0 < \frac{1 - \beta}{\alpha - 1} < \beta$, we have $\mathcal{N}D(\alpha, \beta) \subset \mathcal{N}D(0, \beta) \equiv \mathcal{M}(\beta)$. And for $\alpha = -1$, $zf''(z)/f(z)$ lies in the domain which contains $w = 0$ and is bounded by parabola

$$u = -\frac{v^2}{2(\beta - 1)} + \frac{\beta - 1}{2}.$$
The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were considered by Uralegaddi, Ganigi and Sarangi [3], Nishiwaki and Owa [1], and Owa and Nishiwaki [2].

2 Coefficient inequalities for the classes $\mathcal{M}D(\alpha, \beta)$ and $\mathcal{N'D}(\alpha, \beta)$

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2.1. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|$$

for some $\alpha (\alpha \leq 0)$ and for some $\beta (\beta > 1)$, then $f(z) \in MD(\alpha, \beta)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|$$

for $f(z) \in A$. It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| - \beta \right| + 1 < 1 \quad (z \in \mathbb{U}).$$

We have

$$\left| \frac{zf'(z)}{f(z)} - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| - \beta \right| + 1 = \left| zf'(z) - \alpha e^{i\theta} |zf'(z) - f(z)| - \beta f(z) + f(z) \right|$$

$$= \left| zf'(z) - \alpha e^{i\theta} |zf'(z) - f(z)| - \beta f(z) - f(z) \right|$$

$$= \left| \frac{2 - \beta + \sum_{n=2}^{\infty} (n-\beta+1)a_n z^{n-1} - \alpha e^{i\theta} \sum_{n=2}^{\infty} (n-1)a_n z^{n-1}}{ -(\beta - \sum_{n=2}^{\infty} (n-\beta-1)a_n z^{n-1} + \alpha e^{i\theta} \sum_{n=2}^{\infty} (n-1)a_n z^{n-1})} \right|$$
The last expression is bounded above by 1 if

\[
|2 - \beta| + \sum_{n=2}^{\infty} |n - \beta + 1||a_n| - \alpha \sum_{n=2}^{\infty} (n-1)|a_n| \leq \beta - \sum_{n=2}^{\infty} \alpha (n-1)|a_n|
\]

which is equivalent to our condition

\[
\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|
\]

of the theorem. This completes the proof of the theorem.

By using Theorem 2.1, we have

**Corollary 2.1.** If \( f(z) \in A \) satisfies

\[
\sum_{n=2}^{\infty} n\{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)\}|a_n| \leq \beta - |2 - \beta|
\]

for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \), then \( f(z) \in ND(\alpha, \beta) \)

**Proof.** From \( f(z) \in ND(\alpha, \beta) \) if and only if \( zf'(z) \in MD(\alpha, \beta) \), replacing \( a_n \) by \( na_n \) in Theorem 2.1, we have the corollary.

### 3 Relation for \( MD^*(\alpha, \beta) \) and \( ND^*(\alpha, \beta) \)

By Theorem 2.1, the class \( MD^*(\alpha, \beta) \) is considered as the subclass of \( MD(\alpha, \beta) \) consisting of \( f(z) \) satisfying

\[
\sum_{n=2}^{\infty} \{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)|a_n| \leq \beta - |2 - \beta|
\]

for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \). The class \( ND^*(\alpha, \beta) \) is also considered as the subclass of \( ND(\alpha, \beta) \) consisting of \( f(z) \) which satisfy

\[
\sum_{n=2}^{\infty} n\{|n - \beta + 1| + |n - \beta - 1| - 2\alpha(n-1)\}|a_n| \leq \beta - |2 - \beta|
\]

for some \( \alpha(\alpha \leq 0) \) and for some \( \beta(\beta > 1) \). By the coefficient inequalities for the classes \( MD^*(\alpha, \beta) \) and \( ND^*(\alpha, \beta) \), we see
Theorem 3.1. If $f(z) \in \mathcal{A}$, then
\[ \mathcal{M}D^*(\alpha_1, \beta) \subset \mathcal{M}D^*(\alpha_2, \beta) \]
for some $\alpha_1$ and $\alpha_2 (\alpha_1 \leq \alpha_2 \leq 0)$.

Proof. For $\alpha_1 \leq \alpha_2 \leq 0$, we obtain
\[ \sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha_2(n-1)|a_n| \]
\[ \leq \sum_{n=2}^{\infty} \{|n-\beta+1| + |n-\beta-1| - 2\alpha_1(n-1)|a_n| \}. \]
Therefore, if $f(z) \in \mathcal{M}D^*(\alpha_1, \beta)$, then $f(z) \in \mathcal{M}D^*(\alpha_2, \beta)$. Hence we get the required result. 

By using Theorem 3.1, we also have

Corollary 3.1. If $f(z) \in \mathcal{A}$, then
\[ \mathcal{N}D^*(\alpha_1, \beta) \subset \mathcal{N}D^*(\alpha_2, \beta) \]
for some $\alpha_1$ and $\alpha_2 (\alpha_1 \leq \alpha_2 \leq 0)$.

4 Convolution of the classes $\mathcal{M}D^*(\alpha, \beta)$ and $\mathcal{N}D^*(\alpha, \beta)$

For analytic functions $f_j(z)$ given by
\[ f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2, \cdots, p), \]
the Hadamard product (or convolution) of $f_1(z), f_2(z), \cdots, f_p(z)$ is defined by
\[ (f_1 \ast f_2 \ast \cdots \ast f_p)(z) = z + \sum_{n=2}^{\infty} \left( \prod_{j=1}^{p} a_{n,j} \right) z^n. \]
Thus we have

Theorem 4.1. If $f_1(z) \in \mathcal{M}D^*(\alpha, \beta_1)$ and $f_2(z) \in \mathcal{M}D^*(\alpha, \beta_2)$ for some $\alpha (\alpha \leq 2 - \sqrt{5})$ and $\beta_1, \beta_2 (1 < \beta_1, \beta_2 \leq 2)$, then $(f_1 \ast f_2) \in \mathcal{M}D^*(\alpha, \beta)$, where
\[ \beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}. \]
**Proof.** From (3.1), for \( f(z) \in \mathcal{MD}^*(\alpha, \beta) \) with \( 1 < \beta \leq 2 \), we have
\[
\sum_{n=2}^{\infty} \{(n+1-\beta)+(n-1-\beta)-2\alpha(n-1)\}|a_n| \leq \sum_{n=2}^{\infty} \{(n+1-\beta)+|n-1-\beta|-2\alpha(n-1)\}|a_n|
\]
\[
\leq 2(\beta-1),
\]
that is, if \( f(z) \in \mathcal{MD}^*(\alpha, \beta) \), then
\[
(4.1) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta+\alpha}{\beta-1}|a_n| \leq 1.
\]
Conversely, if \( f(z) \) satisfies
\[
(4.2) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}|a_n| \leq 1,
\]
then \( f(z) \in \mathcal{MD}^*(\alpha, \beta) \). From (4.1), if \( f_1(z) \in \mathcal{MD}^*(\alpha, \beta_1) \), then
\[
(4.3) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta_1+\alpha}{\beta_1-1}|a_{n,1}| \leq 1,
\]
and also if \( f_2(z) \in \mathcal{MD}^*(\alpha, \beta_2) \), then
\[
(4.4) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)-\beta_2+\alpha}{\beta_2-1}|a_{n,2}| \leq 1.
\]
Applying the Schwarz’s inequality, we have the following inequality
\[
(4.5) \quad \sum_{n=2}^{\infty} \sqrt{\frac{(n(1-\alpha)-\beta_1+\alpha)(n(1-\alpha)-\beta_2+\alpha)}{(\beta_1-1)(\beta_2-1)}} \sqrt{|a_{n,1}||a_{n,2}|} \leq 1
\]
by (4.3) and (4.4). From (4.2) and (4.5), if the following inequality
\[
(4.6) \quad \sum_{n=2}^{\infty} \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1}|a_{n,1}||a_{n,2}|
\]
\[
\leq \sum_{n=2}^{\infty} \sqrt{\frac{(n(1-\alpha)-\beta_1+\alpha)(n(1-\alpha)-\beta_2+\alpha)}{(\beta_1-1)(\beta_2-1)}} \sqrt{|a_{n,1}||a_{n,2}|}
\]
is satisfied, then we say that \( f(z) \in \mathcal{MD}^*(\alpha, \beta) \). This inequality holds true if
\[
(4.7) \quad \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1} \sqrt{|a_{n,1}||a_{n,2}|} \leq \sqrt{\frac{(n(1-\alpha)-\beta_1+\alpha)(n(1-\alpha)-\beta_2+\alpha)}{(\beta_1-1)(\beta_2-1)}}
\]
for all \( n \geq 2 \). Therefore, we have
\[
(4.8) \quad \frac{n(1-\alpha)+1-\beta+\alpha}{\beta-1} \leq \frac{(n(1-\alpha)-\beta_1+\alpha)(n(1-\alpha)-\beta_2+\alpha)}{(\beta_1-1)(\beta_2-1)}
\]
which is equivalent to

\[(4.9) \quad \beta \geq 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(n(1 - \alpha) + \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + n(1 - \alpha) - \beta_1 + \alpha}(n(1 - \alpha) - \beta_2 + \alpha)}\]

for all \( n \geq 2 \).

Let \( G(n) \) be the right hand side of the last inequality. Then \( G(n) \) is decreasing for \( n \geq 2 \) for \( \alpha \leq 2 - \sqrt{5} \). Thus \( G(2) \) is the maximum of \( G(n) \) for \( \alpha \leq 2 - \sqrt{5} \). This completes the proof of the theorem. \( \square \)

For the functions \( f(z) \) belonging to the class \( ND^*(\alpha, \beta) \), we also have

**Corollary 4.1.** If \( f_1(z) \in ND^*(\alpha, \beta_1) \) and \( f_2(z) \in ND^*(\alpha, \beta_2) \) for some \( \alpha \) and \( \beta_1, \beta_2 \), \((1 < \beta_1, \beta_2 \leq 2)\) then \((f_1 * f_2)(z) \in ND^*(\alpha, \beta)\), where

\[
\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.
\]

By virtue of Theorem 4.1, we have the following theorem.

**Theorem 4.2.** If \( f_j \in \mathcal{MD}^*(\alpha, \beta_j) \) \((j = 1, 2, \cdots, p)\) for some \( \alpha(\alpha \leq 2 - \sqrt{5}) \) and \( \beta_j(1 < \beta_j \leq 2) \), then \((f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{MD}^*(\alpha, \beta)\), where

\[
\beta = 1 + \frac{A_p}{B_p - C_p D_p + E_p} \quad (p \geq 2),
\]

\[
A_p = \prod_{j=1}^{p} (\beta_j - 1)(2 - \alpha)^{p-1}, \quad B_p = (2 - \alpha)^{p-2} \prod_{j=1}^{p} (\beta_j - 1),
\]

\[
C_p = \sum_{m=1}^{p-2} (2 - \alpha)^{p-m-2}(1 - \alpha)^{m-1}, \quad D_p = \prod_{j=1}^{p-m-1} (\beta_j - 1) \prod_{j=p-m+1}^{p} (2 - \alpha - \beta_j),
\]

and

\[
E_p = (1 - \alpha)^{p-2} \prod_{j=1}^{p} (2 - \alpha - \beta_j).
\]

**Proof.** When \( p = 2 \), we have

\[
\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(2 - \alpha)}{(\beta_1 - 1)(\beta_2 - 1) + (2 - \alpha - \beta_1)(2 - \alpha - \beta_2)}.
\]

Let us suppose that \((f_1 * \cdots * f_k) \in \mathcal{MD}^*(\alpha, \beta_0)\) and \( f_{k+1} \in \mathcal{MD}^*(\alpha, \beta_{k+1})\), where

\[
\beta_0 = 1 + \frac{A_k}{B_k - C_k D_k + E_k} \quad (k \geq 2).
\]
Using Theorem4.1 and replacing $\beta_1$ by $\beta_0$ and $\beta_2$ by $\beta_{k+1}$, we see that
\[
\beta = 1 + \frac{(\beta_0 - 1)(\beta_{k+1} - 1)(2-\alpha)}{(\beta_0 - 1)(\beta_{k+1} - 1) + (2-\alpha-\beta_0)(2-\alpha-\beta_{k+1})}
\]
\[
= 1 + \frac{A_{k+1}}{B_{k+1} - \{B_k(2-\alpha-\beta_{k+1}) + (1-\alpha)C_kD_k(2-\alpha-\beta_{k+1})\} + E_{k+1}}
\]
\[
= 1 + \frac{A_{k+1}}{B_{k+1} - C_{k+1}D_{k+1} + E_{k+1}},
\]
where
\[
C_k^+ = \sum_{m=2}^{k-1} (2-\alpha)^{k-m-1}(1-\alpha)^{m-1}.
\]
This completes the proof of the Theorem.

Finally we have

**Corollary 4.2.** If $f_j \in \mathcal{ND}^*(\alpha, \beta_j) \ (j = 1, 2, \cdots, p)$ for some $\alpha$ and $\beta_j(1 < \beta_j \leq 2)$, then $(f_1 * f_2 * \cdots * f_p) \in \mathcal{ND}^*(\alpha, \beta)$, where
\[
\beta = 1 + \frac{A_p}{B_p - C_p^*D_p + 2^{p-1}E_p} \quad (p \geq 2),
\]
\[
A_p = \Pi_{j=1}^{p}(\beta_j - 1)(2-\alpha)^{p-1}, \quad B_p = (2-\alpha)^{p-2}\Pi_{j=1}^{p}(\beta_j - 1),
\]
\[
C_p^* = \sum_{m=1}^{p-2} 2^m(2-\alpha)^{p-m-2}(1-\alpha)^{m-1}, \quad D_p = \Pi_{j=1}^{p} (\beta_j - 1)\Pi_{l=p-m+1}^{p-1}(2-\alpha-\beta_l),
\]
and
\[
E_p = (1-\alpha)^{p-2}\Pi_{j=1}^{p}(2-\alpha-\beta_j).
\]

**References**


