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Distributivity numbers of $\mathcal{P}(\omega)/\mathrm{fin}$ and its friends

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Abstract

This brief survey on distributivity numbers is an exposition of the talk which I gave at RIMS in October 2005.

1 Distributivity numbers of Boolean algebras

Let $\mathbb{P}$ be a separative partial order. $D \subseteq \mathbb{P}$ is *dense* if for all $p \in \mathbb{P}$ there is $q \leq p$ with $q \in D$. D is *open* if for all $p \in D$, any $q \leq p$ belongs to D. The *distributivity number* (or *height*) of $\mathbb{P}$, $\mathfrak{h}(\mathbb{P})$, is the least size of a family $\mathcal{D}$ of open dense subsets of $\mathbb{P}$ such that $\bigcap \mathcal{D}$ is not dense. Note that $\bigcap \mathcal{D}$ necessarily is open. Equivalently, $\mathfrak{h}(\mathbb{P})$ is the least size of a family $\mathcal{A}$ of maximal antichains of $\mathbb{P}$ which has no common refinement. Here, for maximal antichains $A, B \subseteq \mathbb{P}$, we say that A *refines* B if for all $p \in A$ there is $q \in B$ with $p \leq q$. If $\mathbb{A}$ is an atomless Boolean algebra, we let $\mathbb{A}^+ = \mathbb{A} \setminus \{0\}$ denote the partial order of its positive elements and define $\mathfrak{h}(\mathbb{A}) := \mathfrak{h}(\mathbb{A}^+)$. Similarly for other cardinals.

Fact 1. *$\mathfrak{h}(\mathbb{P})$ is a regular cardinal.* $\square$

$\mathfrak{h}(\mathbb{P})$ is an invariant of $\mathbb{P}$ as a forcing notion, that is, it does not depend on the particular realization of $\mathbb{P}$. Equivalently, $\mathfrak{h}(\mathbb{P}) = \mathfrak{h}(\mathrm{r.o.}(\mathbb{P}))$ where r.o.($\mathbb{P}$) is the *completion* of $\mathbb{P}$, i.e., the unique complete Boolean algebra forcing equivalent with $\mathbb{P}$. For a topological space X, r.o.(X) is the *Boolean algebra of regular open subsets* of X where $O \subseteq X$ is called *regular open* if it is open and $\mathrm{Int}(\mathrm{Cl}(O)) = O$. It is well-known that r.o.(X) is a complete Boolean algebra. If $X = \mathbb{P}$ with the topology introduced above, the mapping $p \mapsto O_p = \{q \in \mathbb{P} : q \leq p\}$ is a *dense embedding* of $\mathbb{P}$ into r.o.($\mathbb{P}$) and thus $\mathbb{P}$ and r.o.($\mathbb{P}$) are indeed forcing

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equivalent. In case $\mathbb{A}$ is an atomless Boolean algebra, there is an alternative description of r.o.$(\mathbb{A}) :=$ r.o.$(\mathbb{A}^+)$: namely, if St$(\mathbb{A})$ is the Stone space of $\mathbb{A}$, then r.o.$(\mathbb{A}) =$ r.o.(St$(\mathbb{A})$).

From the forcing-theoretic point of view, $\mathfrak{h}(\mathbb{P})$ is the minimal cardinal κ such that there are $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ for a function from κ to the ground model V such that $p \Vdash_{\mathbb{P}} \dot{f} \notin V$. Indeed if $\lambda < \mathfrak{h}(\mathbb{P})$ and $\dot{f} : \lambda \to V$, then, letting D_α, $\alpha < \lambda$, be the open dense subset of $\mathbb{P}$ consisting of conditions which decide the value $\dot{f}(\alpha)$, $D = \bigcap_{\alpha<\lambda} D_\alpha$ is dense and for any $p \in D$ there is $f_p \in V$ such that $\Vdash_{\mathbb{P}} \dot{f} = f_p$. Thus $\Vdash_{\mathbb{P}} \dot{f} \in V$. On the other hand, if D_α, $\alpha < \mathfrak{h}(\mathbb{P})$, are open dense such that $D = \bigcap_{\alpha<\mathfrak{h}(\mathbb{P})} D_\alpha$ is not dense and $A_\alpha = \{p_{\alpha,\gamma} : \gamma < \kappa_\alpha\} \subseteq D_\alpha$ are maximal antichains, then, letting $\dot{f} : \mathfrak{h}(\mathbb{P}) \to V$ be the $\mathbb{P}$-name defined by $p_{\alpha,\gamma} \Vdash_{\mathbb{P}} \dot{f}(\alpha) = \gamma$, we see that if $p \in \mathbb{P}$ is such that no $q \leq p$ belongs to D then $p \Vdash_{\mathbb{P}} \dot{f} \notin V$.

If $\mathbb{P}$ is *homogeneous*, that is, if O_p is forcing equivalent with $\mathbb{P}$ for all $p \in \mathbb{P}$ (equivalently, if r.o.$(\mathbb{P}) =$ r.o.(O_p) for all $p \in \mathbb{P}$), then $\mathfrak{h}(\mathbb{P})$ is the least size of a family $\mathcal{D}$ of open dense subsets of $\mathbb{P}$ with $\bigcap \mathcal{D} = \emptyset$. Equivalently, $\mathfrak{h}(\mathbb{P})$ is the least κ such that $\Vdash_{\mathbb{P}} \dot{f} \notin V$ for some $\mathbb{P}$-name $\dot{f} : \kappa \to V$.

We write $\mathbb{P} \lessdot \mathbb{Q}$ if there is a *complete embedding* from $\mathbb{P}$ into $\mathbb{Q}$.

Fact 2. *If* $\mathbb{P} \lessdot \mathbb{Q}$ *then* $\mathfrak{h}(\mathbb{P}) \geq \mathfrak{h}(\mathbb{Q})$. □

We briefly mention two cardinals which are closely related to $\mathfrak{h}(\mathbb{P})$. A *tower* $T \subseteq \mathbb{P}$ is a well-ordered decreasing chain without a lower bound. The *tower number* $\mathfrak{t}(\mathbb{P})$ of $\mathbb{P}$ is the least size of a tower in $\mathbb{P}$. $q \in \mathbb{P}$ *splits* $p \in \mathbb{P}$ if p and q are compatible and there is $r \leq p$ incompatible with q. $S \subseteq \mathbb{P}$ is a *splitting family* if every member of $\mathbb{P}$ is split by a member of S. The *splitting number* $\mathfrak{s}(\mathbb{P})$ of $\mathbb{P}$ is the least size of a splitting family. Unlike $\mathfrak{h}$, $\mathfrak{t}$ and $\mathfrak{s}$ are not invariant under forcing equivalence: e.g. $\mathfrak{t}(\mathbb{A}) = \aleph_0$ for every complete atomless Boolean algebra $\mathbb{A}$. Also the base-matrix theorem [BPS] (see also [BS, Theorem 3.4]) implies that $\mathfrak{s}(\text{r.o.}(\mathcal{P}(\omega)/\text{fin})) = \mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$ while $\mathfrak{h} < \mathfrak{s} := \mathfrak{s}(\mathcal{P}(\omega)/\text{fin})$ is consistent (see below for $\mathcal{P}(\omega)/\text{fin}$).

Fact 3. $\mathfrak{t}(\mathbb{P})$ *is a regular cardinal.* □

Fact 4. $\mathfrak{t}(\mathbb{P}) \leq \mathfrak{h}(\mathbb{P}) \leq \mathfrak{s}(\mathbb{P})$.

Proof. Let D_α, $\alpha < \mathfrak{h}(\mathbb{P})$, be open dense such that there is $p \in \mathbb{P}$ with $\bigcap_{\alpha<\mathfrak{h}(\mathbb{P})} D_\alpha \cap O_p = \emptyset$. Recursively construct $p_\alpha \in D_\alpha \cap O_p$ such that $p_\alpha \geq p_\beta$ for $\alpha \leq \beta$. If there is a limit $\lambda < \mathfrak{h}(\mathbb{P})$ such that p_λ cannot be found, $\{p_\alpha : \alpha < \lambda\}$ is a tower and $\mathfrak{t}(\mathbb{P}) \leq cf(\lambda)$. Otherwise $\{p_\alpha : \alpha < \mathfrak{h}(\mathbb{P})\}$ must be a tower.

Let $\{p_\alpha : \alpha < \mathfrak{s}(\mathbb{P})\}$ be a splitting family. For each α let $A_\alpha \subseteq \mathbb{P}$ be a maximal antichain containing p_α. Clearly the A_α have no common refinement. □

Thus $\mathfrak{t}(\mathbb{P})$ and $\mathfrak{s}(\mathbb{P})$ are useful because they give natural lower and upper bounds of $\mathfrak{h}(\mathbb{P})$, respectively.

2 Products and reduced powers

For separative partial orders $\mathbb{P}$ and $\mathbb{Q}$ consider the *product* $\mathbb{P}\times\mathbb{Q}$ equipped with the product ordering (that is, $(p',q') \leq_{\mathbb{P}\times\mathbb{Q}} (p,q)$ iff $p' \leq_{\mathbb{P}} p$ and $q' \leq_{\mathbb{Q}} q$). Since $\mathbb{P} {<}\circ\, \mathbb{P}\times\mathbb{Q}$ and $\mathbb{Q} {<}\circ\, \mathbb{P}\times\mathbb{Q}$ we see

Fact 5. $\mathfrak{h}(\mathbb{P}\times\mathbb{Q}) \leq \min\{\mathfrak{h}(\mathbb{P}), \mathfrak{h}(\mathbb{Q})\}$. $\square$

Notice that if $\mathbb{A}$ and $\mathbb{B}$ are Boolean algebras, then $\mathbb{A}\times\mathbb{B}$ denotes what is called the *free product* in Boolean algebra theory, namely, $(\mathbb{A}^+\times\mathbb{B}^+)\cup\{\mathbf{0}\}$.

For a Boolean algebra $\mathbb{A}$ let $\mathbb{A}^\omega/\mathrm{fin} := \{[f] : f \in \mathbb{A}^\omega\}$ where $[f] = \{g \in \mathbb{A}^\omega : \forall^\infty n\ (f(n) = g(n))\}$, ordered by $[f] \leq [g]$ if $f(n) \leq g(n)$ holds for almost all n. The *reduced power* $\mathbb{A}^\omega/\mathrm{fin}$ is again a Boolean algebra.

Fact 6. *If* $\mathbb{A} {<}\circ\, \mathbb{B}$ *then* $\mathbb{A}^\omega/\mathrm{fin} {<}\circ\, \mathbb{B}^\omega/\mathrm{fin}$ *(and thus* $\mathfrak{h}(\mathbb{A}^\omega/\mathrm{fin}) \geq \mathfrak{h}(\mathbb{B}^\omega/\mathrm{fin})$*).* $\square$

If $\mathbb{A}$ is the trivial algebra $\{\mathbf{0},\mathbf{1}\}$, we see $\mathbb{A}^\omega/\mathrm{fin} \cong \mathcal{P}(\omega)/\mathrm{fin}$ where $\mathcal{P}(\omega)/\mathrm{fin} := \{[A] : A \subseteq \omega\}$ with $[A] = \{B \subseteq \omega : |A \triangle B| < \aleph_0\}$, ordered by $[A] \leq [B]$ if $|A \setminus B| < \aleph_0$. In particular $\mathfrak{h}(\mathbb{B}^\omega/\mathrm{fin}) \leq \mathfrak{h}$ for any Boolean algebra $\mathbb{B}$ where $\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\mathrm{fin})$.

Stone-Čech remainders. Much of the interest in Boolean algebras of the form $\mathbb{A}^\omega/\mathrm{fin}$ stems from the fact their completion is isomorphic to the regular open algebra $\mathrm{r.o.}(X^*)$ of the *Stone-Čech remainder* X^* of some natural space X. Briefly recall the construction of the *Stone-Čech compactification* βX of a normal space X [En, Section 3.6]. Let βX be the family of all ultrafilters of closed subsets of X. Identify $x \in X$ with $\mathcal{U}(x) = \{A \subseteq X \text{ closed: } x \in A\} \in \beta X$. Clearly $\bigcap \mathcal{U}(x) = \{x\}$. In fact, the maximality of any $\mathcal{U} \in \beta X$ entails that either $\bigcap \mathcal{U} = \{x\}$ for some x and then $\mathcal{U} = \mathcal{U}(x)$ or $\bigcap \mathcal{U} = \emptyset$ and $\mathcal{U}$ is a *free ultrafilter*. Thus $X^* = \beta X \setminus X$ is the space of free ultrafilters of closed sets. For $O \subseteq X$ open let $O^* = \{\mathcal{U} \in \beta X : \exists A \in \mathcal{U} \text{ with } A \subseteq O\}$. Clearly $O^* \cap X = O$. The sets O^*, $O \subseteq X$ open, form a basis of the topology of βX and thus the $O^* \cap X^*$ are a basis of the topology of X^*.

We come to specific examples. First let $X = \omega$, equipped with the discrete topology. $\beta\omega$ is the space of all ultrafilters on ω and ω^* is the space of free ultrafilters. Basic open sets are of the form $O^* = \{\mathcal{U} \in \beta\omega : O \in \mathcal{U}\}$ for $O \subseteq \omega$ and, in fact, every regular open set is of this form so that $\mathrm{r.o.}(\beta\omega) = \mathcal{P}(\omega)$. Basic non-empty open sets of ω^* are of the form $O^* \cap \omega^*$ where $O \subseteq \omega$ is infinite. If $|O_0 \triangle O_1| < \aleph_0$, then clearly $O_0^* \cap \omega^* = O_1^* \cap \omega^*$. Thus a dense subset of $\mathrm{r.o.}(\omega^*)^+$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{fin}^+$ and we obtain

Fact 7. $\mathrm{r.o.}(\omega^*) = \mathrm{r.o.}(\mathcal{P}(\omega)/\mathrm{fin})$. $\square$

Similarly, we get $\mathrm{r.o.}(\beta\omega \times \beta\omega) = \mathrm{r.o.}(\mathcal{P}(\omega) \times \mathcal{P}(\omega))$ and $\mathrm{r.o.}(\omega^* \times \omega^*) = \mathrm{r.o.}(\mathcal{P}(\omega)/\mathrm{fin} \times \mathcal{P}(\omega)/\mathrm{fin})$ etc.

Next, let $X = \mathbb{R}$, equipped with the standard topology. For $s \in 2^{<\omega}$, $n \in \mathbb{Z}$ and $\epsilon > 0$, let

$$O_{s,n,\epsilon} = (n + \sum\{\frac{1}{2^{i+1}} : i < |s| \text{ and } s(i) = 1\} - \epsilon,$$
$$n + \sum\{\frac{1}{2^{i+1}} : i < |s| \text{ and } s(i) = 1\} + \frac{1}{2^{|s|}} + \epsilon)$$

Clearly the $Q_{s,n,\epsilon}$ are a basis of the topology of $\mathbb{R}$ and so the $O^\star_{s,n,\epsilon}$ are a basis of the topology of $\beta\mathbb{R}$. For every infinite partial function $f : \mathbb{Z} \to 2^{<\omega}$ let

$$O_f = \bigcup_{n \in \mathrm{dom}(f)} O_{f(n),n,\epsilon_n}$$

where $\epsilon_n = \min\{\frac{1}{2^{f(i)+5}} : n-1 \leq i \leq n+1\}$ and notice that the $O^\star_f \cap \mathbb{R}^*$ form a basis of regular open sets of the topology of $\mathbb{R}^*$ (indeed every $\mathcal{U} \in \mathbb{R}^*$ contains only unbounded closed sets; otherwise $\bigcap \mathcal{U} \neq \emptyset$ by compactness; thus for bounded $O \subseteq \mathbb{R}$, $O^\star \cap \mathbb{R}^* = \emptyset$, and it is easy to see every unbounded $O \subseteq \mathbb{R}$ contains a set of the form O_f). If $\mathrm{dom}(f) =^* \mathrm{dom}(g)$ and $f(n) = g(n)$ for almost all $n \in \mathrm{dom}(f)$ then $O^\star_f \cap \mathbb{R}^* = O^\star_g \cap \mathbb{R}^*$. Otherwise they are distinct (by choice of the ϵ_n). This means that $\mathrm{r.o.}(\mathbb{R}^*)^+$ has a dense subset isomorphic to $F/\mathrm{fin} := \{[f] : f \in F\}$ where $F = \{f : \mathbb{Z} \to 2^{<\omega} : \mathrm{dom}(f) \text{ is infinite}\}$ and $[f] = \{g \in F : \mathrm{dom}(g) =^* \mathrm{dom}(f) \text{ and } \forall^\infty n \in \mathrm{dom}(g)\ (g(n) = f(n))\}$, ordered by $[f] \leq [g]$ if $\mathrm{dom}(f) \subseteq^* \mathrm{dom}(g)$ and $f(n) \supseteq g(n)$ holds for almost all $n \in \mathrm{dom}(f)$.

Let $\mathbb{C}$ be *Cohen forcing*, that is, the algebra of clopen subsets of the Cantor space 2^ω, ordered by inclusion. Since $\{[s] : s \in 2^{<\omega}\}$ is a dense subset of $\mathbb{C}^+$, $\mathbb{C}^+$ has a dense subset isomorphic to $2^{<\omega}$ ordered by reverse inclusion. Thus $\mathbb{C}^\omega/\mathrm{fin}^+$ has a dense subset isomorphic to F/fin. This shows

Fact 8. $\mathrm{r.o.}(\mathbb{R}^*) = \mathrm{r.o.}(\mathbb{C}^\omega/\mathrm{fin})$. $\square$

The above discussion motivates the investigation of cardinal numbers like $\mathfrak{h} = \mathfrak{h}(\mathrm{r.o.}(\omega^*))$, $\mathfrak{h}_2 := \mathfrak{h}(\mathcal{P}(\omega)/\mathrm{fin} \times \mathcal{P}(\omega)/\mathrm{fin}) = \mathfrak{h}(\mathrm{r.o.}(\omega^* \times \omega^*))$, $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) = \mathfrak{h}(\mathrm{r.o.}(\mathbb{R}^*))$ etc.

Distributivity numbers of products and reduced powers. We know already $\mathfrak{h}_2 \leq \mathfrak{h}$. The following is easy to see

Fact 9. $\mathfrak{t} \leq \mathfrak{h}_2$. $\square$

On the other hand $\mathfrak{h}_2$ may be less than $\mathfrak{h}$.

Theorem 1. (Shelah-Spinas [SS1]) $CON(\mathfrak{h}_2 < \mathfrak{h})$.
In fact $\mathfrak{h}_2 < \mathfrak{h}$ holds in the iterated Mathias model (the ω_2-stage countable support iteration of Mathias forcing over a model of CH). $\square$

In fact Shelah and Spinas also obtained the consistency of $\mathfrak{h}_{n+1} < \mathfrak{h}_n$ for any n [SS2] where $\mathfrak{h}_n := \mathfrak{h}((\mathcal{P}(\omega)/\mathrm{fin})^n)$. Since $\mathfrak{t} \leq \mathfrak{h}_n$ for all n, the consistency of $\mathfrak{t} < \mathfrak{h}_n$ follows immediately.

We know already $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) \leq \mathfrak{h}$. Again, the inequality is consistently strict.

Theorem 2. (Dow [Do]) *CON*($\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \mathfrak{h}$).
In fact $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \mathfrak{h}$ *holds in the iterated Mathias model.* □

The similarity to the Shelah-Spinas result lead to the following

Question 1. (Dow [Do]) *Is* $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) = \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$? *Is* $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) \leq \mathfrak{h}_2$?

Note that $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) \leq \mathfrak{h}_2$ because $\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin} <\!\circ\, \mathcal{P}(\omega)/\text{fin} \times \mathcal{P}(\omega)/\text{fin}$. Upper and lower bounds for $\mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ are given by

Theorem 3. (Balcar-Hrušák [BH]) $\mathfrak{t} \leq \mathfrak{h}(\mathbb{C}^\omega/\text{fin}) \leq \mathsf{add}(\mathcal{M})$. □

Here $\mathsf{add}(\mathcal{M})$ denotes the *additivity of the meager ideal*, that is, the least size of a family of meager sets whose union is not meager.

Balcar and Hrušák also observed [BH] that $\mathfrak{t} < \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ is consistent. Moreover, Dow's Theorem 2 is a Corollary of Theorem 3. Namely, it is well-known (and much easier to prove than Dow's argument for $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) = \aleph_1$) that $\mathsf{add}(\mathcal{M}) = \aleph_1$ in the iterated Mathias model (see, e.g., [BJ]). On the other hand, since $\mathfrak{h} < \mathsf{add}(\mathcal{M})$ in the Hechler model, the consistency of $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \mathsf{add}(\mathcal{M})$ follows. This naturally leads to

Question 2. (Balcar-Hrušák [BH]) *Is* $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \min\{\mathfrak{h}, \mathsf{add}(\mathcal{M})\}$ *consistent?*

Both questions can be answered with basically the same method.

Theorem 4. [Br3] *CON*($\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \min\{\mathfrak{h}, \mathsf{add}(\mathcal{M})\}$).

Theorem 5. [Br3] *CON*($\mathfrak{h}_2 < \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$).
Thus $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) < \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ *is consistent as well.*

Notice that the converse, namely, the consistency of $\mathfrak{h}_2 > \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$, follows from the consistency of $\mathfrak{h}_2 > \mathsf{add}(\mathcal{M})$ established by Shelah and Spinas [SS2] and from Theorem 3.

Sketch of proof. We briefly sketch the proof of Theorem 4. Unlike earlier results on the independence of distributivity numbers ([Do], [SS1], [SS2]), we use a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ of ccc forcing over a model of CH. This is natural because we have to add Cohen reals anyway (in Theorem 4 we want to add increase $\mathsf{add}(\mathcal{M})$, and in Theorem 5 we have to increase $\mathsf{add}(\mathcal{M})$ by Theorem 3).

Roughly speaking, the iteration adds dominating reals (via Hechler forcing $\mathbb{D}$, see [BJ]) in successor stages and limit stages of cofinality ω while we use *Laver forcing* $\mathbb{L}_\mathcal{U}$ with a Ramsey ultrafilter $\mathcal{U}$ at limit stages of cofinality ω_1 (see below for the definition). The Hechler reals (as well as the Laver reals) guarantee that $\mathfrak{b} = \aleph_2$ while the Cohen reals give $\mathsf{cov}(\mathcal{M}) = \aleph_2$. So $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\} = \aleph_2$ holds.

Recall that an ultrafilter $\mathcal{U}$ on ω is *Ramsey* if for all partitions $\langle X_n : n \in \omega \rangle$ of ω either $X_n \in \mathcal{U}$ for some $n \in \omega$ or there is $Y \in \mathcal{U}$ with $|Y \cap X_n| \leq 1$ for all n. $\mathbb{L}_\mathcal{U}$ consists of all trees $T \subseteq \omega^{<\omega}$ such that for all $s \in T$ below the stem (i.e. $s \supseteq \mathrm{stem}(T)$) the set of successor nodes $\{n : s \frown n \in T\}$ belongs to $\mathcal{U}$. $\mathbb{L}_{\mathcal{U}}$ is ordered by inclusion. It is easy to see that the generic *Laver real* $\ell_{\mathcal{U}} := \bigcap\{[T] : T \in G\} \in \omega^\omega$ dominates the ground model reals and that $\mathrm{ran}(\ell_{\mathcal{U}}) \subseteq \omega$ diagonalizes $\mathcal{U}$ (i.e. $\ell_{\mathcal{U}} \subseteq^* U$ for all $U \in \mathcal{U}$). Here G denotes the generic filter. Thus iterating $\mathbb{L}_{\mathcal{U}}$ naturally increases $\mathfrak{b}$ and $\mathfrak{s}$. The effect of $\mathbb{L}_{\mathcal{U}}$ on $\mathfrak{h}$, however, is a more subtle issue and depends very much on the choice of the ultrafilter $\mathcal{U}$. In some situations $\mathfrak{h}$ (and its relatives) stay small (see [Br2, Section 2] for such a construction) and we obtain a natural model for $\mathfrak{h} < \min\{\mathfrak{b}, \mathfrak{s}\}$, the consistency of which was originally obtained by Shelah [Sh1] (see also [Sh2, Theorem VI.8.2]).

We assume $\diamondsuit_{S^2_1}$ holds in the ground model. This means there is a sequence $\langle Z_\alpha : cf(\alpha) = \omega_1, \alpha < \omega_2\rangle$ such that for all $Z \subseteq \omega_2$, the set $\{\alpha < \omega_2 : cf(\alpha) = \omega_1$ and $Z \cap \alpha = Z_\alpha\}$ is stationary. $\diamondsuit_{S^2_1}$ is used for guessing (initial segments of) names for potential witnesses for $\mathfrak{h} = \aleph_1$. Notice that if $\dot{A}$ is a $\mathbb{P}_{\omega_2}$-name for such a witness, then by CH and ccc, $\dot{A}$ can be thought of as an object of size ω_2 and can be coded into a subset of ω_2. Similarly if $\alpha < \omega_2$, $|\alpha| = \aleph_1$, and $\dot{A}$ is a $\mathbb{P}_\alpha$-name for a witness of $\mathfrak{h} = \aleph_1$, $\dot{A}$ can be coded into a subset of α. Thus, if at stage α where $cf(\alpha) = \omega_1$, Z_α codes such a $\mathbb{P}_\alpha$-name $\dot{A} = \{\dot{A}_\beta : \beta < \omega_1\}$, then we construct a Ramsey ultrafilter $\dot{\mathcal{U}}_\alpha$ such that $\Vdash_\alpha \dot{\mathcal{U}}_\alpha \cap \dot{A}_\beta \neq \emptyset$ for all $\beta < \omega_1$ and force with $\dot{\mathbb{Q}}_\alpha = \mathbb{L}_{\dot{\mathcal{U}}_\alpha}$. This destroys the witness $\dot{A}$. Now, if $\dot{A}$ is a $\mathbb{P}_{\omega_2}$-name for a witness for $\mathfrak{h} = \aleph_1$ coded by $Z \subseteq \omega_2$, then $C = \{\alpha < \omega_2 : cf(\alpha) = \omega_1$ and $\dot{A}{\restriction}\alpha$ is a $\mathbb{P}_\alpha$-name for a witness for $\mathfrak{h} = \aleph_1\}$ is ω_1-club. By $\diamondsuit_{S^2_1}$ there is $\alpha \in C$ with $Z \cap \alpha = Z_\alpha$. So Z_α codes $\dot{A}{\restriction}\alpha$ and $\mathbb{L}_{\dot{\mathcal{U}}_\alpha}$ destroys $\dot{A}{\restriction}\alpha$ and also $\dot{A}$. This shows $\mathfrak{h} = \aleph_2$.

The most difficult part of the argument is the proof of $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) = \aleph_1$. We build a witness $\mathcal{F} = \{F_\beta \subseteq \mathbb{C}^\omega : \beta < \omega_1\}$ along the iteration. The main point is that if the Ramsey ultrafilter $\mathcal{U}_\alpha$ is carefully chosen, then $\mathbb{Q}_\alpha = \mathbb{L}_{\mathcal{U}_\alpha}$ does not destroy (the initial segment of) this witness $\mathcal{F}$. This is a technical argument which relies heavily on a *rank analysis* of $\mathbb{L}_{\mathcal{U}_\alpha}$-names. See [Br3] for details. To be able to build the required Ramsey ultrafilter in limit stages of cofinality ω_1, we use the Hechler reals which we added in successor stages. For Hechler forcing it is much easier to see that it preserves (the initial segment of) the witness $\mathcal{F}$. Thus $\mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin}) = \aleph_1$ follows. □

We close this section with some comments and questions on related cardinals. Balcar and Hrušák [BH] proved that $\mathfrak{t}(\mathbb{C}^\omega/\mathrm{fin}) = \mathfrak{t}$ (and thus $\mathfrak{t}(\mathbb{C}^\omega/\mathrm{fin}) < \mathfrak{h}(\mathbb{C}^\omega/\mathrm{fin})$ is consistent as well, see above). But little seems to be known about $\mathfrak{s}(\mathbb{C}^\omega/\mathrm{fin})$ except for the trivial $\mathfrak{s}(\mathbb{C}^\omega/\mathrm{fin}) \leq \mathfrak{s}$.

Problem 1. *Investigate* $\mathfrak{s}(\mathbb{C}^\omega/\mathrm{fin})$*! Investigate* $\mathfrak{x}(\mathbb{C}^\omega/\mathrm{fin})$ *for other cardinal invariants* $\mathfrak{x}$*!*

For a topological space X without isolated points, the *Baire number of* X (also called *Novák number*), $\mathfrak{n}(X)$, is the least size of a family of nowhere dense sets covering X. Let $\mathfrak{n} := \mathfrak{n}(\omega^*)$.

Theorem 6. (Balcar-Pelant-Simon [BPS], see also [BS, Theorem 3.10])

(i) *If* $\mathfrak{h} < \mathfrak{c}$, *then* $\mathfrak{h} \leq \mathfrak{n} \leq \mathfrak{h}^+$.

(ii) *If* $\mathfrak{h} = \mathfrak{c}$, *then* $\mathfrak{c} \leq \mathfrak{n} \leq 2^{\mathfrak{c}}$. $\square$

The analogous result holds for $\mathfrak{h}(\mathbb{R}^*)$ and $\mathfrak{n}(\mathbb{R}^*)$. Also $\mathfrak{n}(\mathbb{R}^*) \leq \mathfrak{n}$ is easy to see, but the following is still open.

Question 3. (van Douwen, see [Do]) *Is* $\mathfrak{n}(\mathbb{R}^*) < \mathfrak{n}$ *consistent?*

3 Further friends of $\mathcal{P}(\omega)/\mathrm{fin}$

We briefly discuss the distributivity number of other structures related to $\mathcal{P}(\omega)/\mathrm{fin}$.

Dense ($\mathbb{Q}$) / nwd. Let Dense($\mathbb{Q}$) denote the family of dense subsets of the rationals $\mathbb{Q}$, and let nwd stand for the nowhere dense sets of rationals. Let $\mathrm{Dense}(\mathbb{Q})/\mathrm{nwd} = \{[A] : A \in \mathrm{Dense}(\mathbb{Q})\}$ where $[A] = \{B : A \triangle B \in \mathrm{nwd}\}$ for $A \in \mathrm{Dense}(\mathbb{Q})$, ordered by $[A] \leq [B]$ if $A \setminus B \in \mathrm{nwd}$. Let $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h}(\mathrm{Dense}(\mathbb{Q})/\mathrm{nwd})$. $\mathfrak{s}_{\mathbb{Q}}$ and $\mathfrak{t}_{\mathbb{Q}}$ are defined similarly. The investigation of Dense($\mathbb{Q}$)/nwd has been started by Balcar, Hernández and Hrušák [BHH].

Theorem 7. (Balcar-Hernández-Hrušák [BHH], Brendle [Br2])

(i) $\mathfrak{t}_{\mathbb{Q}} = \mathfrak{t}$.

(ii) $\mathfrak{s}_{\mathbb{Q}} \leq \min\{\mathfrak{s}, \mathsf{add}(\mathcal{M})\}$. $\square$

Balcar, Hernández and Hrušák [BHH] also proved the consistency of $\mathfrak{t}_{\mathbb{Q}} < \mathfrak{h}_{\mathbb{Q}}$ and of $\mathfrak{h}_{\mathbb{Q}} < \mathfrak{h}$. In fact, by (ii) of Theorem 7, $\mathfrak{s}_{\mathbb{Q}} < \mathfrak{h}$ holds in the iterated Mathias model. Furthermore:

Theorem 8. [Br2]

(i) $CON(\mathfrak{h}_{\mathbb{Q}} < \mathfrak{s}_{\mathbb{Q}})$.

(ii) $CON(\mathfrak{h} < \mathfrak{h}_{\mathbb{Q}})$. $\square$

The argument for the proof of (ii) is similar to the argument for Theorems 4 and 5, see above. The following is still open.

Question 4. [Br2] *Is* $\mathfrak{s}_{\mathbb{Q}} < \min\{\mathfrak{s}, \mathsf{add}(\mathcal{M})\}$ *consistent?*

Partitions of ω. Let (ω) denote the collection of partitions of ω. $(\omega)^\omega$ is the *infinite partitions* of ω (i.e. the partitions into infinitely many blocks), and $(\omega)^c$ is the *non-trivial partitions* of ω. Here, we say $A \in (\omega)$ is trivial if $\{n\} \in A$ for almost all n (equivalently, A has no infinite block and almost all blocks are singletons). Write $A \leq B$ if A is *coarser* than B iff all blocks of A are unions of blocks of B. Say X is a *finite coarsening* of A if X is gotten from A by merging finitely many blocks of A. Write $A \leq^* B$ if there is a finite coarsening X of A such that $X \leq B$. Say $A =^* B$ if $A \leq^* B$ and $B \leq^* A$ iff there is

X which is a finite coarsening of both A and B. Let $[A] = \{B : A =^* B\}$ and set $[A] \leq [B]$ if $A \leq^* B$. $((\omega)^\omega / =^*, \leq)$ is the separative quotient of $((\omega)^\omega, \leq)$. It is called the *dual structure* and we let $\mathfrak{h}_d = \mathfrak{h}((\omega)^\omega / =^*)$ etc. As usual, we work with $((\omega)^\omega, \leq^*)$ instead of $((\omega)^\omega / =^*, \leq)$. It is easy to see that $\mathcal{P}(\omega)/\text{fin} {<\!\circ}\, (\omega)^\omega / =^*$; namely, $h : (\omega)^\omega \to [\omega]^\omega$ given by $h(A) = \{\min(b) : b \in A\}$ induces the projection mapping giving rise to the complete embedding. Thus $\mathfrak{h}_d \leq \mathfrak{h}$. The investigation of cardinal invariants of $((\omega)^\omega, \leq^*)$ has been started by Cichoń, Krawczyk, Majcher-Iwanow and Węglorz [CKMW].

Theorem 9. (Carlson [Mat]) $\mathfrak{t}_d = \aleph_1$. □

Theorem 10. (i) (Halbeisen [Ha]) $CON(\mathfrak{h}_d > \aleph_1)$.
Namely, $\mathfrak{c} = \mathfrak{h}_d = \aleph_2$ holds in the iterated dual Mathias model.

(ii) (Spinas [Sp]) $CON(\mathfrak{h}_d < \mathfrak{h})$.
In fact $\mathfrak{h}_d < \mathfrak{h}$ holds in the iterated Mathias model.

(iii) [Br1] $CON(\mathfrak{h}_d = \aleph_1 + MA + \neg CH)$. □

Note that (iii) strengthens (ii) because MA implies $\mathfrak{t} = \mathfrak{c}$ and, thus, $\mathfrak{h} = \mathfrak{c}$ and $\mathfrak{h}(\mathbb{P}) = \mathfrak{c}$ where $\mathbb{P}$ is any of the partial orders considered in Section 2 or $\mathbb{P} = \text{Dense}(\mathbb{Q})/\text{nwd}$. (The main distinction seems to be that for all $\mathbb{P}$ considered earlier in this paper, $\mathfrak{t}(\mathbb{P}) = \mathfrak{t}$ in ZFC while $\mathfrak{t}_d = \aleph_1$.) On the other hand, Cichoń et al. [CKMW] already observed that MA implies $\mathfrak{s}_d = \mathfrak{c}$ so that $\mathfrak{h}_d < \mathfrak{s}_d$ is consistent as well.

There is another natural structure associated with (ω), which is obtained by turning the order upside down and looking at refining instead of coarsening. Say $A \leq_c B$ if A is *finer* than B iff $B \leq A$. X is a *finite refinement* of A if for some finite $x \subseteq \omega$, $X = \{b \setminus x : b \in A\} \cup \{\{n\} : n \in x\}$. Write $A \leq_c^* B$ if there is a finite refinement X of A such that $X \leq_c B$. Say $A =_c^* B$ if $A \leq_c^* B$ and $B \leq_c^* A$ iff there is X which is a finite refinement of both A and B. Notice that $A \leq_c^* B$ implies $B \leq^* A$ (and equivalence holds for partitions which contain only finite blocks). As usual let $[A] = \{B : A =_c^* B\}$, $[A] \leq [B]$ if $A \leq_c^* B$ and consider the *converse dual structure* $((\omega)^c / =_c^*, \leq)$ which may be identified with $((\omega)^c, \leq_c^*)$. The reason for considering $\leq_c^*$ instead of $\geq^*$ is that the former gives indeed rise to the separative quotient of $((\omega)^c, \leq_c)$ (while $((\omega)^c, \geq^*)$ does not). This structure has been investigated by Majcher-Iwanow [Maj]. Again $\mathcal{P}(\omega)/\text{fin} {<\!\circ}\, (\omega)^c / =^*$; but more is true: $((\omega)^c, \leq_c^*)$ is locally isomorphic to $([\omega]^\omega, \subseteq^*)$ [Maj] so that $\text{r.o.}(\mathcal{P}(\omega)/\text{fin}) = \text{r.o.}((\omega)^c / =_c^*)$. Thus $\mathfrak{h}_c = \mathfrak{h}$ where $\mathfrak{h}_c = \mathfrak{h}((\omega)^c / =_c^*)$. In fact, equality also holds for several other cardinal invariants of the continuum; e.g., $\mathfrak{t}_c = \mathfrak{t}$ and $\mathfrak{s}_c = \mathfrak{s}$, see [BZ] for details.

The General Philosophy behind the results obtained so far is that distributivity numbers are independent unless there is an order relationship for trivial reasons, namely, unless there is a complete embedding between the partial orderings. Indeed, in all cases investigated so far, either $\mathbb{P} {<\!\circ}\, \mathbb{Q}$ or $CON(\mathfrak{h}(\mathbb{P}) < \mathfrak{h}(\mathbb{Q}))$ has been established.

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