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A Strong Form of ψ_{AC}

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Abstract

We formulate a principle, called τ_{AC}, which implies both ψ_{AC} and ϕ_{AC}. We also force τ_{AC} and conclude equiconsistencies of these.

Introduction

In [W], combinatorial principles ψ_{AC} and ϕ_{AC} are introduced. We consider these in ZFC and formulate a stronger principle. We call our stronger one τ_{AC}. This τ_{AC} deals with many stationary subsets of ω_1 at a time. By choosing arrangements of stationary sets, we may conclude ψ_{AC} and ϕ_{AC}.

In §1, we fix notations. In §2, we recap ψ_{AC} and ϕ_{AC} and so forth and define τ_{AC}. We mention immediate implications. In §3, we prepare technical lemmas. This builds on the communication [A] with D. Aspero. In §4, we outline a forcing construction of τ_{AC} and conclude equiconsistencies based on [DD].

§1. Preliminary

1.1 Notation. For a set X, $|X|$ denotes the cardinality of X and for a set Y of ordinals, o.t.(Y) denotes the order-type of $(Y,<)$. For a set A, $[A]^{\omega}$ denotes $\{X \mid X \subseteq A, |X| = \omega\}$.

For a set x, $\mathrm{TC}(x)$ denotes the $\in$-transitive closure of x. For a regular cardinal θ, $H_\theta = \{x \mid \ |\mathrm{TC}(x)| < \theta\}$. *A countable elementary substructure N of H_θ* means $(N,\in)$ is a countable elementary substructure of $(H_\theta,\in)$. Hence we assume no other predicates and functions on H_θ.

A notion of forcing P is *semiproper*, if for all sufficiently large regular cardinals and countable elementary substructures N of H_θ with $P \in N$ (and possibly other parameters are in N), if $p \in P \cap N$, then there exists $q \leq p$ such that for all P-names $\tau \in N$ with $\Vdash_P$"$\tau \in \omega_1^V$", we have $q \Vdash_P$"$\tau \in N$". We call this q *(P,N)-semigeneric*. Equivalently, $q \Vdash_P$"$N[G] \cap \omega_1^V = N \cap \omega_1^V$", where $N[G] = \{\tau[G] \mid \tau$ is a P-name with $\tau \in N\}$.

Clubs and *stationary subsets of* ω_1 have standard meanings.

We consider stronger stationary sets to come up with notions of forcing which are semiproper.

1.2 Definition. Let K be any set with $K \supseteq \omega_1$. For $S \subseteq [K]^{\omega}$, we say S is *semiproper*, if for all sufficiently large regular cardinals θ and all countable elementary substructures N of H_θ with $K \in N$ (and possibly other parameters are in N), there exist countable elementary substructures M of H_θ such that $N \subseteq M$, $N \cap \omega_1 = M \cap \omega_1$ and $M \cap K \in S$.

1.3 Proposition. *Let $S \subseteq [K]^{\omega}$ be semiproper, then S is stationary in $[K]^{\omega}$. In particular, S is cofinal in $[K]^{\omega}$.*

Proof. Let $S \subseteq [K]^{\omega}$ be semiproper. Let $f : {}^{<\omega}K \longrightarrow K$. It suffices to find $X \in S$ which is closed under f. To this end, let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $K, f \in N$. Then since S is semiproper, there exists a countable elemetary substructure M of H_θ with $M \cap K \in S$. Let $X = M \cap K$. Then this X works.

□

§2. Implications

We recap three principles from [W] and [LS].

2.1 Definition. ([W]) Let S be any stationary subset of ω_1. We define $\tilde{S}$. $\gamma \in \tilde{S}$, if $\omega_1 \leq \gamma < \omega_2$, there exists a bijection $\pi : \omega_1 \longrightarrow \gamma$ such that

$$\{\alpha < \omega_1 \mid \text{o.t.}(\{\pi(\beta) \mid \beta < \alpha\}) \in S\}$$

contains a club.

2.2 Definition. ([W]) ψ_{AC} stands for the following statement.

For any stationary costationary subsets S and T, there exist $\gamma < \omega_2$, a bijection $\pi : \omega_1 \longrightarrow \gamma$ and a closed unbounded set $C \subset \omega_1$ such that

$$\{\alpha < \omega_1 \mid \text{o.t.}(\{\pi(\beta) \mid \beta < \alpha\}) \in S\} \cap C = T \cap C.$$

2.3 Definition. ([W]) ϕ_{AC} stands for the following statement.

(1) There is an ω_1 sequence of distinct reals.

(2) Suppose $\langle S_n \mid n < \omega \rangle$ and $\langle T_n \mid n < \omega \rangle$ are sequences of pairwise disjoint subsets of ω_1. Suppose the S_n are stationary and suppose that

$$\omega_1 = \bigcup \{T_n \mid n < \omega\}.$$

Then there exists $\eta < \omega_2$ and a continuous increasing function $F : \omega_1 \longrightarrow \eta$ with cofinal range such that for each $n < \omega$ and $j \in T_n$

$$F(j) \in \widetilde{S_n}.$$

2.4 Definition. ([LS]) *The cofinal bounding (The complete bounding, CB)* means that for any function $f : \omega_1 \longrightarrow \omega_1$, there exist γ with $\omega_1 \leq \gamma < \omega_2$, a bijection $\pi : \omega_1 \longrightarrow \gamma$ and a club C such that for each $\alpha \in C$, $f(\alpha) < \text{o.t.}(\{\pi(\beta) \mid \beta < \alpha\})$.

The following is strongest among these.

2.5 Definition. τ_{AC} holds, if for any system $\langle S_i^j \mid j \leq i < \omega_1 \rangle$ of stationary subsets of ω_1, there exists a continuously $<$-increasing sequence $\langle \gamma_j \mid j \leq \omega_1 \rangle$ of ordinals with $\omega_1 < \gamma_0 < \gamma_{\omega_1} < \omega_2$ and a continuously $\subseteq$-increasing countable sets $\langle X_i \mid i < \omega_1 \rangle$ such that

- $X_i \in [\gamma_i]^\omega$.
- $\bigcup \{X_i \mid i < \omega_1\} = \gamma_{\omega_1}$.
- For all $j \leq i$, we have $\text{o.t.}(X_i \cap \gamma_j) \in S_i^j$.

2.6 Proposition. *τ_{AC} implies both ψ_{AC} and ϕ_{AC}.*

Proof. We show ψ_{AC} gets implied by τ_{AC}. Let both S and T be stationary and costationary. Define S_i^0 by

$$S_i^0 = \begin{cases} S, & \text{if } i \in T \\ \omega_1 \setminus S, & \text{otherwise.} \end{cases}$$

We do not care about other S_i^j. Apply τ_{AC} to this $\langle S_i^j \mid j \leq i < \omega_1 \rangle$. We get a continuously $<$-increasing sequence $\langle \gamma_j \mid j \leq \omega_1 \rangle$ and a continuously $\subseteq$-increasing sequence $\langle X_i \mid i < \omega_1 \rangle$. For each $i < \omega_1$, let $Y_i = X_i \cap \gamma_0$. Then

- $\omega_1 < \gamma_0 < \omega_2$.

- Y_i are continuously $\subseteq$-increasing countable subsets of γ_0 with $\bigcup\{Y_i \mid i < \omega_1\} = \gamma_0$.
- $i \in T$ iff $\text{o.t.}(Y_i) \in S$.

Let π be any bijection $\pi : \omega_1 \longrightarrow \gamma_0$. Then

$$\{i < \omega_1 \mid \{\pi(\beta) \mid \beta < i\} = Y_i\}$$

contains a club C. We conclude

$$\{i < \omega_1 \mid \text{o.t.}(\{\pi(\beta) \mid \beta < i\}) \in S\} \cap C = T \cap C.$$

Next, we show ϕ_{AC} gets implied by τ_{AC}. Let $\langle S_n \mid n < \omega\rangle$ and $\langle T_n \mid n < \omega\rangle$ be given. For each $j \leq i < \omega_1$, define

$$S_i^j = S_n, \text{ if } j \in T_n.$$

Since $\omega_1 = \bigcup\{T_n \mid n < \omega\}$ is a disjoint union, this is well-defined. Apply τ_{AC} to this $\langle S_i^j \mid j \leq i < \omega_1\rangle$. We get a continuously $<$-increasing sequence $\langle \gamma_j \mid j \leq \omega_1\rangle$ and a continuously $\subseteq$-increasing sequence $\langle X_i \mid i < \omega_1\rangle$. Let $\gamma = \gamma_{\omega_1}$ and for each $j < \omega_1$, let $F(j) = \gamma_j$. Then we have

- $\omega_1 < \gamma < \omega_2$.
- $F : \omega_1 \longrightarrow \gamma$ is a continuous increasing function whose range is cofinal in γ.

Want to observe

- For each $n < \omega$ and $j \in T_n$, we have $F(j) \in \tilde{S}_n$.

Fix n, j with $j \in T_n$. Then $\langle X_i \cap \gamma_j \mid i < \omega_1\rangle$ is a continuously $\subseteq$-increasing sequence of countable subsets of $F(j)$ such that $\bigcup\{X_i \cap \gamma_j \mid i < \omega_1\} = F(j)$ and for all i with $j \leq i < \omega_1$, we have $\text{o.t.}(X_i \cap \gamma_j) \in S_i^j = S_n$. Let $\pi : \omega_1 \longrightarrow F(j)$ be any bijection. Since

$$\{i < \omega_1 \mid j \leq i, \{\pi(\beta) \mid \beta < i\} = X_i \cap \gamma_j\}$$

contains a club C and we have

$$C \subseteq \Big\{i < \omega_1 \mid \text{o.t.}(\{\pi(\beta) \mid \beta < i\}) \in S_n\Big\}.$$

Hence $F(j) \in \tilde{S}_n$.

□

The following is communicated by D. Aspero. We provide our proof.

2.7 Proposition. ([A]) *(1) ϕ_{AC} implies CB.*

(2) ψ_{AC} also implies CB.

Proof. For (1): Let $f : \omega_1 \longrightarrow \omega_1$ and $C(f) = \{i < \omega_1 \mid i \text{ is closed under } f\}$. Then $C(f)$ is a club in ω_1. Partition $C(f)$ into ω-many stationary pieces $\langle C(f)_n \mid n < \omega\rangle$. We also partition ω_1 into any $\langle T_n \mid n < \omega\rangle$. Apply ϕ_{AC} to $\langle C(f)_n \mid n < \omega\rangle$ and $\langle T_n \mid n < \omega\rangle$. We have $\eta < \omega_2$ and an increasing continuous function $F : \omega_1 \longrightarrow \eta$ with cofinal range such that for all $n < \omega$ and $j \in T_n$, we have $F(j) \in \widetilde{C(f)_n}$.

Since $\omega_1 \leq F(j)$ and the $F(j)$ are cofinal in η, we may choose $j < \omega_1$ such that $\omega_1 < F(j)$. Let $n < \omega$ be such that $j \in T_n$ and let $\gamma = F(j)$. Then $\omega_1 < \gamma < \omega_2$ holds. Since $\gamma \in \widetilde{C(f)_n} \subset \widetilde{C(f)}$, there exists a bijection $\pi : \omega_1 \longrightarrow \gamma$ such that $\{\alpha < \omega_1 \mid \text{o.t.}(\{\pi(\beta) \mid \beta < \alpha\}) \in C(f)\}$ contains a club C. Let $X_\alpha = \{\pi(\beta) \mid \beta < \alpha\}$ for all $\alpha < \omega_1$. Let

$$D = \{\alpha < \omega_1 \mid \omega_1 \in X_\alpha,\ \omega_1 \cap X_\alpha = \alpha\}.$$

Then D is a club in ω_1. It suffices to show that for all $\alpha \in C \cap D$, $f(\alpha) < \text{o.t.}(X_\alpha)$ hold. But $\alpha < \text{o.t.}(X_\alpha) \in C(f)$, so this is immediate.

For (2): Let $f : \omega_1 \longrightarrow \omega_1$ and $C(f) = \{i < \omega_1 \mid i \text{ is closed under } f\}$. Then $C(f)$ is a club in ω_1. Partition $C(f)$ into two stationary sets S and T. So $C(f) = S \cup T$ and $S \cap T = \emptyset$. Apply ψ_{AC} to (S,T) and (T,S). So for $k = 1,2$, there exist γ_k, C_k, a continuously $\subseteq$-increasing sequence of countable subsets $\langle X^k_\delta \mid \delta < \omega_1\rangle$ of γ_k with $\bigcup\{X^k_\delta \mid \delta < \omega_1\} = \gamma_k$ such that

$$T \cap C_1 = \{\delta \in C_1 \mid \text{o.t.}(X^1_\delta) \in S\},$$

$$S \cap C_2 = \{\delta \in C_2 \mid \text{o.t.}(X^2_\delta) \in T\}.$$

Since we must have $\omega_1 < \gamma_1, \gamma_2$ under this situation, we may assume $\omega_1 < \gamma_1 \leq \gamma_2 < \omega_2$. Let

$$D = C(f) \cap C_1 \cap C_2 \cap \{\delta < \omega_1 \mid X^1_\delta \cap \omega_1 = \delta,\ \omega_1 \in X^1_\delta = X^2_\delta \cap \gamma_1\}.$$

Then D is a club in ω_1. It suffices to show that for all $\delta \in D$, we have

$$f(\delta) < \text{o.t.}(X^2_\delta).$$

Case 1. $\delta \in T$: $\delta < \text{o.t.}(X^1_\delta) \in S \subset C(f)$. Hence $f(\delta) < \text{o.t.}(X^1_\delta) \leq \text{o.t.}(X^2_\delta)$.

Case 2. $\delta \in S$: $\delta < \text{o.t.}(X^1_\delta) \leq \text{o.t.}(X^2_\delta) \in T \subset C(f)$. Hence $f(\delta) < \text{o.t.}(X^2_\delta)$.

□

2.8 Note. ([W]) (1) The Strong Reflection Principle (SRP) implies ψ_{AC}.

(2) ψ_{AC} implies $2^\omega = 2^{\omega_1} = \omega_2$.

(3) The Martin's Maximum (MM) implies ϕ_{AC}.

(4) ϕ_{AC} implies $2^{\omega_1} = \omega_2$.

2.9 Question. (1) ([LS]) It is known Con(CB+CH) and so CB does not imply ψ_{AC}. Separate these principles as much as possible.

(2) Investigate the effects of MM and SRP on τ_{AC}.

§3. Main Lemma

This section builds on the communication [A] by D. Aspero.

3.1 Lemma. *Let κ be a measurable cardinal, θ be a regular cardinal with $\theta \geq (2^\kappa)^+$, N be a countable elementary substructure of H_θ with $\kappa \in N$, $\delta < \omega_1$ and $S \subseteq \omega_1$ be stationary. Then there exists a countable elementary substructure M of H_θ such that*

(1) $N \subseteq M$.

(2) For any $a \in H_\kappa \cap N$, $a \cap N = a \cap M$.

(3) $\delta < \text{o.t.}(M \cap \kappa) \in S$.

Proof. Since $H_\theta \models$ "κ is measurable" and N is an elementary substructure of H_θ with $\kappa \in N$, we may take a normal measure $D \in N$. Take any $s \in \bigcap(N \cap D)$ and define

$$N(s) = \{f(s) \mid f \in N\}.$$

Then $N(s)$ is a countable elementary substructure of H_θ such that (1) $N(s) \cap \kappa$ end-extends $N \cap \kappa$ and s is the least in $(N(s) \cap \kappa) \setminus (N \cap \kappa)$. (2) For any $a \in N \cap H_\kappa$, $a \cap N(s) = a \cap N$ holds.

Now iterate this process to construct a continuously $\subset$-increasing sequence $\langle N_i \mid i < \omega_1\rangle$ of countable elementary substructures of H_θ with $N = N_0$. Notice that $\langle \text{o.t.}(N_i \cap \kappa) \mid i < \omega_1\rangle$ provides a club. Hence we have N_i such that $\delta < \text{o.t.}(N_i \cap \kappa) \in S$. Let $M = N_i$. This M works.

□

3.2 Definition. For the rest of this section, we fix a continuously strictly increasing sequence $\langle \kappa_j \mid j \leq \omega_1\rangle$ of cardinals such that

(1) κ_0 is a measurable cardinal.

(2) For all successor ordinals $j+1$, κ_{j+1} are measurable cardinals.

(3) Hence if $j \leq \omega_1$ is a limit, then $\kappa_j = \sup\{\kappa_{j'} \mid j' < j\}$ is singular.

3.3 Definition. Let $\langle S^j \mid j < \omega_1\rangle$ be any sequence of stationary subsets of ω_1 and $t < \omega_1$. Then let $\phi(\langle S^j \mid j < \omega_1\rangle, t)$ stand for

For all (s, θ, N, δ) such that

- $s < t$,
- θ is a regular cardinal with $\theta \geq (\kappa_{\omega_1})^+$.
- N is a countable elememtary substructure of H_θ with $\langle \kappa_j \mid j \leq \omega_1\rangle, s, t \in N$,
- $\delta < \omega_1$.

There exists a countable elementary substructure M of H_θ such that

- $N \subseteq M$,
- $N \cap \kappa_s = M \cap \kappa_s$,
- For all j with $s+1 \leq j \leq t$, we have $\delta < \text{o.t.}(M \cap \kappa_j) \in S^j$.

3.4 Lemma. *For any sequence $\langle S^j \mid j < \omega_1\rangle$ of stationary subsets of ω_1 and any $t < \omega_1$, we have $\phi(\langle S^j \mid j < \omega_1\rangle, t)$.*

Proof. Fix $\langle S^j \mid j < \omega_1\rangle$ and simply denote $\phi(t)$. We show $\phi(t)$ by induction on $t < \omega_1$. First notice $\phi(0)$ is vacuously true.

$\phi(t) \longrightarrow \phi(t+1)$: Let (s, θ, N, δ) be given as in $\phi(t+1)$. Since $s < t+1$, we consider in two cases.

Case 1. $s = t$: Want a countable elementary substructure M of H_θ such that $N \subseteq M$, $N \cap \kappa_t = M \cap \kappa_t$ and $\delta < \text{o.t.}(M \cap \kappa_{t+1}) \in S^{t+1}$. But this is done by 3.1 Lemma with the measurable κ_{t+1}.

Case 2. $s < t$: Apply $\phi(t)$ with (s, θ, N, δ). Then we have M' such that

- $N \subseteq M'$.
- $N \cap \kappa_s = M' \cap \kappa_s$.
- For all j with $s+1 \leq j \leq t$, we have $\delta < \text{o.t.}(M' \cap \kappa_j) \in S^j$.

Since $\kappa_{t+1} \in (N \subseteq) M'$, we may again apply 3.1 Lemma. So may take a countable elementary substructure M of H_θ such that

- $M' \subseteq M$.
- $M' \cap \kappa_t = M \cap \kappa_t$.
- $\delta < \text{o.t.}(M \cap \kappa_{t+1}) \in S^{t+1}$.

Then this M works.

t is limit, $(\forall \bar{t} < t\ \phi(\bar{t})) \longrightarrow \phi(t)$: Let (s, θ, N, δ) be given as in $\phi(t)$. Fix a $<$-increasing sequence $\langle t_n \mid n < \omega_1\rangle$ such that $t_0 = s$ and $\sup\{t_n \mid n < \omega\} = t$. Notice $t_n \in N \cap \omega_1$.

Now let us take a sufficiently large regular cardinal χ and a countable elementary substructure N^* of H_χ such that N^* contains every thing visible.

- $\langle S^j \mid j < \omega_1\rangle, H_\theta, N, \delta, \langle t_n \mid n < \omega\rangle \in N^*$ and so $\langle \kappa_j \mid j \leq \omega_1\rangle \in N \subset N^*$.

  And

- $N^* \cap \omega_1 \in S^t$.

Let $\langle \delta_n \mid n < \omega\rangle$ be an increasing sequence of ordinals such that $\delta_0 = \delta$ and $\sup\{\delta_n \mid n < \omega\} = N^* \cap \omega_1$. Construct a sequence of countable elementary substructures $\langle M_n \mid n < \omega\rangle$ of H_θ by recursion on n. We first apply $\phi(t_1)$ with (s, θ, N, δ) so that

- $N \subseteq M_0$, $M_0 \in N^*$.
- $N \cap \kappa_s = M_0 \cap \kappa_s$.
- For all j with $s + 1 \leq j \leq t_1$, we have $\delta < \text{o.t.}(M_0 \cap \kappa_j) \in S^j$.

  It is posssible to have $M_0 \in N^*$ by elementarity. Suppose we have constructed M_n so that

- $N \subseteq M_n$, $M_n \in N^*$.
- $N \cap \kappa_s = M_n \cap \kappa_s$.
- For all j with $t_n + 1 \leq j \leq t_{n+1}$, we have $\delta_n < \text{o.t.}(M_n \cap \kappa_j) \in S^j$.

  Want M_{n+1}. By $\phi(t_{n+2})$ with $(t_{n+1}, \theta, M_n, \delta_{n+1})$, we have $M_{n+1} \in N^*$ such that

- $M_n \subseteq M_{n+1}$.
- $M_n \cap \kappa_{t_{n+1}} = M_{n+1} \cap \kappa_{t_{n+1}}$.
- For all j with $t_{n+1} + 1 \leq j \leq t_{n+2}$, we have $\delta_{n+1} < \text{o.t.}(M_{n+1} \cap \kappa_j) \in S^j$.

  Let $M = \bigcup\{M_n \mid n < \omega\}$. We claim this M works. Among others, we provide details for

- For all j with $s + 1 \leq j \leq t$, we have $\text{o.t.}(M \cap \kappa_j) \in S^j$.

  We consider in two cases. If $t_n + 1 \leq j \leq t_{n+1}$, then

$$M \cap \kappa_j = M_{n+1} \cap \kappa_j = M_n \cap \kappa_j$$

and so

$$\text{o.t.}(M \cap \kappa_j) \in S^j.$$

If $j = t$, then

$$\text{o.t.}(M \cap \kappa_t) = \sup\{\text{o.t.}(M \cap \kappa_{t_{n+1}}) \mid n < \omega\} = \sup\{\text{o.t.}(M_n \cap \kappa_{t_{n+1}}) \mid n < \omega\}.$$

While

$$\sup\{\text{o.t.}(M_n \cap \kappa_{t_{n+1}}) \mid n < \omega\} = \sup\{\delta_n \mid n < \omega\} = N^* \cap \omega_1 \in S^t.$$

Hence we are done.

□

3.5 Definition. Let $\langle S_i^j \mid j \leq i < \omega_1\rangle$ be any system of stationary subsets of ω_1 and $i < \omega_1$. Define $S[i]$ by

$$S[i] = \{X \in [\kappa_i]^\omega \mid (\forall j \leq i)\ \text{o.t.}(X \cap \kappa_j) \in S_i^j\}.$$

We also define $S[*]$ by

$$S[*] = \{X \in [\kappa_{\omega_1}]^\omega \mid X \cap \omega_1 < \omega_1, (\forall j \leq X \cap \omega_1)\ \text{o.t.}(X \cap \kappa_j) \in S^j_{X \cap \omega_1}\}.$$

Notice that if $X \in S[*]$, then $X \cap \kappa_{X \cap \omega_1} \in S[X \cap \omega_1]$ holds.

3.6 Lemma. *For any system $\langle S_i^j \mid j \leq i < \omega_1 \rangle$ of stationary subsets of ω_1 and any $i < \omega_1$, $S[i]$ is semiproper. By this we mean;*

For all regular cardinals $\theta \geq (\kappa_{\omega_1})^+$ and all countable elementary substructures N of H_θ with $\langle \kappa_j \mid j \leq \omega_1 \rangle, i \in N$, there exist countable elementary substructures M of H_θ such that $N \subseteq M$, $N \cap \omega_1 = M \cap \omega_1$ and $M \cap \kappa_i \in S[i]$.

Proof. Let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $\langle \kappa_j \mid j \leq \omega_1 \rangle, i \in N$. Want M such that $N \subseteq M$, $N \cap \omega_1 = M \cap \omega_1$ and $M \cap \kappa_i \in S[i]$.

By 3.1 Lemma, we first take N' such that

- $N \subseteq N'$.
- $N \cap \omega_1 = N' \cap \omega_1$.
- o.t.$(N' \cap \kappa_0) \in S_i^0$.

We consider in two cases. If $i = 0$, then let $M = N'$. This M works.

If $0 < i$, then for each $j < \omega_1$, let

$$S^j = \begin{cases} S_i^j, & \text{if } j \leq i \\ \omega_1, & \text{otherwise.} \end{cases}$$

By $\phi(\langle S^j \mid j < \omega_1 \rangle, i)$ with $(0, \theta, N', 0)$, we have a countable elementary substructure M of H_θ such that

- $N' \subseteq M$.
- $N' \cap \kappa_0 = M \cap \kappa_0$.
- For all j with $1 \leq j \leq i$, we have o.t.$(M \cap \kappa_j) \in S^j = S_i^j$.

This M works.

□

3.7 Lemma. *For any system $\langle S_i^j \mid j \leq i < \omega_1 \rangle$ of stationary subsets of ω_1, $S[*]$ is semiproper. By this we mean;*

For all regular cardinals $\theta \geq (\kappa_{\omega_1})^+$ and all countable elementary substructures N of H_θ with $\langle \kappa_j \mid j \leq \omega_1 \rangle \in N$, there exist countable elementary substructures M of H_θ such that $N \subseteq M$, $N \cap \omega_1 = M \cap \omega_1$ and $M \cap \kappa_{\omega_1} \in S[*]$.

Proof. Let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $\langle \kappa_j \mid j \leq \omega_1 \rangle \in N$. Let $\langle t_n \mid n < \omega \rangle$ be a $<$-increasing sequence of ordinals such that $t_0 = 0$ and $\sup\{t_n \mid n < \omega\} = N \cap \omega_1$.

Let χ be a large regular cardinal and N^* be a countable elementary substructure of H_χ such that N^* contains every parameter.

- $\langle S_i^j \mid j \leq i < \omega_1 \rangle$, $H_\theta, N, \langle t_n \mid n < \omega \rangle \in N^*$ and so $\langle \kappa_j \mid j \leq \omega_1 \rangle \in N \subset N^*$ holds.

And

- $N^* \cap \omega_1 \in S_{N \cap \omega_1}^{N \cap \omega_1}$.

Let $\langle \delta_n \mid n < \omega \rangle$ be an increasing sequence of ordinals such that $\delta_0 = 0$ and $\sup\{\delta_n \mid n < \omega\} = N^* \cap \omega_1$. Construct a sequence of countable elementary substructures $\langle M_n \mid n < \omega \rangle$ of H_θ by recursion on n.

We first get M_0 such that

- $N \subseteq M_0$, $M_0 \in N^*$.
- $N \cap \omega_1 = M_0 \cap \omega_1$.

- $\delta_0 < \text{o.t.}(M_0 \cap \kappa_0) \in S^0_{N\cap\omega_1}$.

  It is possible to have $M_0 \in N^*$ by elementarity. Suppose we have constructed M_n such that

- $N \subseteq M_n$, $M_n \in N^*$.
- $N \cap \omega_1 = M_n \cap \omega_1$.
- For all j with $j \leq t_n$, we have $\text{o.t.}(M_n \cap \kappa_j) \in S^j_{N\cap\omega_1}$ and $\delta_n < \text{o.t.}(M_n \cap \kappa_{t_n})$.

Want M_{n+1}. By $\phi(\langle S^j_{N\cap\omega_1} \mid j \leq N \cap \omega_1\rangle^\frown\langle\omega_1, \cdots\rangle, t_{n+1})$ with $(t_n, \theta, M_n, \delta_{n+1})$, we get $M_{n+1} \in N^*$ such that

- $M_n \subseteq M_{n+1}$.
- $M_n \cap \kappa_{t_n} = M_{n+1} \cap \kappa_{t_n}$.
- For all j with $t_n + 1 \leq j \leq t_{n+1}$, we have $\delta_{n+1} < \text{o.t.}(M_{n+1} \cap \kappa_j) \in S^j_{N\cap\omega_1}$.

This completes the construction. Let $M = \bigcup\{M_n \mid n < \omega\}$. Then this M works. Among others, we provide details for

- For all j with $j \leq M \cap \omega_1$, we have $\text{o.t.}(M \cap \kappa_j) \in S^j_{M\cap\omega_1}$.

First note that $N \cap \omega_1 = M \cap \omega_1$. We consider in two cases. If $j \leq t_n$, then

$$M \cap \kappa_j = M_n \cap \kappa_j.$$

And so

$$\text{o.t.}(M \cap \kappa_j) \in S^j_{M\cap\omega_1}.$$

If $j = M \cap \omega_1$, then

$$\text{o.t.}(M \cap \kappa_{M\cap\omega_1}) = \sup\{\text{o.t.}(M \cap \kappa_{t_n}) \mid n < \omega\} = \sup\{\text{o.t.}(M_n \cap \kappa_{t_n}) \mid n < \omega\}.$$

While

$$\sup\{\text{o.t.}(M_n \cap \kappa_{t_n}) \mid n < \omega\} = \sup\{\delta_n \mid n < \omega\} = N^* \cap \omega_1 \in S^{M\cap\omega_1}_{M\cap\omega_1}.$$

Hence we are done.

□

§4. Forcing Construction

We force τ_{AC} by iteration. Here is a single step.

4.1 Definition. Let $\langle \kappa_j \mid j \leq \omega_1\rangle$ be as before. Let $\langle S^j_i \mid j \leq i < \omega_1\rangle$ be any system of stationary subsets of ω_1. We define $p = \langle X^p_i \mid i \leq i^p\rangle \in P$, more precisely, $P(\langle S^j_i \mid j \leq i < \omega_1\rangle)$, if

(1) $i^p < \omega_1$.
(2) $X^p_i \in S[i]$. Namely, $X^p_i \in [\kappa_i]^\omega$ and for all $j \leq i$, $\text{o.t.}(X^p_i \cap \kappa_j) \in S^j_i$.
(3) X^p_i are continuously $\subseteq$-increasing.

For $p, q \in P$, let $q \leq p$, if $q \supseteq p$.

4.2 Lemma. *For any $p \in P$, $t > i^p$ and $\xi \in \kappa_t$, there exists $q \leq p$ such that $i^q = t$ and $\xi \in X^q_t$.*

Proof. By induction on $t < \omega_1$. If $t = 0$, then it is vacuously true.

$\underline{t \longrightarrow t+1}$: Let (p, ξ) be given. Since we assume $i^p < t + 1$, we consider in two cases. If $i^p = t$, then since $S[t+1]$ is cofinal in $[\kappa_{t+1}]^\omega$, we may take $X \in S[t+1]$ with $X^p_t \cup \{\xi\} \subseteq X$. Let $q = p \cup \{(t+1, X)\}$. Then this q works.

If $i^p < t$, then by induction we have $p' \in P$ such that

- $p' \leq p$.
- $i^{p'} = t$.

and, say

- $0 \in X_t^{p'}$.

Then take $X \in S[t+1]$ with $X_t^{p'} \cup \{\xi\} \subseteq X$. Let $q = p' \cup \{(t+1, X)\}$. Then this q works.

<u>t is limit</u>: Let (p, ξ) be given. We assume $i^p < t$. Let $\langle t_n \mid n < \omega \rangle$ be a sequence of ordinals such that $t_0 = i^p$ and $\sup\{t_n \mid n < \omega\} = t$. Since $S[t]$ is stationary in $[\kappa_t]^\omega$, we may take a countable elementary substructure N of H_θ, where θ is a sufficiently large regular cardinal, such that

- $p, P, \langle t_n \mid n < \omega \rangle \in N$.

And

- $N \cap \kappa_t \in S[t]$.

Let $\langle \xi_n \mid n < \omega \rangle$ enumerate $N \cap \kappa_t$ such that $\xi_n \in N \cap \kappa_{t_{n+1}}$. Construct a sequence $\langle p_n \mid n < \omega \rangle$ of conditions of P by recursion on n. Let $p_0 = p$. Suppose we have constructed p_n such that

- $p_n \in N$.
- $i^{p_n} = t_n$.

Want p_{n+1}. By induction we get $p_{n+1} \in N$ such that

- $p_{n+1} \leq p_n$.
- $i^{p_{n+1}} = t_{n+1}$.
- $\xi_n \in X_{t_{n+1}}^{p_{n+1}}$.

This completes the construction. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(t, N \cap \kappa_t)\}$. Then this q works.

□

4.3 Lemma. *P is σ-Baire and semiproper.*

Proof. We show P is semiproper. Let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $P \in N$. We further assume $\langle \kappa_j \mid j \leq \omega_1 \rangle \in N$. Let $p \in P \cap N$. Want $q \leq p$ such that q is (P, N)-semigeneric.

Since $S[*]$ is semiproper, there exists a countable elementary substructure M of H_θ such that

- $N \subseteq M$.
- $N \cap \omega_1 = M \cap \omega_1$.
- $M \cap \kappa_{\omega_1} \in S[*]$.

Hence

$$M \cap \kappa_{M \cap \omega_1} \in S[M \cap \omega_1].$$

Let $\langle p_n \mid n < \omega \rangle$ be any (P, M)-generic sequence with $p_0 = p$. Then let

$$q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa_{N \cap \omega_1})\}.$$

We claim this $q \in P$ works. This is because for all $n < \omega$, $q \leq p_n$ and so q is (P, M)-generic. Hence $q \Vdash_P$ "$M[G] \cap \omega_1^V = M \cap \omega_1^V$". Since $M \cap \omega_1 = N \cap \omega_1$, we have $q \Vdash_P$ "$N[G] \cap \omega_1^V \subseteq M[G] \cap \omega_1^V = N \cap \omega_1^V$". Hence $q \Vdash_P$ "$N[G] \cap \omega_1^V = N \cap \omega_1^V$".

By the above, we may also conclude that P is σ-Baire.

□

4.4 Lemma. *Let G be P-generic over V. Let $\langle \dot{X}_i \mid i < \omega_1 \rangle = \bigcup G$. Then for all $j \le i < \omega_1$, we have* o.t.$(\dot{X}_i \cap \kappa_j) \in S_i^j$ *and* $\bigcup\{\dot{X}_i \mid i < \omega_1\} = \kappa_{\omega_1}$.

Proof. By Lemma 4.3, ω_1 gets preserved. By 4.2 Lemma, we have $\bigcup G$ is of length ω_1 and $\bigcup\{X_i \mid i < \omega_1\} = \kappa_{\omega_1}$.

□

4.5 Theorem. *Let ρ be a regular cardinal such that $\rho = \sup\{\kappa < \rho \mid \kappa$ is measurable$\}$. Then there exists a ρ-stage iteration P_ρ such that*

(1) P_ρ is semiproper and has the ρ-c.c.

(2) In V^{P_ρ}, $\rho = \dot{\omega}_2$ and τ_{AC} holds.

Proof. (Out-line) Let ρ be a regular limit of measurables. Let $\alpha < \rho$ and suppose we have constructed P_α such that P_α is semiproper and $P_\alpha \in H_\rho$. Then we force with some $P(\langle S_i^j \mid j \le i < \omega_1 \rangle)$ by naturally choosing the least sequence $\langle \kappa_j \mid j \le \omega_1 \rangle$ in the intermediate stage V^{P_α}. The system $\langle S_i^j \mid j \le i < \omega_1 \rangle$ is specified to be calculated at some stage $\beta \le \alpha$. This is done as usual by book-keeping every possible system of subsets $\langle S_i^j \mid j \le i < \omega_1 \rangle$ of ω_1 in V^{P_β} for all $\beta \le \alpha$. At the limit stages, we take the simple limit of [M]. This completes the construction of P_ρ. By construction P_ρ is semiproper and ω_1 is preserved. Since P_ρ is a semiproper iteration such that for all $\alpha < \rho$, $|P_\alpha| < \rho$, we conclude ([M]) that P_ρ has the ρ-c.c. Hence by the end, we have dealt with every possible system of stationary subsets of ω_1.

Hence τ_{AC} holds in V^{P_ρ}. Since relevant measurable cardinals are collapsed, we conclude ρ becomes the ω_2 in V^{P_ρ}.

□

The following, possibly except (2), have been known to D. Aspero and others.

4.6 Corollary. ([A] et al) *The following are all equiconsistent.*

(1) There exists a regular limit of measurable cardinals.

(2) τ_{AC} holds.

(3) ψ_{AC} holds.

(4) ϕ_{AC} holds.

(5) CB holds.

Proof. The consistency of (5) implies that of (1) by [DD]. Hence all of these are equiconsistent.

□

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