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The covering number and the uniformity of the ideal $\mathcal{I}_f$

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1 Introduction

For the ideal $\mathcal{SN}$ of strongly measure zero subsets of the real line, the cardinal coefficients have been studied[1]. But its cofinality had not been studied. In general, it may be larger than the continuum. Yorioka studied its cofinality(see [2]). One of his results is that the value of $\mathsf{cof}(\mathcal{SN})$ is equal to the dominating number for ${\omega_1}^{\omega_1}$ under the continuum hypothesis. In the process, he introdused ideals $\mathcal{I}_f$ for $f \in \omega^\omega$. These ideals were used in the proof. We are interested in the ideals $\mathcal{I}_f$ themselves. These ideals are subideals of the null ideal $\mathcal{N}$ and include $\mathcal{SN}$. The properties of these ideals depend on f.

In this paper, we discuss the following contents. In section 3, we show a characterization of $\mathsf{cov}(\mathcal{I}_f) \geq \mathfrak{b}$. In section 4, we define a forcing notion which has the countable chain condition. And with the results of section 3 we show that its ω_2-stage finite support iteration by bookkeeping method lifts up $\mathsf{cov}(\mathcal{I}_f)$ from a ground model with the continuum hypothesis. In section 5, we introduce a sufficient condition not to lift up $\mathsf{cov}(\mathcal{I}_f)$ for forcing notions which satisfy axiom A.

2 Definitions and notation

Throughout this paper, we use the standard terminology for forcing of set theory and cardinal coefficients (see[1]). We regard the set of all reals as the Cantor set 2^ω. We denote by $\mathcal{M}$ and $\mathcal{N}$ the set of all meager subsets of 2^ω and the set of all null subsets of 2^ω respectively.

For functions $f,\ g$ in ω^ω we write "$f \leq g$" to mean that g dominates f everywhere, that is, $f(n) \leq g(n)$ for all $n < \omega$. And we let "$f \leq^* g$" mean that g eventually dominates f, that is, there exists an $n < \omega$ such that $f(m) \leq g(m)$ holds for all $m < \omega$ larger than n. We denote by S the set of all non-decreasing functions d in ω^ω which diverges to infinity and $d(0) = 0$. We denote by $\mathbb{C}$ and $\mathbb{D}$ the Cohen forcing notion and the dominating forcing notion respectively[1]. For each ideal (or family if there is not a problem in particular) $\mathcal{I}$ on 2^ω which contains all singletons, we denote by $\mathsf{add}(\mathcal{I})$, $\mathsf{cov}(\mathcal{I})$, $\mathsf{non}(\mathcal{I})$ and $\mathsf{cof}(\mathcal{I})$ the additivity, covering number, uniformity and cofinality of $\mathcal{I}$ respectively which means that:

1. $\mathsf{add}(\mathcal{I}) = \min\{\,|\mathcal{A}| \mid \mathcal{A} \subset \mathcal{I} \bigcup \mathcal{A} \notin \mathcal{I}\,\}$,
2. $\mathsf{cov}(\mathcal{I}) = \min\{\,|\mathcal{A}| \mid \mathcal{A} \subset \mathcal{I} \bigcup \mathcal{A} = 2^\omega\,\}$,
3. $\mathsf{non}(\mathcal{I}) = \min\{\,|Y| \mid Y \subset 2^\omega Y \notin \mathcal{I}\,\}$,
4. $\mathsf{cof}(\mathcal{I}) = \min\{\,|\mathcal{A}| \mid \mathcal{A} \subset \mathcal{I} \forall B \in \mathcal{I}\, \exists A \in \mathcal{A}\ (B \subset A)\,\}$.

We have that $\mathsf{add}(\mathcal{I}) \leq \mathsf{cov}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$ and $\mathsf{add}(\mathcal{I}) \leq \mathsf{non}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$ for each ideal or family $\mathcal{I}$ on 2^ω which contains all singletons.

We define some notation before we define the ideals $\mathcal{I}_f$ and $\mathcal{K}_f$.

Definition 2.1 *Let f, g be functions in ω^ω.*

1. *We define the order "$\ll$" on ω^ω by*
   $f \ll g$ *iff* $\forall k < \omega\ \exists N < \omega\ \forall n \geq N\ (f(n^k) \leq g(n))$,

2. *We define the order "$\lll$" on ω^ω by*
   $\mathrm{g}_\sigma(n) = |\sigma(n)|$ *for all* $n < \omega$,

3. *For $\sigma \in (2^{<\omega})^\omega$, define the subset $\mathsf{Y}(\sigma) \subset 2^\omega$ by*
   $\mathsf{Y}(\sigma) = \bigcap_{n<\omega} \bigcup_{m \geq n} [\sigma(m)]$, *where* $[s] = \{\, x \in 2^\omega \mid s \subset x \,\}$ *for each* $s \in 2^{<\omega}$.

   Define the subsets $\mathcal{S}(f)$, $\mathcal{T}(f)$ and $\mathcal{U}(f)$ of $(2^{<\omega})^\omega$ by

$$
\begin{aligned}
\mathcal{S}(f) &= \{\, \sigma \in (2^{<\omega})^\omega \mid \mathrm{g}_\sigma \gg f \,\}, \\
\mathcal{T}(f) &= \{\, \sigma \in (2^{<\omega})^\omega \mid \mathrm{g}_\sigma = f \,\}.
\end{aligned}
$$

Definition 2.2 *Let $f \in \omega^\omega$. Define the families $\mathcal{I}_f, \mathcal{J}_f$ and $\mathcal{K}_f$ on 2^ω by*

$$
\begin{aligned}
\mathcal{I}_f &= \{\, X \subset 2^\omega \mid \exists \sigma \in \mathcal{S}(f)\ X \subset \mathsf{Y}(\sigma) \,\}, \\
\mathcal{J}_f &= \{\, X \subset 2^\omega \mid \exists \sigma \in \mathcal{T}(f)\ X \subset \mathsf{Y}(\sigma) \,\}.
\end{aligned}
$$

The following definiton is not necessary for the definition of ideal $\mathcal{I}_f$. But it is the very useful.

Definition 2.3 *Let $f \in \omega^\omega$. For each $d \in \mathsf{S}$, we define the functions $g_d^{(f)}$ and $h_d^{(f)} \in \omega^\omega$ by*
$g_d^{(f)}(n) = f(n^{k+2})$ *if* $n \in [d(k), d(k+1))$
for all $n < \omega$, respectively. If $g \in \omega^\omega$ is $g = g_d^{(f)}$ for some $d \in \omega^{\uparrow\omega}$, then we say "$g$ is generated by d (and f) for $\ll$".

3 $\mathsf{cov}(\mathcal{I}_f)$, $\mathsf{cov}(\mathcal{J}_f)$ and bouding number $\mathfrak{b}$

In this section, we show that the ideal $\mathcal{I}_f$ and the family $\mathcal{J}_f$ are related to bounding number $\mathfrak{b}$ intimately. For each $d \in \mathsf{S}$, $g_d^{(f)} \gg f$ holds where $g_d^{(f)}$ was introdused in chapter 2. In addition, for each $g \gg f$ there exists a $d \in \mathsf{S}$ by the definitions of $g_d^{(f)}$ and $\ll$ such that $g_d^{(f)} \leq^* g$. Therefore, the following hold.

Lemma 3.1 *For each family $\mathcal{F} \subset \omega^\omega$ such that $|\mathcal{F}| < \mathfrak{b}$ and $\forall g \in \mathcal{F}\ (g \gg f)$, there exists $d \in \mathsf{S}$ such that $\forall g \in \mathcal{F}\ (g_d^{(f)} \leq^* g)$.*

Proof of Lemma3.1 Let $\mathcal{F} \subset \omega^\omega$ satisfy $\forall g \in \mathcal{F}\ (g \gg f)$ and $|\mathcal{F}| < \mathfrak{b}$. For each $g \in \mathcal{F}$, there exists $d_g \in \mathsf{S}$ such that $g_{d_g}^{(f)} \leq g$. Since $|\mathcal{F}| < \mathfrak{b}$, the family $\{ d_g \mid g \in \mathcal{F} \}$ is bounded family in ω^ω. So there exists $d \in \mathsf{S}$ which dominates for all functions in $\{ d_g \mid g \in \mathcal{F} \}$. □(Lemma3.1)

Lemma 3.2 *There exists a family $\mathcal{F} \subset \omega^\omega$ such that $|\mathcal{F}| = \mathfrak{b}$ and $\forall g \in \mathcal{F}\ (g \gg f)$ and $\forall h \gg f \exists g \in \mathcal{F}\ (h \not\leq^* g)$.*

Proof of Lemma3.2 Take a unbounded family $\mathcal{B} \subset \mathsf{S}$. Then a family $\left\{ g_d^{(f)} \mid d \in \mathcal{B} \right\}$ is as desired.

□(Lemma3.2)

For all $d \in \mathsf{S}$, $\mathsf{cov}(\mathcal{I}_f) \leq \mathsf{cov}(\mathcal{J}_{g_d^{(f)}})$ holds by $\mathcal{I}_f = \bigcup_{g \gg f} \mathcal{J}_g = \bigcup_{d \in \mathsf{S}} \mathcal{J}_{g_d^{(f)}}$. By this, if $\mathsf{cov}(\mathcal{I}_f)$ is larger than $\mathfrak{b}$, then $\mathsf{cov}(\mathcal{J}_{g_d^{(f)}})$ is larger than $\mathfrak{b}$ for all $d \in \mathsf{S}$. The inverse holds.

Theorem 3.1 $\mathsf{cov}(\mathcal{I}_f) \geq \mathfrak{b}$ *iff* $\mathsf{cov}(\mathcal{J}_{g_d^{(f)}}) \geq \mathfrak{b}$ *for all* $d \in \mathsf{S}$.

Proof of Theorem3.1 $\Longrightarrow$: As above.
$\Longleftarrow$: Assume $\mathsf{cov}(\mathcal{I}_f) < \mathfrak{b}$. There exists a family $\mathcal{F}$ such that $|\mathcal{F}| = \mathsf{cov}(\mathcal{I}_f) < \mathfrak{b}$ and $\bigcup \mathcal{F} = 2^\omega$. For each $X \in \mathcal{F}$, there exists σ_X such that $X \subset \mathsf{Y}(\sigma_X)$. By Lemma3.1, there exists $d \in \mathsf{S}$ such that $\forall X \in \mathcal{F}\ g_d^{(f)} \leq^* g_{\sigma_X}$. For each $X \in \mathcal{F}$, define $\tau_X \in \mathcal{T}(g_d^{(f)})$ by $\tau_X(n) = \sigma_X(n) \restriction g_d^{(f)}(n)$. Then a family $\{ \mathsf{Y}(\tau_X) \mid X \in \mathcal{F} \} \subset \mathcal{J}_{g_d^{(f)}}$ covers 2^ω. □(Theorem3.1)

However, it is easily proved that $\mathsf{cov}(\mathcal{I}_f) \geq \mathfrak{b}$ is independent from ZFC. $\mathsf{cov}(\mathcal{I}_f) = \omega_1$ and $\mathfrak{b} = \mathfrak{c}$ hold in a generic model which is obtained by a forcing notion satisfying Laver property from a ground model with the continuum hypothesis. Also $\mathsf{cov}(\mathcal{I}_f) = \mathfrak{b} = \omega_1$ holds in a generic model which is obtained by the Cohen forcing notion of any weight from a ground model with continuum hypethesis.

4 The forcing notion $\mathbb{P}(d)$ for $d \in \mathsf{S}$ and $\mathsf{cov}(\mathcal{I}_f)$ and $\mathsf{non}(\mathcal{I}_f)$

In this section, we discuss the covering number and the uniformity of ideal $\mathcal{I}_f$ in the model obtained by a certain itaration of the forcing notion $\mathbb{P}(d)$. We define the forcing notion $\mathbb{P}(d)$ for $d \in \mathsf{S}$.

Definition 4.1 *Let $d \in \mathsf{S}$. Define the forcing notion $\mathbb{P}(d)$ by*

$$\mathbb{P}(d) = \left\{ (s,\ F) \in 2^{<\omega} \times \left[\mathcal{T}(g_d^{(f)})\right]^{<\omega} \mid |s| = f(|F|) \right\},$$

$$(s,\ F) \le (s',\ F')$$

$$\iff 1.\ s \supset s' F \supset F'$$

$$2.\ \forall \sigma \in F'\ \forall n \in |F| \setminus |F'|\ \ [s{\restriction}[f(n),\ f(n+1)) \ne \sigma(n+1){\restriction}[f(n),\ f(n+1))].$$

Lemma 4.1 *For all $d \in \mathsf{S}$, the forcing notion $\mathbb{P}(d)$ is σ-linked. So it has the countable chain condition.*

Proof of Lemma4.1 Since $g_d^{(f)}(n+1) - g_d^{(f)}(n) > n$ for all $n < \omega$, holds that $\forall (s,\ F) \in \mathbb{P}(d) \forall F' \in \left[\mathcal{T}(g_d^{(f)})\right]^{<\omega} \exists (t,\ H) \le (s,\ F)\ \ (H = F \cup F')$.

Let $N < \omega$ and $g = g_d^{(f)}$. For each $t \in 2^{g(N)}$, $\psi \in \prod_{n \in [N, 2N)} \left[2^{g(n+1)-g(n)}\right]^{\le N}$, define a subset $B_{t,\psi}$ of $\mathbb{P}(d)$ by

$B_{t,\psi} = \{ (s,\ F) \in \mathbb{P}(d) \mid s = t\psi = \langle \{ \sigma(n+1){\restriction}[g(n),\ g(n+1)) \mid \sigma \in F\} \mid n \in [|F|,\ 2|F|) \rangle \}$.

Clearly $\mathbb{P}(d) = \bigcup_{N<\omega} \bigcup \left\{ B_{t,\ \psi} \ \middle|\ t \in 2^{g(N)} \psi \in \prod_{n \in [N,2N)} \left[2^{g(n+1)-g(n)}\right]^{\le N} \right\}$. We show that for all $N < \omega$, $t \in 2^{g(N)}$ and $\psi \in \prod_{n \in [N,2N)} \left[2^{g(n+1)-g(n)}\right]^{\le N}$, any two distinct conditions in $B_{t,\psi}$ are compatible. Let $(s,\ F)$, $(s',\ F')$ be in $B_{t,\psi}$ and $(s,\ F) \ne (s',\ F')$. By the definition of $B_{t,\psi}$,

$$s = s' = t\,|F| = |F'| = N$$

$$\langle \{ \sigma(n+1){\restriction}[g(n),\ g(n+1)) \mid \sigma \in F\} \mid n \in [|F|,\ 2|F|) \rangle$$

$$= \left\langle \{ \sigma(n+1){\restriction}[g(n),\ g(n+1)) \mid \sigma \in F'\} \mid n \in [|F'|,\ 2|F'|) \right\rangle = \psi.$$

There exists $(u,\ H) \le (s,\ F)$ such that $H = F \cup F'$. Clealy $|F'| < |H| \le 2N$. To prove $(u,\ H) \le (s',\ F')$, let $\sigma \in F'$ and $n \in |H| \setminus |F'|$. Since $(u,\ H) \le (s,\ F)$, $u{\restriction}[g(n),\ g(n+1)) \ne \tau(n+1){\restriction}[g(n),\ g(n+1))$ for all $\tau \in F$, that is, $u{\restriction}[g(n),\ g(n+1)) \notin \psi(n)$.
But $\sigma(n+1){\restriction}[g(n),\ g(n+1)) \in \psi(n)$.

Therefore $u{\restriction}[g(n),\ g(n+1)) \ne \sigma(n+1){\restriction}[g(n),\ g(n+1))$. □(Lemma4.1)

For each $d \in \mathsf{S}$, $\sigma \in \mathcal{T}(g_d^{(f)})$ and $n < \omega$, define the subsets D_σ, $E_n \subset \mathbb{P}(d)$ as follows:

$$D_\sigma = \{ (s,\ F) \in \mathbb{P}(d) \mid \sigma \in F \},$$

$$E_n = \{ (s,\ F) \in \mathbb{P}(d) \mid |F| \ge n \}.$$

Lemma 4.2 *For all $s \in \mathsf{S}$, $\sigma \in \mathcal{T}(g_d^{(f)})$ and $n < \omega$, the subsets D_σ and E_n are dense open sets in $\mathbb{P}(d)$.*

Proof of Lemma4.2 Let $\sigma \in \mathcal{T}(g_d^{(f)})$, $n < \omega$ and $(s,\ F) \in \mathbb{P}(d)$. Take $F' \subset \mathcal{T}(g_d^{(f)})$ such that $\sigma \in F'$ and $|F| \ge n$. There exists $(t,\ H) \le (s,\ F)$ suth that $H = F \cup F'$. Since $\sigma \in H$ and $|H| \ge n$, $(t,\ H) \in D_\sigma$ and $(t,\ H) \in E_n$. □(Lemma4.2)

We are interested in the generic model of $\mathbb{P}(d)$. Let $d \in \mathsf{S}$ and $\dot{G}$ be the canonical generic $\mathbb{P}(d)$-name. Define $\mathbb{P}(d)$-name $\dot{a}_{\dot{G}}$ by

$$\Vdash_{\mathbb{P}(d)} \dot{a}_{\dot{G}} = \bigcup \left\{ s \;\middle|\; \exists F \ (s,\, F) \in \dot{G} \right\} \in 2^{\omega}.$$

Lemma 4.3 *For all $d \in \mathsf{S}$, $\Vdash_{\mathbb{P}(d)} \forall \sigma \in \mathcal{T}(g_d^{(f)}) \cap \mathbf{V} \ \ (\dot{a}_{\dot{G}} \notin \mathsf{Y}(\sigma))$.*

Proof of Lemma4.3 Let $d \in \mathsf{S}$, $\sigma \in \mathcal{T}(g_d^{(f)})$ and $(s,\, F) \in \mathbb{P}(d)$. By Lemma4.2, there exists $(s',\, F') \leq (s,\, F)$ such that $\sigma \in F'$. To prove that $(s',\, F') \Vdash_{\mathbb{P}(d)} \sigma(n) \not\subset \dot{a}_{\dot{G}}$ for all $n > |F'|$, let $n > |F'|$. By Lemma4.2, there exists $(s'',\, F'') \leq (s',\, F')$ such that $|F''| \geq n$. Then $(s'',\, F'') \Vdash_{\mathbb{P}(d)}$ " $s'' \subset \dot{a}_{\dot{G}} s'' \restriction [g_d^{(f)}(n-1),\ g_d^{(f)}(n)) \neq \sigma(n) \restriction [g_d^{(f)}(n-1),\ g_d^{(f)}(n))$ ". Therefore $(s'',\, F'') \Vdash_{\mathbb{P}(d)} \sigma(n) \not\subset \dot{a}_{\dot{G}}$. □(Lemma4.3)

Lemma 4.4 *For all $d \in \mathsf{S}$, $\Vdash_{\mathbb{P}(d)} 2^{\omega} \cap \mathbf{V} \in \mathcal{J}_{g_d^{(f)}}$.*

Proof of Lemma4.4 This is directly followed from the fact that $\mathbb{P}(d)$ adds Cohen reals in $\prod_{n<\omega} 2^{g_d^{(f)}(n)}$. □(Lemma4.4)

To define a finite support iteration of $\mathbb{P}(d)$, let κ be an uncountable reguler cardinal and π be a bijection from κ onto $\kappa \times \kappa$ such that if $\pi(\alpha) = (\beta,\, \gamma)$ then $\beta \leq \alpha$ for all $\alpha < \kappa$. Let π_0 and π_1 be the first and second coordinate of the value of π respectively.

Assume the continuum hypothesis. We define $\mathbb{P}_{\kappa}$ by κ-stage finite support iteration $\left\langle P_{\alpha},\ \dot{Q}_{\alpha} \mid \alpha < \kappa \right\rangle$ as follows:
Assume that P_{β} and the P_{β}-names $\dot{d}_{\xi}^{\beta}$ for $\xi < \kappa$ with $\Vdash_{\beta}$ "$\left\langle \dot{d}_{\xi}^{\beta} \mid \xi < \kappa \right\rangle$ be an enumeration of S" are defined for all $\beta \leq \alpha$ in α-stage. Define $\Vdash_{\alpha} \dot{Q}_{\alpha} \simeq \mathbb{P}\left(\dot{d}_{\pi_1(\alpha)}^{\pi_0(\alpha)}\right) \star \mathbb{D}$.

Theorem 4.1 (CH) $\Vdash_{\mathbb{P}_{\kappa}} \mathfrak{c} = \mathfrak{b} = \kappa \wedge \forall d \in \mathsf{S}\ \mathsf{cov}(\mathcal{J}_{g_d^{(f)}}) = \mathfrak{c}$.
Therefore, it holds that $\Vdash_{\mathbb{P}_{\kappa}} \mathsf{cov}(\mathcal{I}_f) = \mathfrak{c}$ by theorem3.1.

Proof of Theorem4.1 Clearly $\mathfrak{c} = \mathfrak{b} = \kappa$ in $\mathbf{V}[G_{\kappa}]$. Let $d \in \mathsf{S}$, $\lambda < \mathfrak{c}$ and a family $\{ X_{\delta} \mid \delta < \lambda \} \subset \mathcal{J}_{g_d^{(f)}}$ in $\mathbf{V}[G_{\kappa}]$. There exists $\alpha < \kappa$ such that X_{δ} is coded by $\sigma_{\delta} \in \mathcal{T}(g_d^{(f)})$ for each $\delta < \lambda$ in $\mathbf{V}[G_{\alpha}]$. By Lemma4.3, $\{ \mathsf{Y}(\sigma_{\delta}) \mid \delta < \lambda \}$ does not cover 2^{ω} in $\mathbf{V}[G_{\alpha+1}]$. Hence $\{ X_{\delta} \mid \delta < \lambda \}$ does not cover 2^{ω} in $\mathbf{V}[G_{\kappa}]$. □(Theorem4.1)

Theorem 4.2 (CH) $\Vdash_{\mathbb{P}_{\omega_2}} \mathsf{non}(\mathcal{I}_f) = \mathfrak{c}$

Proof of Theorem4.2 Clearly by Lemma4.4. □(Theorem4.2)

5 Property E and $\mathsf{cov}(\mathcal{I}_f) = \omega_1$

In this section, we introduce a certain property for forcing notions which satisfy axiom A. A forcing notion with this property does not add a real which is not covered by all elements of $\mathcal{S}(f)$ in ground model. This property is preserved in an iterated forcing. So the countable support iteration of forcing notions with this property does not lift up $\mathsf{cov}(\mathcal{I}_f)$. For example, the infinitely equal forcing notion $\mathbb{EE}$ satisfies this property.

Definition 5.1 *Let forcing notion P satisfy axiom A by the fusion orders $\langle \leq_n \mid n < \omega \rangle$. P has property E if there exists $\varphi \in \omega^{P \times \omega}$ such that*

(1) *for all $p \in P$ and $n < \omega$, if $p \Vdash_P \dot{a} \in \mathbf{V}$ then there exist $q \leq_n p$ and a finite set B such that $|B| \leq \varphi(p, n)$ and $q \Vdash_P \dot{a} \in B$,*

(2) *for all $p, q \in P$ and $n < \omega$, if $q \leq_n p$ then $\varphi(q, n) = \varphi(p, n)$.*

Lemma 5.1 *Suppose that the axiom A forcing notion P has property E. Then $\Vdash_P$ "$2^\omega \subset \bigcup \{ \mathsf{Y}(\tau) \mid \tau \in \mathcal{T}(g) \cap \mathbf{V} \}$" for all strictly increasing function $g \in \omega^\omega$. Therefore, $\Vdash_P$ "$2^\omega \subset \bigcup \{ \mathsf{Y}(\tau) \mid \tau \in \mathcal{S}(f) \cap \mathbf{V} \}$".*

Proof of Lemma5.1 Let $p \in P$ satisfy $p \Vdash_P \dot{x} \in 2^\omega$ and $g \in \omega^\omega$ be strictly increasing. By induction on $j < \omega$, define three sequences $\langle p_j \in P \mid j < \omega \rangle$, $\langle m_j < \omega \mid j < \omega \rangle$ and $\langle A_j \mid j < \omega \rangle$ as follows:

(i) $p_0 = p$,

(ii) $p_{j+1} \leq_j p_j$,

(iii) $m_j = \sum_{i<j} \varphi(p_i, i)$,

(iv) $A_j \subset 2^{g(m_j + \varphi(p_j, j))}$,

(v) $|A_j| \leq \varphi(p_j, j)$,

(vi) $p_{j+1} \Vdash_P \dot{x} \restriction g(m_j + \varphi(p_j, j)) \in A_j$,

for all $j < \omega$. For each $j < \omega$, let $\left\{ s^j_l \mid l < \varphi(p_j, j) \right\}$ be a enumration of A_j. There exists $q \in P$ such that $\forall j < \omega\ q \leq_j p_j$.

We define $\sigma \in (2^{<\omega})^\omega$ by for each $n < \omega$, $\sigma(n) = s^j_l \restriction g(n)$ where $n = m_j + l$. To prove that $q \Vdash_P \dot{x} \in \mathsf{Y}(\sigma)$, let $n < \omega$. There exists $j < \omega$ such that $m_j \geq n$. Since $q \Vdash_P \dot{x} \restriction (m_j + \varphi(p_j, j)) \in A_j$, there exist $q' \leq q$ and $l < \varphi(p_j, j)$ such that $q' \Vdash_P \dot{x} \restriction (m_j + \varphi(p_j, j)) = s^j_l \supset \sigma(m_j + l)$. □(Lemma5.1)

Let $\delta \leq \omega_2$. Let $P_\delta = \left\langle P_\alpha,\ \dot{Q}_\alpha \mid \alpha < \delta \right\rangle$ be a δ-stage countable support iteration such that $\dot{Q}_\alpha$ is defined by the forcing notion with property E for all $\alpha < \delta$. For $n < \omega$ and $F \in [\delta]^{<\omega}$, $p \in P_\delta$ is $(n,\ F)$-good if there exists $h \in \omega^F$ such that $p \restriction \gamma \Vdash_\gamma \dot{\varphi_\gamma}(p(\gamma), n) \leq h(\gamma)$ for all $\gamma \in F$ where $\dot{\varphi_\gamma}$ is P_γ-name for the function φ appeared in the definition of property E for $\dot{Q}_\gamma$.

Lemma 5.2 *Let $\delta \le \omega_2$. For all $n < \omega$ and $F \in [\delta]^{<\omega}$, the set $\{ p \in P_\delta \mid p$ is $(n,\ F)$-good $\}$ is $(n,\ F)$-dense open in P_δ.*

Proof of Lemma5.2 Since the property E implies the strongly ω^ω-bounding, we can prove easily by induction on $\delta \le \omega_2$. □(Lemma5.2)

By the lemma above, we may suppose only the condition that is $(n,\ F)$-good. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and p with $(n,\ F)$-good, define $h_{p,n,F} \in \omega^F$ by

(a) $p{\restriction}\gamma \Vdash_\gamma \dot{\varphi_\gamma}(p(\gamma), n) \le h_{p,n,F}(\gamma)$ for all $\gamma \in F$,

(b) if $q \le_{n,F} p$ then $h_{q,n,F}(\gamma) \le h_{p,n,F}(\gamma)$ for all $\gamma \in F$.

Lemma 5.3 *Let $\delta \le \omega_2$. There exists $\tilde{\varphi_\delta} \in \omega^{P_\delta \times \omega \times [\delta]^{<\omega}}$ such that*

(1) *for all $n < \omega$, $F \in [\delta]^{<\omega}$ and p with $(n,\ F)$-good, if $p \Vdash_\delta \dot{a} \in \mathbf{V}$ then there exist $q \le_{n,F} p$ and a finite set B such that $|B| \le \tilde{\varphi_\delta}(p, n, F)$ and $q \Vdash_\delta \dot{a} \in B$,*

(2) *for all $p, q \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$, if $q \le_{n,F} p$ then $\tilde{\varphi_\delta}(q, n, F) \le \tilde{\varphi_\delta}(p, n, F)$.*

Proof of Lemma5.3 We prove by induction on $\delta \le \omega_2$. For each $n < \omega$, $F \in [\delta]^{<\omega}$ and p with $(n,\ F)$-good, we define $\tilde{\varphi_\delta}(p, n, F)$ as follows:
Case 1 : δ is limit ordinal.

Let $\alpha = \max(F)+1$. Then $F \subset \alpha$. By induction hypothesis, there exists $\tilde{\varphi_\alpha} \in \omega^{P_\alpha \times \omega \times [\alpha]^{<\omega}}$ such that (1) and (2). So we define $\tilde{\varphi_\delta}(p, n, F)$ by $\tilde{\varphi_\alpha}(p{\restriction}\alpha, n, F)$.

We show that (1) and (2). (1): Let $p \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $p \Vdash_\delta \dot{a} \in \mathbf{V}$. Suppose $\alpha = \max(F) + 1$. Since $p{\restriction}\alpha \Vdash_\alpha$ "$\dot{b} \in \mathbf{V} \dot{f} \in P_{\alpha\delta} \dot{f} \Vdash_{\alpha\delta} \dot{a} = \dot{b}$" for some P_α-name $\dot{b}$ and $\dot{f}$, there exist $r \le_{n,F} p{\restriction}\alpha$, finite set B and $g \in P_{\alpha\delta}$ such that $|B| \le \tilde{\varphi_\alpha}(p{\restriction}\alpha, n, F) = \tilde{\varphi_\delta}(p, n, F)$ and $r \Vdash_\alpha$ "$\dot{b} \in B \wedge \dot{f} = g$". Let $q = r \cup g$. Then $q \le_{n,F} p$ and $q \Vdash_\delta \dot{a} \in B$.
(2): Let $p, q \in P_\delta$, $n < \omega$ and $F \in [\delta]^{<\omega}$ satisfy $q \le_{n,F} q$. Suppose $\alpha = \max(F) + 1$. Then since $q{\restriction}\alpha \le_{n,F} p{\restriction}\alpha$,

$$\begin{aligned} \tilde{\varphi_\delta}(q, n, F) &= \tilde{\varphi_\alpha}(q{\restriction}\alpha, n, F) \\ &\le \tilde{\varphi_\alpha}(p{\restriction}\alpha, n, F) \\ &= \tilde{\varphi_\delta}(p, n, F) \end{aligned}$$

Case 2 : $\delta = \gamma + 1$.
In the case of $F \subset \gamma$, we define in the same way as the case of that δ is limit ordinal.
Suppose $\gamma \in F$.

By induction hypothesis, there exists $\tilde{\varphi_\gamma}$ such that for all $p' \in P_\gamma$, $n' < \omega$ and $F' \in [\gamma]^{<\omega}$, if $p{\restriction}\gamma \Vdash_\gamma \dot{a} \in \mathbf{V}$, there exist $r \le_{n,F\cap\gamma} p'$ and B such that $|B| \le \tilde{\varphi_\gamma}(p{\restriction}\gamma, n, F\cap\gamma)$ and $r \Vdash_\gamma \dot{a} \in B$. So we define $\tilde{\varphi_\delta}(p, n, F)$ by $\tilde{\varphi_\gamma}(p{\restriction}\gamma, n, F \cap \gamma) \cdot h_{p,n,F}(\gamma)$.

We show that (1) and (2). (1): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and p with $(n,\ F)$-good. In tha case of $F \subset \gamma$, we can show in the same way as the case of that δ is limit ordinal.

Assume $\gamma \in F$. Also there exist P_γ-names $\dot{q}$ and $\dot{B}$ such that $p{\restriction}\gamma \Vdash_\gamma$ "$\dot{q} \leq_n p(\gamma) \wedge \dot{B} \subset \mathbf{V} \wedge \left|\dot{B}\right| \leq \dot{\varphi}_\gamma(p(\gamma), n) \wedge \dot{q} \Vdash_{\dot{Q}_\gamma} \dot{a} \in \dot{B}$". By p is $(n,\ F)$-good, $p{\restriction}\gamma$ is $(n,\ F\cap\gamma)$-good and $p{\restriction}\gamma \Vdash_\gamma \left|\dot{B}\right| \leq \dot{\varphi}_\gamma(p(\gamma), n) \leq h_{p,n,F}(\gamma)$. Let $\left\langle \dot{b}_j \mid j < h_{p,n,F}(\gamma) \right\rangle$ be a sequence of P_γ-names for a enumeration of $\dot{B}$. That is $p{\restriction}\gamma \Vdash_\gamma \left\{ \dot{b}_j \mid j < h_{p,n,F} \right\} = \dot{B} \subset \mathbf{V}$. By induction on $j < h_{p,n,F}$, we construct two sequences $\langle r_j \mid j < h_{p,n,F} \rangle$ and $\langle B_j \mid j < h_{p,n,F} \rangle$ such that (let $r_{-1} = p{\restriction}\gamma$)

(*a*) $r_j \leq_{n,F\cap\gamma} r_{j-1}$ for all $j < h_{p,n,F}(\gamma)$,
(*b*) $|B_j| \leq \tilde{\varphi}_\gamma(r_{j-1}, n, F\cap\gamma) \leq \tilde{\varphi}_\gamma(p{\restriction}\gamma, n, F\cap\gamma)$ for $j < h_{p,n,F}$,
(*c*) $r_j \Vdash_\gamma \dot{b}_j \in B_j$ for all $j < h_{p,n,F}$

Let $q = r_{h_{p,n,F}(\gamma)-1} \cup \{(\gamma,\ \dot{q})\}$ and $B = \bigcup\{ B_j \mid j < h_{p,n,F}(\gamma) \}$. Clearly $q \Vdash_\delta \dot{a} \in B$ and

$$\begin{aligned} |B| &\leq \sum_{j<h_{p,n,F}} |B_j| \\ &\leq \sum_{j<h_{p,n,F}} \tilde{\varphi}_\gamma(p{\restriction}\gamma, n, F\cap\gamma) \\ &= \tilde{\varphi}_\gamma(p{\restriction}\gamma, n, F\cap\gamma)\cdot h_{p,n,F}(\gamma) \\ &= \tilde{\varphi}_\delta(p, n, F). \end{aligned}$$

(2): Let $n < \omega$, $F \in [\delta]^{<\omega}$ and p, q satisfy $(n,\ F)$-good and $q \leq_{n,F} p$. In the case of $F \subset \gamma$, we can show in the same way as the case of that δ is limit ordinal. Suppose $\gamma \in F$. Then

$$\begin{aligned} \tilde{\varphi}_\delta(q, n, F) &= \tilde{\varphi}_\gamma(q{\restriction}\gamma, n, F\cap\gamma)\cdot h_{q,n,F}(\gamma) \\ &\leq \tilde{\varphi}_\gamma(p{\restriction}\gamma, n, F\cap\gamma)\cdot h_{p,n,F}(\gamma) \\ &= \tilde{\varphi}_\delta(p, n, F). \end{aligned}$$

□(Lemma5.3)

Theorem 5.1 $\Vdash_{P_{\omega_2}} 2^\omega \subset \bigcup\{ \Upsilon(\tau) \mid \tau \in \mathcal{T}(g)\cap\mathbf{V}\}$, *for all strictly increasing function* $g \in \omega^\omega$. *therefore*, $\Vdash_{P_{\omega_2}} 2^\omega \subset \bigcup\{ \Upsilon(\tau) \mid \tau \in \mathcal{S}(f)\cap\mathbf{V}\}$.

Proof of Theorem5.1 By Lemma5.3, we can show in the same way as Lemma5.1.

□(Theorem5.1)

Corollary 5.4 (CH) $\Vdash_{P_{\omega_2}}$ "$\mathsf{cov}(\mathcal{I}_f) = \mathsf{cov}(\mathcal{J}_g) = \omega_1$" *for all strictly increasing function* $g \in \omega^\omega$.

6 The diagram of cardinal coefficients of $\mathcal{I}_f$

In this section, we give the resuls for the cardinal coefficients of ideal $\mathcal{I}_f$ of the forcing notions that we studied. Let κ be a uncountable regular cardinal. We express the parts which we do not yet understand in '?'.

forcing notions	add	cov	non	cof	$\mathfrak{b}$	$\mathfrak{d}$	$\mathfrak{c}$
$\mathbb{O}(f)_\kappa$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	κ
$\mathbb{P}_\kappa$	?	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	?	?	κ
$\mathbb{C}_\kappa$	ω_1	ω_1	$\mathfrak{c}$	$\mathfrak{c}$	ω_1	$\mathfrak{c}$	κ
$\mathbb{EE}_{\omega_2}$	ω_1	ω_1	$\mathfrak{c}$	$\mathfrak{c}$	ω_1	ω_1	ω_2
$\mathbb{S}_{\omega_2}$	ω_1	ω_1	ω_1	ω_1	ω_1	ω_1	ω_2

$\mathbb{O}(f)_\kappa$: the κ-stage finite support iteration of the forcing notion $\mathbb{O}(f)$ introduced by T.Yorioka,
$\mathbb{P}_\kappa$: the κ-stage finite support iteration of the forcing notion $\mathbb{P}(d)$ by bookkeeping method,
$\mathbb{C}_\kappa$: the Cohen forcing notion which adds κ many Cohen reals,
$\mathbb{EE}_{\omega_2}$: the ω_2-stage countable support iteration of the infinitely equal forcing notion, (the infinitely equal forcing notion has property E),
$\mathbb{S}_{\omega_2}$: the ω_2-stage countable support iteration of the Sacks forcing notion.

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