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Reconstruction of a Current from Tracer Observation and the Inverse Scattering

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Abstract

An inverse problem to reconstruct an oceanic current from the data of an observable property called tracer is disscussed in connection with the inverse scattering for Schödinger operators with energy dependent potentials. It shows that the inverse scattering problems for the time-independent Klein-Gordon equation, for the radial time-independent Schödinger equation, and the inverse spectral problem in an advection-diffusion can be treated in a unified manner.

1 Introduction

The steady-state conservation of the concentration ϕ of observable properties in the ocean called tracers, such as salinity, potential temperature, oxygen and so on, can be written by the time-independent advection-diffusion equation

$$\boldsymbol{v}\cdot\nabla\phi=\nabla\cdot(\kappa\nabla\phi), \tag{1}$$

where $\boldsymbol{v}$ is the velocity field, assumed to be nondivergent: $\nabla\cdot\boldsymbol{v}=0$, and κ is the diffusivity of the tracer. In a two-dimensional, simple model where the vertical velocity can be ignored and the diffusivity can be considered to be constant, the inverse problem to recover the velocity $\boldsymbol{v}=(0,v(x))$ from the data of tracer observed at an appropriate depth level has been studied by the author [5]. It established a recovery procedure of the velocity $v(x)$ from the observed data, and as a consequence, proved the uniqueness of the velocity realizing the prescribed data. It also suggested that the inverse problem is closely related with the inverse scattering for Schödinger operators with energy dependent potentials. The purpose of the present note is to make clear how our *inverse spectral problem* is related to the inverse scattering problem, with emphasis on the role of an integral version of a transformation kernel.

2 Energy Dependent Scattering Theory

To start from scratch, we begin with a review of the inverse scattering problem for some Schrödinger equations with energy-dependent potentials. As a generalization of the time-independent Klein-Gordon equation, Cornille [1] considered the equation

$$f'' + [k^2 - (U(x) + 2E\,Q(x))]f = 0 \qquad (0 \le x < \infty,\ '' = \tfrac{d^2}{dx^2}), \tag{2}$$

where k is the wave number, $E = (k^2 + m^2)^{\frac{1}{2}}$ is the energy, Q and U are static potentials for a particle (anti-particle) of mass m. He derived a Marchenko equation for (2) and suggested the validity of the Marchenko method for the relativistic inverse problem.

For $m = 0$ corresponding to a particle of zero mass such as the photon, equation (2) becomes

$$f'' + [k^2 - (U(x) + 2k\,Q(x))]f = 0 \tag{3}$$

and, as a special case $U(x) = -Q(x)^2$, it gives

$$f'' + (k - Q)^2 f = 0, \tag{4}$$

which is the time-independent Klein-Gordon equation; it can be deduced from the quantization $p = -i\hbar\frac{d}{dx}$ ($\hbar$ denotes the Plank constant) for the momentum p in the relativistic energy equality

$$(E - Q)^2 = (pc)^2 + (mc^2)^2$$

with $m = 0$, after a scale-transformation.

Jaulent-Jean [4] and Jaulent [2, 3] treated the inverse scattering problem for equation (3) and have established a procedure by which the potentials U and Q are recovered from the scattering data, under the assumption that U, Q are real-valued, differentiable functions belonging to $L^1(0, \infty)$ together with the derivatives and that there are no bound states. They derived a nonlinear differential equation for $\int_x^\infty Q(r)dr$ from the Marchenko equation and, based upon it, proved the potentials are uniquely determined from the scattering data.

3 A Simple Recovery Formula

Recently the author [6] has improved the Jaulent-Jean procedure in the following two aspects: firstly, we require no auxiliary differential equations for the unknown potentials; we can recover the potentials directly from the solution of a Marchenko equation, and secondly, we require no differentiability assumptions on potentials; we just assume

$$(1 + x)\,U(x), Q(x) \in L^1(0, \infty), \tag{5}$$

$$Q(x) \text{ is continuous and bounded.} \tag{6}$$

We pick out some ingredients of [6]. Under assumption (5), for each k in $\operatorname{Im} k \leq 0$, equation (3) admits a unique solution $f(x,k)$ with the asymptotics as $x \to \infty$

$$f(x,k) = e^{-ikx}[1+o(1)], \quad f'(x,k) = -ike^{-ikx}[1+o(1)]. \tag{7}$$

The solution $f(x,k)$ can be expressed as

$$f(x,k) = f(x,0)e^{-ikx} + ik\int_x^\infty K(x,t)e^{-ikt}dt \tag{8}$$

in terms of a continuous, bounded function $K(x,t)$ defined on $0 \leq x \leq t < \infty$ with the derivative $K_t(x,t)$ that belongs to $L^1(x,\infty)$ as a function of t.

The function $K(x,t)$, which is referred to as the transformation kernel, is connected with the potential Q through the formula

$$f(x,0) + K(x,x) = e^{-i\int_x^\infty Q(r)dr}, \tag{9}$$

and, under the additional assumption (6), the derivative $K(x,t)$ is connected with the potentials U, Q through the formula

$$2K_t(x,x)\, e^{i\int_x^\infty Q(r)dr} = \int_x^\infty [U(r) + Q(r)^2]dr + i\,Q(x). \tag{10}$$

The inverse problem we are concerned with is: to recover $U(x)$ and $Q(x)$ in (3) from the scattering data

$$S(k) := \frac{f(0,k)}{\overline{f(0,k)}} \tag{11}$$

on the real axis, where $\overline{f(0,k)}$ denotes the complex conjugate of $f(x,k)$. Under the assumption $f(0,0) \neq 0$, the data $S(k)$ can be expressed as

$$S(k) = C + \int_{-\infty}^\infty F(t)e^{-ikt}dt \tag{12}$$

in terms of a complex constant C with the absolute value 1 and a function $F(t) \in L^1(\boldsymbol{R})$. This is due to the Wiener-Lévy theorem. The pair $(C, F(t))$ is uniquely determined from $S(k)$, in view of the Riemann-Lebesgue theorem and the uniqueness theorem of the Fourier transform.

It can be proved that if

$$\operatorname{ind} S(k) := \arg S(k)|_{-\infty}^{\infty} = 0, \tag{13}$$

in other words, if $f(0,k)$ has no zeros in $\operatorname{Im} k < 0$, then the transformation kernel $K(x,t)$ and $F(t)$ are connected by a Marchenko equation of the following form:

$$K(x,t) + \int_x^\infty \overline{K(x,r)}\,F(r+t)dr + f(x,0)\int_x^\infty F(r+t)dr = 0. \tag{14}$$

This equation can be solved as

$$K(x,t) = f(x,0)\Delta(x,t), \tag{15}$$

where $\Delta(x,t)$ is the solution of

$$\Delta(x,t) + \int_x^\infty \overline{\Delta(x,r)}\, F(r+t)dr + \int_x^\infty F(r+t)dr = 0. \tag{16}$$

From (9), (10), (15), we readily deduce

$$2\,\frac{\Delta_t(x,x)}{1+\Delta(x,x)} = \int_x^\infty \left[U(r) + Q(r)^2\right] dr + i\,Q(x). \tag{17}$$

This gives a recovery formula of U, Q from the solution $\Delta(x,t)$ of the Marchenko equation (16) ; $Q(x)$ is determined from the imaginary part of it and, in turn, $U(x)$ is determined by taking the derivative of the real part of it.

4 Particular Cases

There are three special cases where:

(I) $Q(x) = 0$. In this case, (9) combined with (15) yields $f(x,0)(1 + \Delta(x,x)) = 1$. Therefore (17) is rewritten as

$$U(x) = -2\frac{d}{dx}K_t(x,x).$$

This is the well-known formula (see [7, p.224], note that $K_t(x,t)$ in our terminology is no other than $K(x,t)$ there) in the original Marchenko theory concerning the (nonrelativistic) inverse scattering problem. When $Q(x) \equiv 0$, the transformation kernel $K(x,t)$ and the function $F(t)$ in (12) are real-valued because the scattering data $S(k)$ has the symmetric relation: $S(-k) = \overline{S(k)}$. Hence the original Marchenko equation is deduced from (14) by differentiating it and performing an integration by parts. In this way we can reproduce the inverse scattering theory to restore the interaction potential $U(x)$ in classical quantum mechanics when there are no bound states.

(II) $U(x) = -Q(x)^2$. In this case, formula (16) reads as

$$Q(x) = -2i\,\frac{\Delta_t(x,x)}{1+\Delta(x,x)}.$$

This gives a recovery formula of the potential in the Klein-Gordon equation (4) from the solution $\Delta(x,t)$ of (16), originally from the scattering data $S(k)$, which in $k > 0$ and $k < 0$ describes the scattering of the particle and the anti-particle, respectively. With regard to (17), the Klein-Gordon equation is a specific one in the sense that the left-hand side of it is pure-imaginary.

(III) $U(x) = 0$. In this case, $Q(x)$ is determined directly from (16) as

$$Q(x) = -i\frac{d}{dx}\log(1 + K(x,x)). \tag{18}$$

Here we require neither assumptions (6) nor (13) because the former is necessary only for (10) and the latter is automatically satisfied in this case (see [5, Theorem 3.2]). The problem to reconstruct a current from tracer observation is corresponding to the inverse scattering problem in this case.

5 Reconstruction of a Current

When $\boldsymbol{v} = (0, v(x))$, $\kappa = 1$, advection-diffusion equation (1) becomes:

$$\phi_{xx} + \phi_{yy} - v(x)\phi_y = 0. \tag{19}$$

We consider this equation in the region $\Omega := [0,\infty) \times \boldsymbol{R}$ under the Dirichlet condition at the infinity : $\phi(x,y) \to 0$ as $x + |y| \to \infty$. In our modeling of the concentration of a tracer in the ocean, the coordinate x denotes the depth, y denotes the horizontal position, and $v(x)$ is the horizontal velocity at depth x. The reason for $v(x)$ being independent of y is because of the nondivergence condition $\nabla \cdot \boldsymbol{v} = 0$. Our inverse problem here is to reconstruct the velocity $v(x)$ from the concentration $a(y)$ of the tracer and the flux $b(y)$ at a fixed depth, say $x = 0$. Mathematically it is no other than to recover the coefficient $v(x)$ of the elliptic equation (19) from the boundary data $a(y) := \phi(0,y)$, $b(y) := -\phi_x(0,y)$.

The relation between this inverse problem and the inverse scattering problem for the energy dependent Schrödinger equation (3) becomes clear by taking the Fourier transform of (19) with respect to the variable y; in fact, for the Fourier transform of $\phi(x,y)$

$$\Phi(x,k) := \int_{-\infty}^{\infty} \phi(x,y)e^{-ky}dy,$$

defined on the imaginary axis $\mathrm{Im}\, k = 0$, equation (19) is reduced to

$$\Phi'' + [k^2 - kv(x)]\Phi = 0, \tag{20}$$

which is of the form (3) with $U(x) = 0$, $2Q(x) = v(x)$. In addition, the boundary condition $\phi(0,y) = a(y)$, $-\phi_x(0,y) = b(y)$ is reduced to

$$\Phi(0,k) = A(k), \ \ \Phi'(0,k) = -B(k), \tag{21}$$

where $A(k)$, $B(k)$ are the Fourier transform of $a(y)$, $b(y)$ respectively, which are functions on the imaginary axis.

At first glance the inverse scattering theory in §3 does not seem to apply to our problem because of the difference between the scattering data $S(k)$ and data (21); the former is given on the real axis, while the latter, on the imaginary axis. However, in some inverse problems containing ours, the data corresponding to (21) are also appropriate to recover potentials in the energy dependent Schrödinger operator, as will be shown in the next section.

6 Data on the Imaginary Axis

We consider the inverse problem to recover potentials in (3) from the data

$$A(k) := f(0,k), \quad B(k) = -f'(0,k) \tag{22}$$

on the imaginary axis. Here $f(x,k)$ is the solution of (3) defined in §3 by the asymptotics (7). Throughout this section we suppose that $f'(0,0) = 0$. Notice that in the case $U(x) = 0$ this condition is satisfied because $f(x,0) \equiv 1$ in this case.

For real k, the function $\overline{f(x,k)}$ is the solution of (3) because U, Q are real-valued. In view of the asymptotics (7) and that for $\overline{f(x,k)}$, the Wronskian of $f(x,k)$ and $\overline{f(x,k)}$ is identically equal to $2ik$. This leads to the identity

$$\operatorname{Im} \frac{f'(0,k)}{kf(0,k)} = -\frac{1}{|f(0,k)|^2} \tag{23}$$

for real k, provided that $f(0,0) \neq 0$.

As is seen from (8) and (9), the function $f(0,k)$ is expressed as

$$f(0,k) = e^{-i\int_0^\infty Q(r)dr} + \int_0^\infty K_t(0,t)e^{-ikt}dt$$

for each k in the lower half-plane. Moreover we obtain

$$\frac{f'(0,k)}{k} = -ie^{-i\int_0^\infty Q(r)dr} + i\int_0^\infty K_x(0,t)e^{-ikt}dt,$$

provided that $K_x(0,t) \in L^1(0,\infty)$. Accordingly, if $f(0,k)$ has no zeros in the lower half-plane then, by means of the Paley-Wiener theorem and the convolution theorem, we find that there exists a function $F_1(t) \in L^1(0,\infty)$ such that

$$\frac{f'(0,k)}{kf(0,k)} = -i - \int_0^\infty F_1(t)e^{-ikt}dt. \tag{24}$$

Since the right-hand side of this equality is holomorphic in the lower half-plane, the function $F_1(t)$ is uniquely determined from the function in the left-hand side, i.e.,

$$-\frac{B(k)}{kA(k)}$$

on the lower part of the imaginary axis if the set of k where $A(k)$ does not vanish has accumulation points there.

By (23), (24) and the Wiener-Lévy theorem, we find that, for real k, the function $|f(0,k)|^2$ is expressed as

$$|f(0,k)|^2 = 1 + \int_{-\infty}^{\infty} G(t)e^{-ikt}dt$$

in terms of $G(t) \in L^1(\boldsymbol{R})$. By the Wiener-Hopf factorization, this is rewritten as

$$\frac{f(0,k)}{1+\int_0^\infty G_-(t)e^{-ikt}dt} = \frac{1+\int_{-\infty}^0 G_+(t)e^{-ikt}dt}{\overline{f(0,k)}}$$

in a unique manner. The function on the left-hand (right-hand) side of this identity is holomorphic and bounded in the lower (the upper) half plane, and is extended continuously up to the imaginary axis. Hence, by Morera's theorem, it has a continuation to the whole complex plane as an entire, bounded function. In view of Liouville's theorem, this implies that the function must be a constant, which we denote by C_0. Since $A(0) = f(0,0) = C_0\left(1+\int_0^\infty G_-(t)dt\right)$, the constant C_0 is uniquely determined from $G_-(t)$ and $A(0)$.

From these expressions, we arrive at

$$\frac{f(0,k)}{\overline{f(0,k)}} = C_0^2 \frac{1+\int_{-\infty}^0 G_-(t)e^{\lambda t}dt}{1+\int_0^\infty G_+(t)e^{\lambda t}dt}. \tag{25}$$

Hence the scattering data $S(k)$ in (11) is uniquely determined from the data $A(k), B(k)$ in (22), provided that the set of k where $A(k)$ does not vanish has non-zero accumulation points on the imaginary axis.

In this way the problem to recover potentials in (3) from the data (22) is reduced to the inverse scattering problem discussed in §§2–4, under the assumption $f'(0,0) = 0$ and $K_x(x,\cdot) \in L^1(x,\infty)$. These conditions are satisfied automatically in the case of $U(x) = 0$.

Besides the quantum scattering theory, equation (3) is expected to be applicable for some inverse spectral problems. Our problem to reconstruct an oceanic flow from the data of a tracer provides such an inverse spectral problem.

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