Second order Nonlinear Difference Equations
whose Eigenvalues are 1

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1 Introduction

At first we consider the following second order nonlinear difference equation,

\[
\begin{cases}
  u(t + 1) = U(u(t), v(t)), \\
  v(t + 1) = V(u(t), v(t)),
\end{cases}
\]

(1.1)

where \( U(u, v) \) and \( V(u, v) \) are entire functions for \( u \) and \( v \). We suppose that the equation (1.1) admits an equilibrium point \( (u^*, v^*) = (0, 0) \). Furthermore we suppose that \( U \) and \( V \) are written in the following form

\[
\begin{pmatrix}
  u(t + 1) \\
  v(t + 1)
\end{pmatrix} = M
\begin{pmatrix}
  u(t) \\
  v(t)
\end{pmatrix} + \begin{pmatrix}
  U_1(u(t), v(t)) \\
  V_1(u(t), v(t))
\end{pmatrix},
\]

where \( U_1(u, v) \) and \( V_1(u, v) \) are higher order terms of \( u \) and \( v \). Let \( \lambda_1, \lambda_2 \) be characteristic values of matrix \( M \). For some regular matrix \( P \) which decided by \( M \), put

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix} = P
\begin{pmatrix}
  x \\
  y
\end{pmatrix},
\]

then we can transform the system (1.1) into the following simultaneous system of first order difference equations (1.2):

\[
\begin{cases}
  x(t + 1) = X(x(t), y(t)), \\
  y(t + 1) = Y(x(t), y(t)),
\end{cases}
\]

(1.2)

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where $X(x, y)$ and $Y(x, y)$ are supposed to be holomorphic and expanded in a neighborhood of $(0, 0)$ in the form

$$
\begin{aligned}
X(x, y) &= \lambda_1 x + \sum_{i+j \geq 2} c_{ij} x^i y^j = \lambda_1 x + X_1(x, y), \\
Y(x, y) &= \lambda_2 y + \sum_{i+j \geq 2} d_{ij} x^i y^j = \lambda_2 y + Y_1(x, y),
\end{aligned}
$$

(1.3)

or

$$
\begin{aligned}
X(x, y) &= \lambda x + y + \sum_{i+j \geq 2} c_{ij}' x^i y^j = \lambda x + X_1'(x, y), \\
Y(x, y) &= \sum_{i+j \geq 2} d_{ij}' x^i y^j = \lambda y + Y_1'(x, y),
\end{aligned}
$$

(1.4)

where $\lambda = \lambda_1 = \lambda_2$.

In this note we consider analytic solutions of difference system (1.2), making use of Theorems in [1] and [4]. We will seek an analytic solution of (1.2) under the conditions $\lambda_1 = \lambda_2 = 1$ and definition (1.3). Further we suppose that

$$
\begin{aligned}
X(x, y) &= x + \sum_{i+j \geq 2} c_{ij} x^i y^j = x + X_1(x, y), \\
Y(x, y) &= y + \sum_{i+j \geq 2} d_{ij} x^i y^j = y + Y_1(x, y),
\end{aligned}
$$

(1.5)

where $X_1(x, y) \not\equiv 0$ or $Y_1(x, y) \not\equiv 0$. For the case $|\lambda_1| \neq 1$ or $|\lambda_2| \neq 1$, we obtained analytic general solutions of (1.2) in [5] and [6]. For a long time we could not treat the equation (1.2) under the condition $|\lambda_1| = |\lambda_2| = 1$, because it is difficult to have an analytic solution of the equation (1.2). For analytic solutions of a nonlinear first order difference equations, Kimura [1] and Yanagihara [7] studied the cases in which the absolute value of the eigenvalue equal to 1.

Next we consider a functional equation

$$
\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)),
$$

(1.6)

where $X(x, y)$ and $Y(x, y)$ are holomorphic functions in $|x| < \delta_1$, $|y| < \delta_1$. We assume that $X(x, y)$ and $Y(x, y)$ are expanded there as in (1.5).

As far as $\frac{dx}{dt} \neq 0$, an existence of solutions of (1.2) is equivalent to an existence of solution $\Psi$ of (1.6). Furthermore we can reduce (1.2) to the following first order difference equation

$$
x(t + 1) = X(x(t), \Psi(x(t))),
$$

(1.7)

Hereafter we consider $t$ to be a complex variable, and concentrate on the difference system (1.2). Our aim in this paper is to show the following Theorem 1.

**Theorem 1** Suppose $X(x, y)$ and $Y(x, y)$ are expanded in the forms (1.5) such that $X_1(x, y) \neq 0$ or $Y_1(x, y) \neq 0$. 
(1) We define domains $D_1(\kappa_0, R_0)$ by
\[ D_1(\kappa_0, R_0) = \{ t : |t| > R_0, |\arg[t]| < \kappa_0 \}, \] (1.8)
where $\kappa_0$ is any constant such that $0 < \kappa_0 \leq \frac{\pi}{4}$ and $R_0$ is sufficiently large number which may depend on $X$ and $Y$. Further define
\[ D^*(\kappa, \delta) = \{ x ; |\arg[x]| < \kappa, 0 < |x| < \delta \}, \] (1.8)
where $\delta$ is a small constant and $\kappa$ is a constant such that $\kappa = 2\kappa_0$, i.e., $0 < \kappa \leq \frac{\pi}{2}$. Suppose that $k c_{20} = d_{11} < 0$ for some $k \in \mathbb{N}$, $k \geq 2$, and $A = c_{20}$, then we have a formal solution $x(t)$ of (1.2) the following form
\[ \frac{1}{At} \left( 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)^{-1}, \] (1.10)
where $\hat{q}_{jk}$ are constants which are defined by $X$ and $Y$.
(2) Suppose $R_1 = \max(R_0, 2/(|A|\delta))$, then there is a solution $x(t)$ of (1.2) such that $x(t) \in D^*(\kappa, \delta)$ for $t \in D_1(\kappa_0, R_1)$, which the solution satisfying the following conditions:
(i) $x(t)$ is holomorphic in $D_1(\kappa_0, R_1)$.
(ii) $x(t)$ is expressible in the form
\[ x(t) = -\frac{1}{At} \left( 1 + b(t, \frac{\log t}{t}) \right)^{-1}, \] (1.11)
where $b(t, \eta)$ is holomorphic for $t \in D_1(\kappa_0, R_1)$, $|\eta| < r$, and in the expansion $b(t, \eta) \sim \sum_{k=1}^\infty b_k(t) \eta^k$, $b_k(t)$ is asymptotically develop-able into $b_k(t) \sim \sum_{j+k \geq 1} b_{jk} t^{-j}$, as $t \to \infty$ through $D_1(\kappa_0, R_1)$, where $b_{jk}$ are constants which are defined by $X$ and $Y$.

2 Proof of Theorem 1

In [1], Kimura considered the following first order difference equation
\[ w(t + \lambda) = F(w(t)), \] (D1)
where $F$ is represented in a neighborhood of $\infty$ by a Laurent series
\[ F(z) = z \left( 1 + \sum_{j=1}^\infty b_j z^{-j} \right), \quad b_1 = \lambda \neq 0. \] (2.1)
He defined the following domains
\[ D(\epsilon, R) = \{ t : |t| > R, |\arg[t] - \theta| < \frac{\pi}{2} - \epsilon, \text{or} \quad \text{Im}(e^{i(\theta-\epsilon)}t) > R, \] \[ \text{or} \quad \text{Im}(e^{i(\theta+\epsilon)}t) < -R \}, \] (2.2)
\[
D(\epsilon, R) = \{ t : |t| > R, |\arg[t] - \theta - \pi| < \frac{\pi}{2} - \epsilon \text{ or } \text{Im}(e^{-i(\theta + \pi - \epsilon)}t) > R \\
\text{or } \text{Im}(e^{-i(\theta + \pi + \epsilon)}t) < -R \},
\]

where \( \epsilon \) is an arbitrarily small positive number and \( R \) is a sufficiently large number which may depend on \( \epsilon \) and \( F \), \( \theta = \arg \lambda \), (in this present paper, we consider the case \( \lambda = 1 \) in (D1)). He proved the following theorems A and B.

**Theorem A.** Equation (D1) admits a formal solution of the form

\[
t \left( 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)
\]

containing an arbitrary constant, where \( \hat{q}_{jk} \) are constants defined by \( F \).

**Theorem B.** Given a formal solution of the form (2.4) of (D1), there exists a unique solution \( w(t) \) satisfying the following conditions:

(i) \( w(t) \) is holomorphic in \( D(\epsilon, R) \),

(ii) \( w(t) \) is expressible in the form

\[
w(t) = t \left( 1 + b(t, \log t) \right),
\]

where the domain \( D(\epsilon, R) \) is defined by (2.2) and \( b(t, \eta) \) is holomorphic for \( t \in D(\epsilon, R) \), \( |\eta| < 1/R \), and in the expansion \( b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t) \eta^k \), \( b_k(t) \) is asymptotically developable into \( b_k(t) \sim \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \), as \( t \to \infty \) through \( D(\epsilon, R) \), where \( \hat{q}_{jk} \) are constants which are defined by \( X \) and \( Y \).

Also there exists a unique solution \( \hat{w} \) which is holomorphic in \( \hat{D}(\epsilon, R) \) and satisfies a condition analogous to (ii), where the domain \( \hat{D}(\epsilon, R) \) is defined by (2.2).

In Theorem A and B, he defined the function \( F \) as in (2.1). But in our method, we can not have a Laurent series of the function \( F \). Hence we derive following Propositions.

In the following, \( A \) denotes the constant \( A = c_{20} \) in Theorem 1, where \( c_{20} \) is the coefficient in (1.5). 

**Proposition 2.** Suppose \( \tilde{F}(t) \) is holomorphic and expanded asymptotically in \( \{ t ; -1/(At) \in D^*(\kappa, \delta), A < 0 \} \) as

\[
\tilde{F}(t) \sim t \left( 1 + \sum_{j=1}^{\infty} b_j t^{-j} \right), \quad b_1 = \lambda \neq 0,
\]

where \( D^*(\kappa, \delta) \) is defined in (1.9). Then the equation

\[
\psi(\tilde{F}(t)) = \psi(t) + \lambda
\]

(2.6)
has a formal solution
\[ \psi(t) = t \left( 1 + \sum_{j=1}^{\infty} q_j t^{-j} + q \frac{\log t}{t} \right), \]  
(2.7)

where \( q_1 \) can be arbitrarily prescribed while other coefficients are uniquely determined by \( b_j \), \( (j = 1, 2, \cdots) \), independently of \( q_1 \).

**Proposition 3.** The equation (2.6) has a solution \( w = \psi(t) \), which is holomorphic in \( \{ t; -1/(At) \in D^*(\kappa/2, \delta/2), A < 0 \} \) and has asymptotic expansion (2.7) there.

These Propositions are proved as in [1] pp. 212–222. Since \( A = c_{20} < 0 \) and \( \kappa_0 = \kappa/2 \), we see that \( x = -1/(At) \in D^*(\kappa/2, \delta/2) \) equivalent to \( t \in D_1(\kappa/2, 2/(|A|\delta)) = D_1(\kappa_0, 2/(|A|\delta)), \) where \( D_1(\kappa_0, R_0) \) is defined in (1.8). Further, as in [1] pp.206 and pp.228–232, we have following Proposition 4.

**Proposition 4.** Suppose a function \( \phi \) is the inverse of \( \psi \) such that \( w = \psi^{-1}(t) = \phi(t) \). Then we have \( \phi \circ \psi(w) = w, \psi \circ \phi(t) = t \), furthermore \( \phi \) is holomorphic and asymptotically expanded in \( \{ t; t \in D_1(\kappa_0, 2/(|A|\delta)) \} \) as
\[ \phi(t) \sim t \left\{ 1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right\}. \]  
(2.8)

This function \( \phi(t) \) is a solution of difference equation of \((D1)\).

In [4], we proved the following theorem C.

**Theorem C.** Suppose \( X(x,y) \) and \( Y(x,y) \) are defined in (1.5). Then
(1) if \( kc_{20} \neq d_{11} \) for any \( k \in \mathbb{N}, k \geq 2 \), then the formal solution \( \Psi(x) \) of (1.6) of the following form
\[ \Psi(x) = \sum_{m=1}^{\infty} a_m x^m, \]  
(2.9)
is identical to 0, i.e., \( a_1 = a_2 = \cdots = 0 \).
(2) if \( kc_{20} = d_{11} \) for some \( k \in \mathbb{N}, k \geq 2 \), then we have a formal solution \( \Psi(x) \) of (1.6) such the following form
\[ \Psi(x) = \sum_{m=k}^{\infty} a_m x^m, \]  
(2.10)
i.e., \( a_1 = a_2 = \cdots = a_{k-1} = 0 \).
(3) suppose
\[ kc_{20} = d_{11} < 0 \]  
for some \( k \in \mathbb{N}, k \geq 2 \).  
(2.11)
For any $\kappa$, $0 < \kappa \leq \frac{\pi}{2}$ and small $\delta$, we define the following domain $D^*(\kappa, \delta)$,

$$D^*(\kappa, \delta) = \{x ; |\arg[x]| < \kappa, 0 < |x| < \delta\}.$$  \hspace{1cm} (1.10)

there is a constant $\delta > 0$ and a solution $\Psi(x)$ of (1.6), which is holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that

$$\Psi(x) \sim \sum_{j=k}^{\infty} a_j x^j.$$  \hspace{1cm} (2.12)

**Proof of Theorem 1.** We prove (1) of Theorem 1. We assume that $kc_2 = q_1 < 0$ for some $k \in \mathbb{N}$, we suppose that $R_0 > R$ and $\kappa_0 < \frac{\pi}{4} - \epsilon$. Since $\theta = \arg[\lambda] = \arg[1] = 0$, we have

$$D_1(\kappa_0, R_0) \subset D(\epsilon, R).$$  \hspace{1cm} (2.13)

From Theorem C, for a $x \in D^*(\kappa, \delta)$ we have a solution $\Psi(x)$ of (1.6) which is holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that

$$\Psi(x) \sim \sum_{j=k}^{\infty} a_j x^j.$$  \hspace{1cm} (2.12)

On the other hand putting $A = c_{20}$ and $w(t) = -\frac{1}{Ax(t)}$ in (1.7), then we have

$$w(t + 1) = -\frac{1}{AX\left(-\frac{1}{Aw(t)}, \Psi\left(-\frac{1}{Aw(t)}\right)\right)}.$$  \hspace{1cm} (2.14)

If we can have $-\frac{1}{Aw} = x \in D^*(\kappa, \delta)$, then making use of Theorem C, we have a solution $\Psi(x)$ of (1.6) such that $\Psi(x) = \Psi\left(-\frac{1}{Aw}\right) \sim \sum_{m=k}^{\infty} a_m \left(-\frac{1}{Aw}\right)^m$, $(k \geq 2)$.

Further from (1.5), we have

$$\frac{1}{AX(x, \Psi(x))} \sim w \left[1 + c_{20} \frac{1}{A} w^{-1} + \sum_{k \geq 2} \tilde{c}_k (w)^{-k}\right],$$  \hspace{1cm} (2.15)

where $\tilde{c}_k$ are defined by $c_i$ and $a_k$ $(i + j \geq 2, i \geq 1, k \geq 2)$. From (2.15) and definition of $A$, we can write (2.15) into the following form (2.16),

$$w(t + 1) = \tilde{F}(w(t)) \sim w(t) \left\{1 + w(t)^{-1} + \sum_{k \geq 2} \tilde{c}_k (w(t))^{-k}\right\}.$$  \hspace{1cm} (2.16)

On the other hand, putting $\lambda = 1$ and $m = 1$ in (2.1), i.e. $\theta = 0$, then making use of the Theorem A, we have the following first order difference equation (D1, $\lambda = 1$)

$$w(t + 1) = F(w(t)) = w(t) \left(1 + w(t)^{-1} + \sum_{j=2}^{\infty} b_j w(t)^{-j}\right),$$  \hspace{1cm} (D1, $\lambda = 1$)
admits a formal solution of the form 

\[ t \left( 1 + \sum_{j+k \geq 1} q_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)^{-1}. \]

Similarly for the first order difference equation (2.16), making use of Proposition 2, we have a formal solution (2.17) of it such that,

\[ w(t) = t \left( 1 + \sum_{j+k \geq 1} b_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right), \quad (2.17) \]

where \( b_{jk} \) are defined by \( \tilde{F} \) in (2.16).

From \( x(t) = \frac{1}{ Aw(t)} \), we have a formal solution of (1.2) such that,

\[ x(t) = -\frac{1}{At} \left( 1 + \sum_{j+k \geq 1} b_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right)^{-1} \quad (2.18) \]

Conversely, if we have a formal function \( x(t) \) such that in (2.18) exist in the domain \( D^*(\kappa, \delta) \), then we can prove that the formal function (2.18) is a formal solution of (1.2), as \( t \to \infty \) through \( D_1(\kappa_0, R_0) \). At first we take a small \( \delta > 0 \). For sufficiently large \( R \), since \( R_0 > R \), we can have

\[ |x(t)| = \left| \frac{1}{At} \left| 1 + \sum_{j+k \geq 1} b_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right|^{-1} \right| < \frac{1}{|A|R} (1 + 1) < \delta. \quad (2.19) \]

for \( t \in D_1(\kappa_0, R_0) \). Since \( A = c_{20} < 0 \), if we take sufficiently large \( R_0 \), then we have

\[ |\arg x(t)| = \left| \frac{1}{At} \left| 1 + \sum_{j+k \geq 1} b_{jk} t^{-j} \left( \frac{\log t}{t} \right)^k \right|^{-1} \right| < \frac{a}{|A|R} (1 + 1) < \delta. \]

Hence we have \(-\kappa_0 - \kappa_0 \leq \arg [x(t)] \leq \kappa_0 + \kappa_0 \). From the assumption of \( \kappa = 2\kappa_0 \), we have

\[ |\arg [x(t)]| < \kappa \leq \frac{\pi}{2} \quad (2.20) \]

From (2.19) and (2.20), we have that \( x(t) \in D^*(\kappa, \delta) \) for a some \( \kappa \), \( (0 \leq \kappa \leq \frac{\pi}{2}) \).

Hence we have \( x(t) \) is a formal solution of (1.2) and holomorphic in \( D_1(\kappa_0, R_0) \). Therefore we see that the function \( x(t) \) in the (2.18) is a formal solution of (1.2).

Next we prove (2). Suppose that \( R_1 = \max(R_0, 2/(|A|\delta)) \), making use of Proposition 4, then we have a holomorphic solution \( w(t) \) of (2.16) for \( t \in D_1(\kappa_0, R_1) \), i.e., we have a solution \( x(t) \) of (1.2) for \( t \) at there, in which satisfying following conditions:

(i) \( x(t) \) is holomorphic in \( D_1(\kappa_0, R_1) \),

(ii) \( w(t) \) is expressible in the form

\[ x(t) = \frac{1}{At} \left( 1 + b \left( t, \frac{\log t}{t} \right) \right)^{-1}, \quad (2.21) \]
where $b(t, \eta)$ is holomorphic for $t \in D_1(\kappa_0, R_1)$, $|\eta| < r$, and in the expansion

$$b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t)\eta^k; \quad b_k(t) \text{ is asymptotically develop-able into } b_k(t) \sim \sum_{j+k \geq 1}^{\infty} b_{jk}t^{-j},$$

as $t \to \infty$ though $D_1(\kappa_0, R_1)$. \[\square\]

Finally, we have a solution $u(t), v(t)$ of (1.1) by the transformation

$$
\begin{pmatrix}
  u(t) \\
  v(t)
\end{pmatrix} = P
\begin{pmatrix}
  x(t) \\
  \Psi(x(t))
\end{pmatrix}.
$$

References


