1 Introduction

We consider the following nonlinear Sturm-Liouville problem

\begin{align}
-u''(t) + f(u(t)) &= \lambda u(t), \quad t \in I := (0, 1), \\
u(t) &> 0, \quad t \in I, \\
u(0) &= u(1) = 0,
\end{align}

where \( \lambda > 0 \) is an eigenvalue parameter. We assume that \( f(u) \) satisfies the following conditions (A.1)–(A.3).

(A.1) \( f(u) \) is a function of \( C^1 \) for \( u \geq 0 \) satisfying \( f(0) = f'(0) = 0 \).

(A.2) \( g(u) := f(u)/u \) is strictly increasing for \( u \geq 0 \) (\( g(0) := 0 \)).

(A.3) \( g(u) \to \infty \) as \( u \to \infty \).

The typical examples of \( f(u) \) which satisfy (A.1)–(A.3) are

\( f(u) = u^p, \quad f(u) = u^{p+2}/(1 + u^2), \quad f(u) = u^p e^u \quad (p > 1). \)

We know from [1] that for each given \( \alpha > 0 \), there exists a unique solution \( (\lambda, u) = (\lambda(\alpha), u_\alpha) \in R_+ \times C^2(I) \) with \( \|u_\alpha\|_2 = \alpha \). Furthermore, The set \( \{(\lambda(\alpha), u_\alpha) : \alpha > 0\} \) gives all solutions and is an unbounded curve of class \( C^1 \) in \( R_+ \times L^2(I) \) emanating from \( (\pi^2, 0) \).
The purpose here is to study precisely the global structure of this bifurcation branch in $\mathbb{R}_+ \times L^2(I)$. To do this, we establish several types of precise asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to \infty$ under some additional conditions on $f$.

We know from [1] that for $t \in I$,

$$g^{-1}(\lambda - \pi^2) \sin \pi t \leq u_\lambda(t) \leq g^{-1}(\lambda).$$

In particular, put $t = \frac{1}{2}$. Then as $\lambda \to \infty$

$$g^{-1}(\lambda - \pi^2) \leq \|u_\lambda\|_\infty \leq g^{-1}(\lambda).$$

Therefore, for $\lambda \gg 1$,

$$\lambda = g(\|u_\lambda\|_\infty) + O(1).$$

For instance, let $f(u) = u^p$. Then since $g(u) = f(u)/u = u^{p-1}$, for $\lambda \gg 1$

$$\lambda = \|u_\lambda\|_\infty^{p-1} + O(1).$$

Furthermore, we know that as $\lambda \to \infty$

$$\frac{u_\lambda(t)}{g^{-1}(\lambda)} \to 1$$

uniformly on any compact set in $I$. Then we obtain

$$\alpha = \|u_\alpha\|_2 = \left( \int_I g^{-1}(\lambda)^2 dt \right)^{1/2} (1 + o(1)) = g^{-1}(\lambda)(1 + o(1)).$$

This implies that, in many cases,

$$\lambda(\alpha) = g(\alpha) + o(g(\alpha)).$$

For instance, let $f(u) = u^p$. Then for $\alpha \gg 1$,

$$\lambda(\alpha) = \alpha^{p-1} + o(\alpha^{p-1}).$$

This asymptotic formula has been improved as follows.

**Theorem 1 [6].** Let $f(u) = u^p$ ($p > 1$). Further, let an arbitrary $n \in \mathbb{N}_0$ be fixed. Then as $\alpha \to \infty$

$$\lambda(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}} C_1^{k+2} \alpha^{(k-1)(1-p)/2} + o(\alpha^{n(1-p)/2}),$$
where

\[ C_1 = (p + 3) \int \sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1} s^{p+1}} ds \]

and \( a_k(p) \) (deg \( a_k(p) \leq k + 1 \)) is the polynomial determined inductively by \( a_0, a_1, \ldots, a_{k-1} \).

For instance, we have

\[ a_0(p) = 1, \quad a_1(p) = \frac{(5-p)(9-p)}{24}, \quad a_2(p) = \frac{(3-p)(5-p)(7-p)}{24}. \]

We also obtain the information about the slope of the boundary layer of \( u_\alpha \) for \( \alpha \gg 1 \).

**Theorem 2** [6]. Let \( f(u) = u^p \) (\( p > 1 \)). Further, let an arbitrary \( n \in \mathbb{N}_0 \) be fixed. Then as \( \alpha \to \infty \)

\[
u_\alpha'(0)^2 = \nu_\alpha'(1)^2 = \frac{p-1}{p+1} \alpha^{p+1} + C_1 \alpha^{(p+3)/2} + \sum_{k=0}^{n} \frac{2 A_k(p)}{(p-1)^{k+1}} C_1^{k+2} \alpha^{2+k(1-p)/2} + o(\alpha^{2+n(1-p)/2}),
\]

where \( A_k(p) \) (deg \( A_k(p) \leq k + 1 \)) is the polynomial determined by \( a_0, a_1, \ldots, a_{k-1} \).

For instance,

\[ A_0(p) = 1, \quad A_1(p) = \frac{(9-p)(13-p)}{48}, \quad A_2(p) = \frac{(5-p)(7-p)(9-p)}{48}. \]

So it is natural to consider the following problem. Consider \( f(u) \) which satisfies \((A.1)-(A.3)\). Then is the following formula valid or not for \( \alpha \gg 1 \)?

\[ \lambda(\alpha) = g(\alpha) + B_1 g(\alpha)^{1/2} + \cdots, \]

where \( B_1 \) is a constant. To treat this problem, we assume additional conditions. Let \( f(u) = u^p h(u) \) (\( p > 1 \)). Assume that \( h(u) \) is \( C^2 \) function for \( u \geq 0 \). Besides, \( h(u) \) satisfies the following conditions \((B.1)-(B.4)\).

**B.1** As \( u \to \infty \)

\[ \frac{uh'(u)}{h(u)} \to 0. \]

Furthermore, there exists a constant \( C_0 \geq 0 \) such that as \( u \to \infty \)

\[ uh'(u) \to C_0. \]
There exist constants $C > 0$ and $\delta > 0$ such that for $u \gg 1$

$$|h'(u) + uh''(u)| \leq Cu^{-(1+\delta)}. \tag{1.14}$$

For $0 \leq a \leq 1$ and $u \gg 1$

$$\frac{h(au)}{h(u)} \leq C. \tag{1.15}$$

Furthermore, for a fixed $0 < a \leq 1$, as $u \to \infty$

$$\frac{h(au)}{h(u)} \to 1. \tag{1.16}$$

(a) $u^{p+1}|h'(u)|$ is non-decreasing for $u \geq 0$ or,

(b) $u^{p+1}|h'(u)|$ is bounded for $u \geq 0$.

The typical examples of $h$ are: (i) $h(u) \equiv 1$, (ii) $h(u) = \log(u+1)$, (iii) $h(u) = u^{2}/(1+u^{2})$.

**Theorem 3 [7].** Let $p > 1$ be fixed. Assume that $f(u) := u^{p}h(u)$ satisfies (A.1)-(A.3) and (B.1)-(B.4). Then as $\alpha \to \infty$

$$\lambda(\alpha) = \alpha^{p-1}h(\alpha) + \frac{1}{p+1}C_{0}\alpha^{p-1} + (p+3)\alpha^{(p-1)/2}\sqrt{h(\alpha)}(1+o(1)). \tag{1.17}$$

**Remark 4.** (i) For $\alpha \gg 1$, by (B.2), we see that

$$C_{0} = \alpha h'(\alpha)(1+o(1)). \tag{1.18}$$

Therefore, as $\alpha \to \infty$

$$\frac{\alpha^{p-1}C_{0}}{\alpha^{p-1}h(\alpha)} = \frac{\alpha^{p}h'(\alpha)(1+o(1))}{\alpha^{p-1}h(\alpha)} \to 0. \tag{1.19}$$

So we find that the leading term of $\lambda(\alpha)$ in Theorem 3 is $\alpha^{p-1}h(\alpha)$.

(ii) If $C_{0} \neq 0$, then the second term of $\lambda(\alpha)$ is $C_{0}\alpha^{p-1}/(p+1)$. Therefore, our conjecture (1.11) is valid if and only if $C_{0} = 0$. We note that, if $h(u) = \log(u+1)$, then $C_{0} = 1$. Further, if $h(u) = u^{2}/(1+u^{2})$, then $C_{0} = 0$.

Now we consider the case where $f(u) = u^{p}e^{u}$ ($p > 1$).

**Theorem 5 [8].** Assume that $f(u) = u^{p}e^{u}$ ($p > 1$) in (1.1). Then as $\alpha \to \infty$

$$\lambda(\alpha) = \alpha^{p-1}e^{\alpha} + \frac{\pi}{4}\alpha^{(p+1)/2}e^{\alpha/2}(1+o(1)).$$
2 Sketch of the proof of Theorem 3

We begin with notations and the fundamental properties of $\lambda(\alpha)$ and $u_{\alpha}$. Let $F(u) := \int_{0}^{u} f(s)ds$. Let $\| \cdot \|_{q}$ ($1 \leq q \leq \infty$) denote the usual $L^{q}$-norm. $C$ denotes various positive constants independent of $\alpha \gg 1$. It is known by [1] that (1.1)–(1.3) has a unique solution $u_{\alpha}$ for a given $\alpha > 0$ and the mapping $\alpha \mapsto u_{\alpha} \in C^{2}(I)$ is $C^{1}$ for $\alpha > 0$. By (1.4) and (1.5), for $\alpha \gg 1$

(2.1) $\lambda(\alpha) = \alpha^{p-1}h(\alpha) + o(\alpha^{p-1}h(\alpha))$, 
(2.2) $u_{\alpha}(t) = \|u_{\alpha}\|_{\infty}(1 + o(1)) = \alpha(1 + o(1)), \; t \in I$.

We put

(2.3) $\lambda_{1}(\alpha) := \lambda(\alpha) - \alpha^{p-1}h(\alpha)$,
(2.4) $\gamma(\alpha) := \|u_{\alpha}'\|_{2}^{2} + 2 \int_{I} F(u_{\alpha}(t)) dt$.

To show Theorem 3, we find $\lambda_{1}(\alpha)$ when $\alpha \gg 1$. To do this, we define the second term $\gamma_{1}(\alpha)$ of $\gamma(\alpha)$, which plays important roles, as follows.

(2.5) $\gamma_{1}(\alpha) := \gamma(\alpha) - \frac{2}{p+1} \alpha^{p+1}h(\alpha)$.

The rough idea of the proof is as follows.

(i) We obtain three estimates in Lemmas 2.1, 2.3 and 2.4.
(ii) We establish the relationship between $\lambda_{1}(\alpha)$ and $\gamma_{1}(\alpha)$ in Lemma 2.2.
(iii) We derive the first order differential equation for $\gamma_{1}(\alpha)$ by using (i) and (ii). Then by solving it, we obtain the asymptotic formula for $\lambda_{1}(\alpha)$.

Lemma 2.1. $\|u_{\alpha}'\|_{2}^{2} = 2C_{1}(1 + o(1))\alpha^{(p+3)/2}\sqrt{h(\alpha)}$ for $\alpha \gg 1$.

Lemma 2.2. For $\alpha > 0$

(2.6) $\frac{d\gamma_{1}(\alpha)}{d\alpha} = 2\alpha\lambda_{1}(\alpha) - \frac{2}{p+1}\alpha^{p+1}h'\alpha)$.

Lemma 2.3. For $\alpha \gg 1$

$$\int_{0}^{\|u_{\alpha}\|_{\infty}} s^{p+1}h'(s) ds = \int_{I} \left( \int_{0}^{u_{\alpha}(t)} s^{p+1}h'(s) ds \right) dt + o\left( \alpha^{(p+3)/2}\sqrt{h(\alpha)} \right).$$
Lemma 2.4. For $\alpha \gg 1$

\begin{equation}
\int_0^{||u_{\alpha}||_{\infty}} s^{p+1} h'(s) \, ds = \int_0^\alpha s^{p+1} h'(s) \, ds + o\left(\alpha^{(p+3)/2} \sqrt{h(\alpha)}\right).
\end{equation}

Proof of Theorem 3. By simple calculation, we have

\[
\frac{2}{p+1} \lambda(\alpha) \alpha^2 - \gamma(\alpha) = -\frac{p-1}{p+1} ||u_{\alpha}'||_2^2 + \frac{2}{p+1} \int_0^{u_{\alpha}(t)} \left(\int_0^{u_{\alpha}(s)} s^{p+1} h'(s) \, ds\right) \, dt.
\]

By this, Lemmas 2.1, 2.3 and 2.4,

\[
\frac{2}{p+1} \lambda_{1}(\alpha) \alpha^2 - \gamma_{1}(\alpha) = -\frac{2(p-1)}{p+1} C_1 \alpha^{(p+3)/2} \sqrt{h(\alpha)}(1 + o(1)) + \frac{2}{p+1} \int_0^{\alpha} s^{p+1} h'(s) \, ds.
\]

By integration by parts,

\[
\int_0^{\alpha} s^{p+1} h'(s) \, ds = \frac{1}{p+1} \alpha^{p+2} h'(\alpha) - \frac{1}{p+1} R(\alpha),
\]

where

\begin{equation}
R(\alpha) := \int_0^{\alpha} s^{p+1} (h'(s) + sh'(s)) \, ds.
\end{equation}

By this and Lemma 2.2,

\begin{equation}
\frac{1}{p+1} \alpha \gamma_{1}'(\alpha) - \gamma_{1}(\alpha) = -\frac{2(p-1)}{p+1} C_1 \alpha^{(p+3)/2} \sqrt{h(\alpha)}(1 + o(1)) - \frac{2}{(p+1)^2} R(\alpha).
\end{equation}

Now we put $\gamma_{1}(\alpha) = \eta(\alpha) \alpha^{p+1}$. Then for $\alpha \gg 1$, we obtain

\begin{equation}
\eta'(\alpha) = -2(p-1) C_1 \alpha^{-(p+1)/2} \sqrt{h(\alpha)}(1 + o(1)) - \frac{2}{p+1} R(\alpha) \alpha^{-p-2} := \eta_1'(\alpha) + \eta_2'(\alpha),
\end{equation}

where

\begin{align}
\eta_1(\alpha) &= (1 + o(1)) \int_\alpha^\infty 2(p-1) C_1 s^{-(p+1)/2} \sqrt{h(s)} \, ds, \\
\eta_2(\alpha) &= \frac{2}{p+1} \int_\alpha^\infty R(s) s^{-(p+2)} \, ds.
\end{align}
Then it is easy to show that for $\alpha \gg 1$

(2.13) \[ \eta_1(\alpha)\alpha^{p+1} = 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)}(1 + o(1)). \]

We next calculate $\eta_2(\alpha)$. By (B.2), we have

(2.14) \[ |R(\alpha)| \leq \int_0^\alpha s^{p+1}|h'(s) + sh''(s)|ds \leq C \int_0^\alpha s^{p-\delta}ds \leq C\alpha^{p+1-\delta}. \]

By this, we easily see that $\eta_2(\alpha)$ is well defined. Then by integration by parts and simple calculation, we have

(2.15) \[
\eta_2(\alpha) = \frac{2}{p+1} \int_\alpha^\infty R(s)s^{-(p+2)}ds = \frac{2}{p+1} \left[ -\frac{1}{p+1} s^{-(p+1)}R(s) \right]_\alpha^\infty + \frac{2}{(p+1)^2} \int_\alpha^\infty (h'(s) + sh''(s))ds
\]
\[
= \frac{2}{(p+1)^2} R(\alpha)\alpha^{-(p+1)} + \frac{2}{(p+1)^2} \int_\alpha^\infty (h'(s) + sh''(s))ds
\]
\[
= \frac{2}{(p+1)^2} R(\alpha)\alpha^{-(p+1)} + \frac{2}{(p+1)^2} (C_0 - \alpha h'(\alpha)).
\]

Therefore,

(2.16) \[
\gamma_1(\alpha) = (\eta_1(\alpha) + \eta_2(\alpha))\alpha^{p+1} = 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)}(1 + o(1)) + \frac{2}{(p+1)^2} (R(\alpha) + C_0\alpha^{p+1} - \alpha^{p+2}h'(\alpha)).
\]

By this and Lemma 2.1, we obtain

(2.17) \[
\frac{2}{p+1} \lambda_1(\alpha) = \gamma_1(\alpha) - \frac{p-1}{p+1} \|u'_\alpha\|_2^2 + \frac{2}{p+1} \int_0^\alpha s^{p+1}h'(s)ds + o \left( \alpha^{(p+3)/2}\sqrt{h(\alpha)} \right)
\]
\[
= 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)} - \frac{2C_1(p-1)}{p+1} \alpha^{(p+3)/2}\sqrt{h(\alpha)}
\]
\[
+ \frac{2}{(p+1)^2} C_0\alpha^{p+1} + o \left( \alpha^{(p+3)/2}\sqrt{h(\alpha)} \right).
\]

By this, we obtain

(2.18) \[
\lambda_1(\alpha) = \frac{1}{p+1} C_0\alpha^{p-1} + (p+3)C_1\alpha^{(p-1)/2}\sqrt{h(\alpha)} + o \left( \alpha^{(p-1)/2}\sqrt{h(\alpha)} \right).
\]

Thus the proof is complete.
References


