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ファジィ微分方程式の変分方程式について

On Variational Equations of Fuzzy Differential Equations

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Abstract

We introduce a parametric representation of fuzzy numbers with bounded supports as well as we consider a normed space including the set of fuzzy numbers, where the addition in the normed space is the same one due to the extension principle but the difference and scalar products are not the same as those of the principle. In this article we treat the Fréchet differetial in a Banach space of fuzzy numbers and we dicuss variational equations of fuzzy differential equations in order to get improved results on the stability analysis of fuzzy differential equations.

1 Introduction

Let $I = [0,1]$. Denote a set of fuzzy numbers with bounded supports by $\mathcal{F}_b^{st}$ as follows (*e.g.* [15, 16]): The following definition means that a fuzzy number can be identified with a membership function.

Definition 1.1 *Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by*

$$\mathcal{F}_b^{st} = \{\mu : \mathbf{R} \to I \textit{ satisfying } \text{(i)-(iv)} \textit{ below}\}.$$

(i) *μ has a unique number $m \in \mathbf{R}$ such that $\mu(m) = 1$ (normality);*

(ii) *$supp(\mu) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ is bounded in $\mathbf{R}$ (bounded support);*

(iii) *μ is strictly fuzzy convex on $supp(\mu)$ as follows:*

(a) *if $supp(\mu) \neq \{m\}$, then*

$$\mu(\lambda\xi_1 + (1-\lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$$

for $\xi_1, \xi_2 \in supp(\mu)$ with $\xi_1 \neq \xi_2$ and $0 < \lambda < 1$;

(b) *if $supp(\mu) = \{m\}$, then $\mu(m) = 1$ and $\mu(\xi) = 0$ for $\xi \neq m$;*

(iv) *μ is upper semi-continuous on $\mathbf{R}$.*

μ is called a membership function if $\mu \in \mathcal{F}_b^{st}$. Fuzzy numbers are identified by membership functions. In what follows we denote the α−cut sets of μ by

$$\mu_\alpha = L_\alpha(\mu) = \{\xi \in \mathbf{R} : \mu(\xi) \geq \alpha\}$$

for $\alpha \in (0,1]$. By the extension principle due to Zadeh, the binary operation between fuzzy numbers is nonlinear. It does not necessarily hold that $(k_1 + k_2)\mu = k_1\mu + k_2\mu$ for a membership function $\mu \in \mathcal{F}_b^{st}$ and $k_i \in \mathbf{R}, i = 1, 2$ with $k_1 + k_2 > 0, k_1 < 0 < k_2$.

We introduce the following parametric representation of $\mu \in \mathcal{F}_b^{st}$ as

$$x_1(\alpha) = \min L_\alpha(\mu), \quad x_2(\alpha) = \max L_\alpha(\mu)$$

for $0 < \alpha \leq 1$ and

$$x_1(0) = \min supp(\mu), \quad x_2(0) = \max supp(\mu).$$

From the strict fuzzy convexity it can be seen that a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve over $\mathbf{R}^2$ and $x_1(\alpha) \leq x_2(\alpha)$ for $\alpha \in I$ (see [17].)

In Section 2 we show that the set of fuzzy numbers $\mathcal{F}_b^{st}$ construct a linear space by the Puri-Ralescue's method and consider the completion of a normed space induced by the linear space.

In Section 3 we discuss differentiation and integration of fuzzy functions. In the case of differentiation our representation of fuzzy numbers is enable to calculate addition, scalar product and difference without difficulties, but it is not easy to calculate the difference by the extension principle. Moreover we define the integral of fuzzy functions by calculating end-points of $\alpha-$cut sets.

In Section 4 we treat two ways in analyzing stability of fuzzy differential equations: One is parametric method and the other is fuzzy differential inclusions. Finally we introduce various types of results on variational equations of ordinary differential equations and we discuss the significancy of variational equations of fuzzy differential equations in Section 5.

2 Induced Normed Space of Fuzzy Numbers

Let $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be an $\mathbf{R}-$valued function. The corresponding binary operation of two fuzzy numbers $x, y \in \mathcal{F}_b^{st}$ to $g(x,y) : \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} \to \mathcal{F}_b^{st}$ is calculated by the extension principle of Zadeh. The membership function $\mu_{g(x,y)}$ of g is as follows:

$$\mu_{g(x,y)}(\xi) = \sup_{\xi = g(\xi_1, \xi_2)} \min(\mu_1(\xi_1), \mu_2(\xi_2))$$

Here $\xi, \xi_1, \xi_2 \in \mathbf{R}$ and μ_1, μ_2 are membership functions of x, y, respectively. From the extension principle, it follows that, in case where $g(x,y) = x + y$,

$$\begin{aligned}
&\mu_{x+y}(\xi) \\
&= \max_{\xi = \xi_1 + \xi_2} \min_{i=1,2} (\mu_i(\xi_i)) \\
&= \max\{\alpha \in I : \xi = \xi_1 + \xi_2,\ \xi_i \in L_\alpha(\mu_i), i = 1,2\} \\
&= \max\{\alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}.
\end{aligned}$$

Thus we get

$$x + y = (x_1 + y_1, x_2 + y_2).$$

In the similar way we have

$$x - y = (x_1 - y_2, x_2 - y_1).$$

Denote a metric by

$$d(x,y) = \sup_{\alpha \in I} \max(|x_1(\alpha) - y_1(\alpha)|, |x_2(\alpha) - y_2(\alpha)|)$$

for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_b^{st}$.

Theorem 2.1 *$\mathcal{F}_b^{st}$ is a complete metric space in $C(I)^2$.*

Proof See [17].

According to the extension principle of Zadeh, for respective membership functions μ_x, μ_y of $x, y \in \mathcal{F}_b^{st}$ and $\lambda \in \mathbf{R}$, the following addition and a scalar product are given as follows :

$$\begin{aligned}\mu_{x+y}(\xi) &= \sup\{\alpha \in [0,1] : \\ &\qquad \xi = \xi_1 + \xi_2,\ \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y)\}; \\ \mu_{\lambda x}(\xi) &= \begin{cases} \mu_x(\xi/\lambda) & (\lambda \neq 0) \\ 0 & (\lambda = 0,\ \xi \neq 0) \\ \sup_{\eta \in \mathbf{R}} \mu_x(\eta) & (\lambda = 0,\ \xi = 0) \end{cases}\end{aligned}$$

In [12] they introduced the following equivalence relation $(x,y) \sim (u,v)$ for $(x,y),(u,v) \in \mathcal{F}_b^{st} \times \mathcal{F}_b^{st}$, *i.e.*,

$$(x,y) \sim (u,v) \Longleftrightarrow x + v = u + y. \tag{2.1}$$

Putting $x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2)$ by the parametric representation, the relation (2.1) means that the following equations hold.

$$x_i + v_i = u_i + y_i \quad (i = 1, 2)$$

Denote an equivalence class by $\langle x, y\rangle = \{(u,v) \in \mathcal{F}_b^{st} \times \mathcal{F}_b^{st} : (u,v) \sim (x,y)\}$ for $x, y \in \mathcal{F}_b^{st}$ and the set of equivalence classes by

$$(\mathcal{F}_b^{st})^2/\sim = \{\langle x,y\rangle : x, y \in \mathcal{F}_b^{st}\}$$

such that one of the following cases (i) and (ii) hold:

(i) if $(x,y) \sim (u,v)$, then $\langle x,y\rangle = \langle u,v\rangle$;

(ii) if $(x,y) \not\sim (u,v)$, then $\langle x,y\rangle \cap \langle u,v\rangle = \emptyset$.

Then $(\mathcal{F}_b^{st})^2/\sim$ is a linear space with the following addition and scalar product

$$\langle x,y\rangle + \langle u,v\rangle = \langle x+u, y+v\rangle \tag{2.2}$$

$$\lambda\langle x,y\rangle = \begin{cases} \langle \lambda x, \lambda y\rangle & (\lambda \geq 0) \\ \langle (-\lambda)y, (-\lambda)x\rangle & (\lambda < 0) \end{cases} \tag{2.3}$$

for $\lambda \in \mathbf{R}$ and $\langle x,y\rangle, \langle u,v\rangle \in (\mathcal{F}_b^{st})^2/\sim$. They denote a norm in $(\mathcal{F}_b^{st})^2/\sim$ by

$$\| \langle x,y\rangle \| = \sup_{\alpha \in I} d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)).$$

Here d_H is the Hausdorff metric is as follows:

$$d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)) = \max\Big(\sup_{\xi \in L_\alpha(\mu_x)} \inf_{\eta \in L_\alpha(\mu_y)} |\xi - \eta|,\ \sup_{\eta \in L_\alpha(\mu_y)} \inf_{\xi \in L_\alpha(\mu_x)} |\xi - \eta|\Big)$$

It can be easily seen that $\| \langle x,y\rangle \| = d(x,y)$. Note that $\| \langle x,y\rangle \| = 0$ in $(\mathcal{F}_b^{st})^2/\sim$ if and only if $x = y$ in $\mathcal{F}_b^{st}$.

3 Fuzzy Differential and Fuzzy Integral

In this section we consider fuzzy function in a Banach space induced by the normed space $(\mathcal{F}_b^{st})^2/\sim$. It can be seen that for $x, y \in \mathcal{F}_b^{st}$

$$\langle x,y\rangle = \langle x,0\rangle + \langle 0,y\rangle = \langle x,0\rangle - \langle y,0\rangle.$$

Denoting a set of fuzzy numbers by

$$X_0 = \{\langle x,0\rangle \in (\mathcal{F}_b^{st})^2/\sim : x, 0 \in \mathcal{F}_b^{st}\},$$

which is a Banach space (see *e.g.*, [17]). Then we have $(\mathcal{F}_b^{st})^2/\sim = X_0 - X_0$.

Denote the completion of $(\mathcal{F}_b^{st})^2/\sim$ by X. Let J be an interval in $\mathbf{R}$. In what follows we consider a function $f : J \to X$ as $f = \langle (f_1, f_2), 0\rangle$. Here f has the parametric representation of $f = (f_1, f_2)$, where $f_i(t,\alpha)$ for $i = 1, 2$ are the end-points of the $\alpha-$cut set of f In this section we give definitions of differentiation and integration of fuzzy functions.

A fuzzy function $f : J \to X$ is said to be differentiable at $t_0 \in J$, if there exists an $\eta \in X$ such that for any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying

$$\| \frac{f(t) - f(t_0)}{t - t_0} - \eta \| < \varepsilon$$

for $t \in J$ and $0 < |t - t_0| < \delta$. Denote $\eta = f'(t_0) = \frac{df}{dt}(t_0)$. f is differentiable on J if f is differentiable at any $t \in J$. In the similar way higher order derivatives of f are defined by $f^{(k)} = (f^{(k-1)})'$ for $k = 2, 3, \cdots$. (Cf. [7, 8])

In [12] they define the embedding $j : \mathcal{F}_b^{st} \to X$ such that $j(u) = \langle u, 0\rangle$. The function $f : J \to \mathcal{F}_b^{st}$ is called differentiable in the sense of Puri-Ralescu, if $j(f(\cdot))$ is differentiable. Suppose that f is differentiable at $t \in J$ in the above sense, denoted the differential $f'(t) \in \mathcal{F}_b^{st}$. Then we have $\frac{d}{dt}(j(f(t))) = \langle f'(t), 0\rangle$, *i.e.*, f is differentiable in the sense of Puri-Ralescu. In [9, 12] H-difference and H-differentiation of f is treated as follows. Suppose that for $f(t+h), f(t) \in \mathcal{F}_b^{st}$, there exists $g \in \mathcal{F}_b^{st}$ such that $f(t+h) = f(t) + g$, then g is called to the H-difference, denoted $f(t+h) - f(t)$. The function f is called H-differentiable at $t \in J$ if there exists an $\eta \in \mathcal{F}_b^{st}$ such that both $\lim\limits_{h\to+0} \dfrac{f(t+h) - f(t)}{h}$ and $\lim\limits_{h\to+0} \dfrac{f(t) - f(t-h)}{h}$ exist and equal to η. If f is H-differentiable, then $f'(t) = \eta$.

Proposition 3.1 *If f is differentiable at t_0, then f is continuous at t_0.*

Theorem 3.1 *Denote a parametric representation of f by $f = \langle (f_1, f_2), 0\rangle$. Here f_1, f_2 are functions defined on $I \times J$ to $\mathbf{R}$ and the left-, right-end point of the $\alpha-$cut set $L_\alpha(f(t))$. If f is differentiable at t_0, then it follows that there exist $\frac{\partial}{\partial t}f_1(t,\alpha), \frac{\partial}{\partial t}f_2(t,\alpha)$ and that*

$$f'(t_0) = (\frac{\partial}{\partial t}f_1, \frac{\partial}{\partial t}f_2)(t_0).$$

Theorem 3.2 *It follows that $f'(t) \equiv 0$ if and only if $f(t) \equiv const \in X$.*

In the following definition we give one of integrals of fuzzy functions.

Definition 3.1 *Let $J = [a, b]$ and f be a mapping from J to X. Divide the interval J such that $a = t_0 < t_1 < \cdots < t_n = b$ and $\tau_i \in [t_{i-1}, t_i]$ for $i = 1, 2, \cdots, n$. f is integrable over J if there exists the limit $\lim\limits_{|\Delta|\to 0} \sum\limits_{i=1}^{n} f(\tau_i)\Delta_i$, where $\Delta_i = t_i - t_{i-1}$, $|\Delta| = \max\limits_{1\le i\le n} \Delta_i$. Define*

$$\int_a^b f(s)ds = \lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(\tau_i)\Delta_i.$$

Proposition 3.2 *Let f be integrable over J. Then the following statements* (i)-(ii) *hold.*

(i) *f is bounded on J, i.e., there exists an $M > 0$ such that $\| f(t) \| \leq M$ for $t \in J$.*

(ii) *If $f(t) \in X$ for $t \in J$, then $\int_a^t f(s)ds \in X$ for $t \in J$.*

Proposition 3.3 *If f is continuous on $[a, b]$ then f is integrable over the interval.*

Theorem 3.3 *Let $f : J \to X$ with $f = \langle (f_1, f_2), 0\rangle$ be integrable over $[a, b]$. Then it follows that*

$$\int_a^b f(s)ds = \langle (\int_a^b f_1(s)ds, \int_a^b f_2(s)ds), 0\rangle$$

Conversely, if f_1, f_2 are continuous on $[a, b] \times I$, then f is integrable over $[a, b]$.

Proposition 3.4 *Let f be continuous on the interval $[a, b]$.*
Denote $F(t) = \displaystyle\int_a^t f(s)ds$. Then the following properties (i) *and* (ii) *hold.*

(i) *F is differentiable on $[a, b]$ with $F(t) \in X$ and $F' = f$;*

(ii) *For $t_1, t_2 \in [a, b]$ and $t_1 \leq t_2$, we have $\int_{t_1}^{t_2} f(s)ds = F(t_2) - F(t_1)$.*

Proposition 3.5 *Let f is continuous on $[a, b]$. Then it follows that*

$$\| \int_a^b f(s)ds \| \leq \int_a^b \| f(s) \| ds.$$

Theorem 3.4 *Let $f : [a, b] \to X$ be continuous on $[a, b]$ and differentiable on (a, b), Then it follows that there exists a number $c \in (a, b)$ such that*

$$\| f(b) - f(a) \| \leq (b - a) \| f'(c) \| .$$

Definition 3.2 *Let $f : J \to X^n$ such that $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))^T$. f is differentiable on J if each f_i is differentiable on J for $i = 1, 2, \cdots, n$. Define the derivative $f'(t) = (f_1'(t), f_2'(t), \cdots, f_n'(t))^T$.*

Let $f : [a, b] \to X^n$ such that $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))^T$. f is integrable over $[a, b]$ if f_i is integrable over $[a, b]$ for $i = 1, 2, \cdots, n$. Define the integral

$$\int_a^b f(s)ds = (\int_a^b f_1(s)ds, \int_a^b f_2(s)ds, \cdots, \int_a^b f_n(s)ds)^T.$$

It can be easily proved that similar theorems and propositions concerning to X^n−valued functions to ones in this section hold.

4 Stability of Fuzzy Differential Equations and Inclusions

In [18] they discuss exponential decay problems, *e.g.*, machine replacement and oil well extraction, etc. They analyze optimization problems for each oil well to determine its optimal replacement schedule. Denote the quality remaining in the well at time t by $x(t)$ and denote the rate of oil extraction by $D > 0$. Then they get the following rate of oil extraction $x'(t) = -Dx$ with $x(0) = \nu$. Then $x(t) = \nu e^{-Dt}$.

In what follows we consider the rate of oil extraction D as a constant fuzzy number $D = (D_1, D_2) \in \mathcal{F}_b^{st}$, where $D_1(\alpha)$ is the left end-point of the α−cut set and $D_1(\alpha) > 0$ for $\alpha \in I$. Then we assume that the oil quality $x(t) = (x_1(t), x_2(t)) \in \mathcal{F}_b^{st}$ is a fuzzy function which means the quality remaining in the well at time t and $\nu \in \mathcal{F}_b^{st}$. Consider an initial value problem of fuzzy differential equation

$$\frac{dx}{dt}(t) = -(Dx), \quad x(0) = \nu. \tag{4.4}$$

The above problem has a unique solution

$$x(t) = \nu + \int_0^t (-(Dx(s)))ds.$$

See [11].

It follows that as long as $x_1(t) \geq 0$, by the extension of principle

$$\begin{aligned} \frac{d}{dt}(x_1(t), x_2(t)) &= -(D_1, D_2)(x_1, x_2) \\ &= -(D_1 x_1, D_2 x_2) \\ &= (-D_2 x_2, -D_1 x_1). \end{aligned}$$

Then we have two ordinary differential equations such as

$$x_1'(t) = -D_2 x_2, \; x_2'(t) = -D_1 x_1$$

with $x(0) = (\nu_1, \nu_2) \in \mathcal{F}_b^{st}$. Therefore

$$\begin{aligned} x_1(t) &= \frac{(\nu_1 + \nu_2\sqrt{\frac{D_2}{D_1}})e^{-\sqrt{D_1 D_2}t}}{2} + \frac{(\nu_1 - \nu_2\sqrt{\frac{D_2}{D_1}})e^{\sqrt{D_1 D_2}t}}{2}, \\ x_2(t) &= \frac{(\nu_1\sqrt{\frac{D_1}{D_2}} + \nu_2)e^{-\sqrt{D_1 D_2}t}}{2} - \frac{(\nu_1\sqrt{\frac{D_1}{D_2}} - \nu_2)e^{\sqrt{D_1 D_2}t}}{2} \end{aligned}$$

for $t \geq 0$. Then we get the unstable result of solution $x = (x_1, x_2)$ such that

$$\lim_{t \to +\infty} d(x(t), 0) = +\infty,$$

where $0 \in \mathbf{R}$, as well as it follows that

$$\lim_{t \to +\infty} \sup_{\alpha \in I} |\sqrt{D_1(\alpha)}x_1(t, \alpha) + \sqrt{D_2(\alpha)}x_2(t, \alpha)| = 0.$$

(see [14]). In this case of $x' = -Dx$ by the method of parametric representation, the equation leads to the unstable result.

In what follows we introduce the idea of fuzzy differential inclusions in [2, 3, 6, 11] In analyzing the equation $x' = -Dx$ via the inclusions method, we find that the same equation is stable in the similar way to the theory of ordinary differential equations.

Example. Consider an initial value problem of fuzzy differential equation (4.4). According to the idea of fuzzy differential inclusions in which a family of differential inclusions plays an important role in finding some kind of fuzzy sets of (4.4) (See [1]). Let $F(\xi, \alpha) = [-D_2(\alpha)\xi, -D_1(\alpha)\xi] \subset \mathbf{R}$ defined on $\mathbf{R} \times I$ to the set of compact and convex sets K_C^1 in $\mathbf{R}$. Then one can solve the following differential inclusions

$$\xi_\alpha'(t) \in F(\xi_\alpha, \alpha), \; \xi_\alpha(0) \in L_\alpha(\nu),$$

where $L_\alpha(\nu) = [\nu_1(\alpha), \nu_2(\alpha)]$ for $\alpha \in I$, which means that differential inequalities

$$\begin{aligned} &-D_2(\alpha)\xi_\alpha(t) \leq \xi_\alpha'(t) \leq -D_1(\alpha)\xi_\alpha(t) \\ &\nu_1(\alpha) \leq \xi_\alpha(0) \leq \nu_2(\alpha) \end{aligned}$$

for $\alpha \in I$. Then we emphasize that the function ξ_α is $\mathbf{R}$–valued function defined on $\mathbf{R}$ without information on the grade of fuzzy number x, so $\xi_\alpha(t)$ is a real numbers but not fuzzy number. By basic calculation we get $\xi_\alpha(0)e^{-D_2(\alpha)t} \leq \xi_\alpha(t) \leq \xi_\alpha(0)e^{-D_1(\alpha)t}$ with $\xi_\alpha(0) \in L_\alpha(\nu)$. Therefore we have
$\xi_\alpha(t) \in [\nu_1(\alpha)e^{-D_2(\alpha)t}, \nu_2(\alpha)e^{-D_1(\alpha)t}]$ for $\alpha \in I, t \in \mathbf{R}$, which is called a solution set denoted by

$S_\alpha(L_\alpha(\nu), t) = [\nu_1(\alpha)e^{-D_2(\alpha)t}, \nu_2(\alpha)e^{-D_1(\alpha)t}]$. The solution set $S_\alpha(L_\alpha(\nu), t)$ is the α–cut set of the parametric representation of a fuzzy number $(\nu_1 e^{-D_2 t}, \nu_2 e^{-D_1 t})$. Thus we get a fuzzy solution of (4.4) as
$x(t) = (\nu_1 e^{-D_2 t}, \nu_2 e^{-D_1 t})$ for $t \in \mathbf{R}$.

In classical analysis of the initial value problem (4.4) we observe the unstability of solutions by the method of parametric representation of fuzzy numbers. By applying differential inclusions to fuzzy differential equations(FDE) the same results of FDE as those in theory of ordinary differential equations. Much richer properties in fuzzy differential inclusions is significant but, in considering K_C^1–valued function $F(\xi, \alpha)$, one treats each fuzzy number $x(t) \in \mathcal{F}_b^{st}$ as a real number $x(t) \in \mathbf{R}$. Finally, we get solution sets which are the α–cut sets of a fuzzy set. By treating many practical modeling of real systems with uncertainty we can get better conclusions on comparison between fuzzy differential inclusions and the parametric representation of fuzzy numbers.

5 Variational Equations

In order to discuss the asymptotic behaviors of solutions to ordinary differential equations(ODE) the variational equation of ODE plays important roles in analyzing parametric dependence of solutions to ODE (see [19]). Consider an ODE

$$y' = f(t, y) \tag{ODE}$$

provided that there exists the Jacobian matrix $\frac{\partial f}{\partial y}$. The following equation $y' = \frac{\partial f}{\partial y}(t, \phi(t; \tau, \eta))y$ is called a variational equation of (ODE). Here $\phi(t; \tau, \xi)$ is a solution of (ODE) with $y(\tau) = \eta$.

One tries to derive the properties of the solutions $x(t)$ to

$$x' = f(t, x) + h(t, x) \tag{P}$$

from the corresponding to properties of the solutions to (P). In [13] Vlasov's theorem is as follows:

(i) Suppose that for all η and for $t \geq \tau$, the $n \times n$matrix y_η satisfies $\| y_\eta(t; \tau, \eta) \| \leq a(\tau)$ with a continuous function $a(\tau)$;

(ii) Suppose that $\| h(t, x) \| \leq p(t)q(\| x \|)$ in which $p(t)$ is continuous, $\int_0^\infty p(t)dt < \infty$, and $q(r) > 0$ ia a non-decreasing function with $\int_0^\infty \frac{r}{q(r)} = \infty$;

(iii) Suppose that $\int_0^\infty p(t)a(t)dt < \infty$.

If the above conditions (i) - (iii) hold , then the boundedness of solutions to (ODE) implies the same to (P).

Let X, Y be Banach spaces and S an open subset of X. Let $f : S \to Y$ be such that

$$f(u + h) = f(u) + f'(u)h + w(u, h)$$

for every $h \in X$ with $u + h \in S$, where $f'(u) : X \to Y$ is a linear operator and $\lim_{h \to 0} \dfrac{\| w(u, h) \|}{\| h \|} = 0$.
Then $f'(u)h$ is called the Fréchet differential of f at u with increment h, $f'(u)$ is called the Fréchet derivative of f at u and f is called Fréchet differentiable at $u \in S$. In the case that a function $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ has the Jacobian matrix $\frac{\partial f}{\partial y}(t, y)$, then f is Fréchet differentiable at $u \in \mathbf{R}^n$

and the Fréchet derivative $f' = \frac{\partial f}{\partial y}(t,y)$. Kartsatos[10] dealt with the existence and uniqueness of solutions to the following problem:

$$x' = F(t,x) + f(t) \tag{5.5}$$
$$Ux = r \tag{5.6}$$

Theorem 8.24 in [10] is as follows:

(i) Let $\mathbf{R}_+ = [0,\infty).F : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $F(\mathbf{R}_+ \times M)$ is bounded for every bounded set $M \subset \mathbf{R}^n$. Moreover there exists the Jacobian matrix $F_x(t,x)$ which is continuous on $\mathbf{R}_+ \times \mathbf{R}^n$;

(ii) For every bounded set $M \subset \mathbf{R}^n, F_x(\mathbf{R}_+ \times M)$ is bounded and for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\| F_x(t,u_1) - F_x(t,u_2) \| < \varepsilon$ for $(t,u_1,u_2) \in \mathbf{R}_+ \times M \times M$;

(iii) Suppose that the operator $U : C_n^1(\mathbf{R}_+) \to \mathbf{R}^n$ is continuous and Fréchet differentiable at every $x_0 \in C_n^1(\mathbf{R}_+)$. Here $C_n^1(\mathbf{R}_+)$ is a set of continuously differentiable functions from $\mathbf{R}_+$ to $\mathbf{R}^n$;

(iv) $S \subset C_n^1(\mathbf{R}_+)$ is any open set. For every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\| [U'(x_1) - U'(x_2)]h \| \leq \varepsilon \| h \|_\infty$ for every $x_1, x_2 \in S, h \in C_n^1(\mathbf{R}_+)$. Here $\| \cdot \|_\infty$ is the sup norm in $C_n^1(\mathbf{R}_+)$;

(v) Let f_0 be continuous on $\mathbf{R}_+$ and $r_0 \in \mathbf{R}^n$. Let $x_0 \in C_n^1(\mathbf{R}_+)$ be a solution to

$$x_0' = F(t, x_0(t)) + f_0(t) \tag{5.7}$$
$$Ux_0 = r_0 \tag{5.8}$$

for $t \in \mathbf{R}_+$. Suppose that the following linear problem

$$x' = F_x(t, x_0(t))x \tag{5.9}$$
$$U'(x_0)x = 0 \tag{5.10}$$

has only the zero solution in $C_n^1(\mathbf{R}_+)$;

(vi) Suppose that

$$\sup_{t\in\mathbf{R}_+} \int_0^t \| X(t)X^{-1}(s) \| \, ds < \infty$$

where $X(t)$ is the fundamental matrix of $x' = F_x(t, x_0(t))x$.

If the above conditions (i) - (vi) hold, then there exist numbers $\alpha, \beta > 0$ such that for every $(f, r) \in C_n^1(\mathbf{R}_+) \times \mathbf{R}^n$ with $\| (f - f_0, r - r_0) \| \leq \beta$, there exists a unique solution $x \in C_n^1(\mathbf{R}_+)$ to ((5.6), (5.6)) such that $\| x \| \leq \alpha$.

In [4] the Jacobian matrix plays an important role in proving the Brauwer's fixed point theorem in finite dimensional linear space.

In analyzing ordinary differential equations, the variational equation plays a significant role in the above results. In the similar way it is expected that analysis of the variational equation of fuzzy differential equations leads to various results on asymptotic behaviors of solutions of fuzzy differential equations(FDE). When we consider the varitional equation of (FDE), it is need to calculate the Fréchet derivative of (FDE). Let X, Y be Banach spaces of fuzzy numbers. Let S be an open subset of X. Let a fuzzy function $f : S \to Y$ be such that

$$f(u+h) = f(u) + f'(u)h + w(u,h)$$

for every $h \in X$ with $u + h \in S$, where $f'(u) : X \to Y$ is an operator and $\lim_{h\to 0} \frac{\| w(u,h) \|}{\| h \|} = 0$. Then $f'(u)h$ is called the Fréchet differential of f at u with increment h, $f'(u)$ is called the Fréchet derivative of f at u and f is called Fréchet differentiable at $u \in S$. In the case of Fréchet differential of fuzzy function, it is necessary to consider the product $f'(u)h$ with an operator $f'(u)$ and a fuzzy number h. As mentioned in Section 4 there are two ways in analyzing the stability of solutions to

(FDE). One is the parametric representation method, in which the equation $x' = -x$ is unstable and the other fuzzy differential inclusions, where the same equation implies the stability. It is possible that analyzing the variational equations of (FDE) will find a suitable method for stability theory of (FDE).

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