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Robust $\mathcal{D}$-stability of linear difference equations

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Abstract

We study robustness of $\mathcal{D}$-stability of linear difference equations under multi-perturbation and affine perturbation of coefficient matrices via the concept of $\mathcal{D}$-stability radius. Some explicit formulae are derived for these $\mathcal{D}$-stability radii. The obtained results include the corresponding ones established earlier in [3], [4], [9], [10] as particular cases.

1 Introduction and Preliminaries

Let $\mathcal{D} := D(\alpha, r)$ be a open disk centered at $(\alpha, 0)$ with radius r in the complex plane. A linear discrete-time (time-invariant) system is called $\mathcal{D}$-stable if its characteristic equation has only roots in $\mathcal{D}$. In this paper, we study the robustness of $\mathcal{D}$-stability of linear discrete-time systems of the form

$$x(k+1) = A_\nu x(k) + A_{\nu-1}x(k-1) + \cdots + A_0 x(k-\nu), \quad k \in \mathbb{N}, k \geq \nu \tag{1}$$

under parameter perturbation of the coefficient matrices via the concept of $\mathcal{D}$-stability radius. It is important to note that the problems of computing of $D(0,1)$-stability radii (or simpler, stability radii) of linear discrete-time systems have been studied during the last twenty years by many mathematical researchers, see e.g. [2]-[5], [9]-[11]. In particular, the problems of computing of stability radii of linear discrete-time systems of the form (1) under single perturbations, affine perturbations

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and multi-perturbations have just been studied in the recent time, see [5], [9], [10]. It is also worth noticing that (robust) $\mathcal{D}$-stability problems of linear discrete-time systems have been received much attention from researchers for a long time. Some sufficient conditions for the (robust) $\mathcal{D}$-stability of the system (1) under parameter perturbations were proposed in [1], [6], [8], [13]-[15]. However, to the best of our knowledge, there is not any formula for the $\mathcal{D}$-stability radii of the system (1) under multi-perturbations or affine-perturbations in the case of $\mathcal{D} = D(\alpha, r)$. In the present paper, using our recent new results on the problems of computing of stability radii (see e.g. [10]), we can compute the $D(\alpha, r)$-stability radii of the system (1) under multi-perturbations and affine perturbations. The obtained results are the extensions of the corresponding results of [3], [4], [9], [10].

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and n, l, q be positive integers. Inequalities between real matrices or vectors will be understood componentwise. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}_+^{l\times q}$. If $P \in \mathbb{K}^{l\times q}$ we define $|P| = (|p_{ij}|)$. For any matrix $A \in \mathbb{K}^{n\times n}$ the *spectral radius* of A is denoted by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the set of all eigenvalues of A. A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in \mathbb{K}^n$. Every p-norm on $\mathbb{K}^n, 1 \leq p \leq \infty$, is monotonic. Throughout the paper, the norm $\|M\|$ of a matrix $M \in \mathbb{K}^{l\times q}$ is always understood as the operator norm defined by $\|M\| = \max_{\|y\|=1} \|My\|$, where $\mathbb{K}^q$ and $\mathbb{K}^l$ are provided with some monotonic vector norms.

2 $\mathcal{D}$-stability radii of linear discrete-time systems

Let $\mathcal{D} = D(\alpha, r)$ be the open disk centered at $(\alpha, 0)$ with radius r in the complex plane. Consider a dynamical system described by a linear discrete-time system of the form

$$x(k+1) = Ax(k), \quad k \in \mathbb{N}, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is a given matrix. The system (2) is called $\mathcal{D}$-stable if $\sigma(A) \subset \mathcal{D}$.

It is important to note that, the system (2) is asymptotically stable in the Lyapunov' s sense in the case of $\mathcal{D} = D(0,1)$ and is strong stable in the case of $\mathcal{D} = D(0, r), 0 < r < 1$. We now assume that the system (2) is $\mathcal{D}$-stable and the system matrix A is subjected to one of the following perturbation types

$$A \longrightarrow A + \textstyle\sum_{i=1}^N D_i\Delta_i E_i, \quad \text{(multi-perturbation)}, \tag{3}$$

$$A \longrightarrow A + \textstyle\sum_{i=1}^N \delta_i B_i, \quad \text{(affine perturbation)}. \tag{4}$$

Here $D_i \in \mathbb{R}^{n\times l_i}, E_i \in \mathbb{R}^{q_i\times n}, B_i \in \mathbb{R}^{n\times n}, i \in \underline{N} := \{1,2,...,N\}$ are given matrices defining the *structure* of perturbations and $\Delta_i \in \mathbb{K}^{l_i\times q_i}$, $\delta_i \in \mathbb{K}$ $(i \in \underline{N})$ unknown disturbance matrices and scalars, respectively. For class of multi-perturbations of the form (3), we always assume that the linear space $\mathbf{\Delta}_{\mathbb{K}} = \mathbb{K}^{l_1\times q_1} \times ... \times \mathbb{K}^{l_N\times q_N}$ of all perturbation families $\Delta = (\Delta_1, ..., \Delta_N)$, with $\Delta_i \in \mathbb{K}^{l_i\times q_i}$, is endowed with the norm $\gamma(\Delta) = \gamma(\Delta_1, ..., \Delta_N) = \sum_{i=1}^{N} \|\Delta_i\|$, where the norms $\|\Delta_i\|$ are operator norms on $\mathbb{K}^{l_i\times q_i}$, induced by given monotonic vector norms on the spaces $\mathbb{K}^{l_i}, \mathbb{K}^{q_i}, i \in \underline{N}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$.

Definition 2.1. *Let the linear discrete time system (2) be $\mathcal{D}-$stable.*

(a) The complex, real $D(\alpha,r)$-stability radius of the system (2) with respect to multi-perturbations of the form (3) are defined, respectively, by

$$r_{\mathbb{C}}(A,(D_i,E_i)_{i\in\underline{N}};\mathcal{D}) = \inf\{\gamma(\Delta):\ \Delta \in \mathbf{\Delta}_{\mathbb{C}},\ \sigma(A+\textstyle\sum_{i=1}^{N} D_i\Delta_iE_i,) \not\subset \mathcal{D}\},$$
$$r_{\mathbb{R}}(A,(D_i,E_i)_{i\in\underline{N}};\mathcal{D}) = \inf\{\gamma(\Delta) : \Delta \in \mathbf{\Delta}_{\mathbb{R}},\ \sigma(A+\textstyle\sum_{i=1}^{N} D_i\Delta_iE_i,) \not\subset \mathcal{D}\}.$$

(b) The complex, real $D(\alpha,r)$-stability radius of the system (2) with respect to affine perturbations of the form (4) are defined, respectively, by

$$r_{\mathbb{C}}(A,(B_i)_{i\in\underline{N}};\mathcal{D}) = \inf\{\max_{i\in\underline{N}}|\delta_i| :\ \delta_i \in \mathbb{C}, i \in \underline{N},\quad \sigma(A+\textstyle\sum_{i=1}^{N}\delta_iB_i) \not\subset \mathcal{D}\},$$
$$r_{\mathbb{R}}(A,(B_i)_{i\in\underline{N}};\mathcal{D}) = \inf\{\max_{i\in\underline{N}}|\delta_i| : \delta_i \in \mathbb{R}, i \in \underline{N},\quad \sigma(A+\textstyle\sum_{i=1}^{N}\delta_iB_i) \not\subset \mathcal{D}\}.$$

As noted in Introduction, the problems of computing of the stability radii (i.e. $D(0,1)$-stability radii) of the system (2) have been studied during the last twenty years and have got the full results, see e.g. [3], [12], [4], [10]. We list here the interesting results for the class of positive systems (i.e. A is a nonnegative matrix).

Theorem 2.2. [4] *Let the system (2) be $D(0,1)-$stable and positive. Suppose the system matrix A is subjected to affine-perturbations (4), where $B_i \in \mathbb{R}_+^{n\times n}, i \in \underline{N}$. Then*

$$r_{\mathbb{C}}(A,(B_i)_{i\in\underline{N}};D(0,1)) = r_{\mathbb{R}}(A,(B_i)_{i\in\underline{N}};D(0,1)) = \frac{1}{\rho(\sum_{i=1}^{N} B_i(I_n-A)^{-1})}.$$

Theorem 2.3. [10] *Let the system (2) be $D(0,1)-$stable and positive. Assume that the matrix A is subjected to parameter multi-perturbations (3). If $D_i = D \in \mathbb{R}_+^{n\times l}$ and $E_i \in \mathbb{R}_+^{q_i\times n}$ for every $i \in \underline{N}$ or $E_i = E \in \mathbb{R}_+^{q\times n}$ and $D_i \in \mathbb{R}_+^{n\times l_i}$ for every $i \in \underline{N}$, then $r_{\mathbb{C}}(A,(D_i,E_i)_{i\in\underline{N}};D(0,1)) = r_{\mathbb{R}}(A,(D_i,E_i)_{i\in\underline{N}};D(0,1)) = \frac{1}{\max_{i\in\underline{N}}\|E_i(I_n-A)^{-1}D_i\|}$.*

The following theorem extends the above results to the general case of $\mathcal{D} = D(\alpha,r)$.

Theorem 2.4. *Let the system (2) be $D(\alpha,r)-$stable and $A \geq \alpha I_n$. (i) If the matrix A is subjected to multi-perturbations (3), where $D_i = D \in \mathbb{R}_+^{n\times l}$ and $E_i \in \mathbb{R}_+^{q_i\times n}$ for every $i \in \underline{N}$ or $E_i = E \in \mathbb{R}_+^{q\times n}$, and $D_i \in \mathbb{R}_+^{n\times l_i}$ for every $i \in \underline{N}$, then*

$r_{\mathbb{C}}(A,(D_i,E_i)_{i\in \underline{N}};D(\alpha,r)) = r_{\mathbb{R}}(A,(D_i,E_i)_{i\in \underline{N}};D(\alpha,r)) = \frac{1}{\max_{i\in \underline{N}}\|E_i((\alpha+r)I_n-A)^{-1}D_i\|}$.

(ii) If the matrix A is subjected to affine-perturbations (4), where $B_i \in \mathbb{R}_+^{n\times n}, i\in \underline{N}$, then $r_{\mathbb{C}}(A,(B_i)_{i\in \underline{N}};D(\alpha,r)) = r_{\mathbb{R}}(A,(B_i)_{i\in \underline{N}};D(\alpha,r)) = \frac{1}{\rho\left(\sum_{i=1}^{N} B_i((\alpha+r)I_n-A)^{-1}\right)}$.

Proof. The proof is based on Theorems 2.2, 2.3 and the fact that the system $x(k+1) = Ax(k), k\in\mathbb{N}$ is $D(\alpha,r)$−stable if and only if the system $x(k+1) = (A-\alpha I_n)x(k), k\in\mathbb{N}$ is $D(0,1)$−stable. For sake of space, it is omitted here. □

The following is an extension of the main result of [7].

Corollary 2.5. *Let $P(z) := I_n z^{\nu+1} - A_\nu z^\nu - \ldots - A_0$ be a given polynomial matrix. Assume that $|\alpha| < r, |\alpha| + r \le 1$ and $\|[A_0A_1\ldots A_\nu]\|_\infty < (r-|\alpha|)^{\nu+1}$. Then all the roots of the equation $\det P(z) = 0$ lie inside the disk $D(\alpha,r)$.*

3 $\mathcal{D}$-stability radii of linear discrete time-delay systems

Consider a dynamical system described by a linear discrete time-delay system of the form (1), where $A_i \in \mathbb{R}^{n\times n}, i\in\overline{\nu} := \{0,1,2,\ldots,\nu\}$, are given matrices. For the linear discrete time-delay system (1), we consider the stable region $\mathcal{D} = D(\alpha,r), |\alpha| < r, r+|\alpha| \le 1$, see e.g. [8], [13], [14]. We associate the system (1) with the following polynomial matrix $P(z) := (z^{\nu+1}I_n - A_\nu z^\nu - A_{\nu-1}z^{\nu-1} - \ldots - A_0),\ z\in\mathbb{C}$. Denote by $\sigma((A_i)_{i\in\overline{\nu}}) := \{z\in\mathbb{C} : \det P(z) = 0\}$ the set of all roots of the characteristic equation of the linear discrete time-delay system (1). Then $\sigma((A_i)_{i\in\overline{\nu}})$ is called the *spectral set* of the linear discrete time-delay system (1) and $\rho((A_i)_{i\in\overline{\nu}}) := \max\{|s| : s\in\sigma((A_i)_{i\in\overline{\nu}})\}$ is called *spectral radius* of the linear discrete time-delay system (1). Recall that the system (1) is said to be $\mathcal{D}$-stable if $\sigma((A_i)_{i\in\overline{\nu}}) \subset \mathcal{D}$. We now assume that the system (1) is $\mathcal{D}$-stable and the coefficient matrices $A_i, i\in\overline{\nu}$ are subjected to parameter perturbations

$$A_i \longrightarrow A_i + \sum_{j=1}^{N} D_{ij}\Delta_{ij}E_{ij}, \qquad \text{(multi-perturbation)} \tag{5}$$

$$A_i \longrightarrow A_i + \sum_{j=1}^{N} \delta_{ij}B_{ij}, \qquad \text{(affine-perturbation)} \tag{6}$$

where $D_{ij}\in\mathbb{R}^{n\times l_{ij}}, E_{ij}\in\mathbb{R}^{q_{ij}\times n},\ (i\in\overline{\nu}, j\in\underline{N} := \{1,2,\ldots,N\})$; $B_{ij}\in\mathbb{R}^{n\times n},\ (i\in\overline{\nu}, j\in\underline{N})$ are given matrices defining the *structure* of perturbations and $\Delta_{ij}\in\mathbb{K}^{l_{ij}\times q_{ij}},\ (i\in\overline{\nu}, j\in\underline{N})$; $\delta_{ij}\in\mathbb{K}, (i\in\overline{\nu}, j\in\underline{N})$ are perturbation matrices, perturbation scalars, respectively. For the class of multi-perturbations of the form

(5), we define $\tilde{\Delta} := (\Delta_0, \Delta_1, \dots, \Delta_\nu)$, where $\Delta_i := (\Delta_{i1}, \Delta_{i2}, \dots, \Delta_{iN}) \in \mathbb{K}^{l_{i1}\times q_{i1}} \times \dots \times \mathbb{K}^{l_{iN}\times q_{iN}}, i \in \overline{\nu}$. Then the size of each perturbation $\tilde{\Delta}$ is measured by $\gamma(\tilde{\Delta}) := \sum_{i=0}^{\nu}\sum_{j=1}^{N}\|\Delta_{ij}\|$. With the class of affine perturbations of the form (6), we denote $\delta := ((\delta_{01}, \dots, \delta_{0N}); \dots; (\delta_{\nu 1}, \dots, \delta_{\nu N})) \in \mathbb{K}^{\nu N}$ and the size of each perturbation δ is measured by $\gamma(\delta) = \max_{i\in\overline{\nu}, j\in\underline{N}} |\delta_{ij}|$.

Definition 3.1. *Let the linear discrete time-delay system (1) be $\mathcal{D}$-stable.*

(a) The complex, real $D(\alpha, r)$-stability radius of the system (1) with respect to multi-perturbations of the form (5) is defined, respectively, by

$$r_{\mathbb{C}}^{m}(\mathcal{D}) = \inf\{\gamma(\tilde{\Delta}) \ : \ \tilde{\Delta} := (\Delta_0, \Delta_1, \dots, \Delta_\nu), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \dots, \Delta_{iN}) \in \mathbb{C}^{l_{i1}\times q_{i1}} \times \dots \times \mathbb{C}^{l_{iN}\times q_{iN}}, i \in \overline{\nu}, \ \sigma\left(\left(A_i + \textstyle\sum_{j=1}^{N} D_{ij}\Delta_{ij}E_{ij}\right)_{i\in\overline{\nu}}\right) \not\subset \mathcal{D}\},$$

$$r_{\mathbb{R}}^{m}(\mathcal{D}) = \inf\{\gamma(\tilde{\Delta}) \ : \ \tilde{\Delta} := (\Delta_0, \Delta_1, \dots, \Delta_\nu), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \dots, \Delta_{iN}) \in \mathbb{R}^{l_{i1}\times q_{i1}} \times \dots \times \mathbb{R}^{l_{iN}\times q_{iN}}, i \in \overline{\nu}, \ \sigma\left(\left(A_i + \textstyle\sum_{j=1}^{N} D_{ij}\Delta_{ij}E_{ij}\right)_{i\in\overline{\nu}}\right) \not\subset \mathcal{D}\}.$$

(b) The complex, real $D(\alpha, r)$-stability radius of the system (1) with respect to affine perturbations of the form (6) is defined, respectively, by

$$r_{\mathbb{C}}^{a}(\mathcal{D}) = \inf\{\gamma(\delta) \ : \delta \in \mathbb{C}^{(\nu+1)N} \ \sigma\left((A_i + \textstyle\sum_{j=1}^{N}\delta_{ij}B_{ij})_{i\in\overline{\nu}}\right) \not\subset \mathcal{D}\},$$

$$r_{\mathbb{R}}^{a}(\mathcal{D}) = \inf\{\gamma(\delta) \ : \delta \in \mathbb{R}^{(\nu+1)N} \ \sigma\left((A_i + \textstyle\sum_{j=1}^{N}\delta_{ij}B_{ij})_{i\in\overline{\nu}}\right) \not\subset \mathcal{D}\}.$$

In particular case of $\mathcal{D} = D(0,1)$, the problems of computing of the stability radii of the linear discrete-time systems (1) under single perturbations, affine perturbations and multi-perturbations have been done recently by ourselves (see [5], [9], [10]). We summarize here some existing results of these problems. Recall that the system (1) is positive if and only if system matrices $A_0, A_1, \dots, A_\nu$ are nonnegative.

Theorem 3.2. [9] *Suppose the linear discrete time-delay system (1) is $D(0,1)$-stable, positive and the system matrices $A_i, i \in \overline{\nu}$ are subjected to affine perturbations of the form (6) where $B_{ij} \in \mathbb{R}_{+}^{n\times n}, i \in \overline{\nu}, j \in \underline{N}$. Then, $r_{\mathbb{C}}^{a}(D(0,1)) = r_{\mathbb{R}}^{a}(D(0,1)) = \frac{1}{\rho\left(P(1)^{-1}B\right)}$, where $B := \sum_{j=1}^{N} B_{0j} + \sum_{j=1}^{N} B_{1j} + \dots + \sum_{j=1}^{N} B_{\nu j}$.*

Remark 3.3. *In the proof of Theorem 3.2, we showed that the real perturbation $\delta := \left((\delta_{01}, \dots, \delta_{0N}); \dots; (\delta_{\nu 1}, \dots, \delta_{\nu N})\right) \in \mathbb{R}^{(\nu+1)N}; \quad \delta_{ij} = \frac{1}{\rho\left(P(1)^{-1}B\right)}, (i \in \overline{\nu}, j \in \underline{N})$ is a minimal size destabilizing perturbation. This fact will be used in the sequel.*

Theorem 3.4. [10] *Let the linear discrete time-delay system (1) be positive, $D(0,1)$-stable. Assume that the system matrices $A_i, i \in \underline{m}$ are subjected to the multi-perturbations of the form (5) where $D_{ij} := D \in \mathbb{R}_{+}^{n\times l}, E_{ij} \in \mathbb{R}_{+}^{q_{ij}\times n}$ for all $i \in \overline{\nu}, j \in \underline{N}$ or $E_{ij} := E \in \mathbb{R}_{+}^{q\times n}, D_{ij} \in \mathbb{R}_{+}^{n\times l_{ij}}$ for all $i \in \overline{\nu}, j \in \underline{N}$. Then, $r_{\mathbb{C}}^{m}(D(0,1)) = r_{\mathbb{R}}^{m}(D(0,1)) = \frac{1}{\max\{\|E_{ij}P(1)^{-1}D_{ij}\| : i\in\overline{\nu}, j\in\underline{N}\}}$.*

Theorem 3.5. *Let the linear discrete time-delay system (1) be $D(\alpha, r)$-stable. Suppose the coefficient matrices $A_i, i \in \overline{\nu}$ are subjected to the multi-perturbations (5), where $D_{ij} := D \in \mathbb{R}_+^{n\times l}, E_{ij} \in \mathbb{R}_+^{q_{ij}\times n}(i \in \overline{\nu}, j \in \underline{N})$ or $D_{ij} \in \mathbb{R}_+^{n\times l_{ij}}, E_{ij} := E \in \mathbb{R}_+^{q\times n}(i \in \overline{\nu}, j \in \underline{N})$. If $\alpha \leq 0$ and $A_0, A_1, ..., A_{\nu-1}, (A_\nu - \alpha I_n) \in \mathbb{R}_+^{n\times n}$, then*

$$r_{\mathbb{C}}^m(D(\alpha,r)) = r_{\mathbb{R}}^m(D(\alpha,r)) = \frac{1}{\max_{i\in\ \overline{\nu},\ j\in \underline{N}} \|(\alpha+r)^i E_{ij} P(\alpha+r)^{-1} D_{ij}\|}.$$

Proof. Assume $D_{ij} := D \in \mathbb{R}_+^{n\times l}, E_{ij} \in \mathbb{R}_+^{q_{ij}\times n}(i \in \overline{\nu}, j \in \underline{N})$. Consider the companion matrix of the polynomial matrix $P(z) = (z^{\nu+1}I_n - A_\nu z^\nu - A_{\nu-1}z^{\nu-1} - \ldots - A_0)$:

$$A_c := \begin{bmatrix} 0 & I_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_n & \ldots & 0 & 0 \\ .. & .. & .. & \ldots & .. & .. \\ 0 & 0 & 0 & \ldots & 0 & I_n \\ A_0 & A_1 & .. & \ldots & .. & A_\nu \end{bmatrix} \in \mathbb{R}^{(\nu+1)n\times(\nu+1)n},$$

and similarly $A_c(\tilde{\Delta})$ for the perturbed polynomial matrix $P_{\tilde{\Delta}}(z) := z^{\nu+1}I_n - \sum_{i=0}^{\nu}(A_i + \sum_{j=1}^N D_{ij}\Delta_{ij}E_{ij})z^i$, where $\tilde{\Delta} := (\Delta_0, \Delta_1, \ldots, \Delta_\nu), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in \mathbb{K}^{l\times q_{i1}} \times \ldots \times \mathbb{K}^{l\times q_{iN}}, i \in \overline{\nu}$. Then the matrix $A_c(\tilde{\Delta})$ can be represented by the following form $A_c(\tilde{\Delta}) = A_c + \sum_{j=1}^N \tilde{D}\Delta_{0j}\tilde{E}_{0j} + \sum_{j=1}^N \tilde{D}\Delta_{1j}\tilde{E}_{1j} + \ldots + \sum_{j=1}^N \tilde{D}\Delta_{\nu j}\tilde{E}_{\nu j}$, where

$$\tilde{D}_{ij} = \tilde{D} := [0, ..., 0, D^T]^T \in \mathbb{R}^{(\nu+1)n\times l};\ \tilde{E}_{0j} := [E_{0j}, 0, ..., 0] \in \mathbb{R}^{q_{0j}\times(\nu+1)n},$$

$$\tilde{E}_{1j} := [0, E_{1j}, 0, ..., 0] \in \mathbb{R}^{q_{1j}\times(\nu+1)n}, ..., \tilde{E}_{\nu j} := [0, ..., 0, E_{\nu j}] \in \mathbb{R}^{l_{\nu j}\times(\nu+1)n},$$

for every $i \in \overline{\nu},\ j \in \underline{N}$. It follows from the equality $\det P_{\tilde{\Delta}}(z) = \det\,(zI_{(\nu+1)n} - A_c(\tilde{\Delta}))$ that $\sigma((A_i + \sum_{j=1}^N D_{ij}\Delta_{ij}E_{ij})_{i\in\overline{\nu}}) = \sigma(A_c(\tilde{\Delta}))$. So, we get $r_{\mathbb{C}}^m(D(\alpha,r)) = r_{\mathbb{C}}(A_c, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i\in\overline{\nu},j\in\underline{N}}; D(\alpha,r)); r_{\mathbb{R}}^m(D(\alpha,r)) = r_{\mathbb{R}}(A_c, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i\in\overline{\nu},j\in\underline{N}}; D(\alpha,r))$. By the assumption $\alpha \leq 0,\ A_0, A_1, ..., A_{\nu-1}, (A_\nu - \alpha I_n) \in \mathbb{R}_+^{n\times n}, D_{ij} \in \mathbb{R}_+^{n\times l}, E_{ij} \in \mathbb{R}_+^{q_{ij}\times n}(i \in \overline{\nu}, j \in \underline{N})$, we have $A_c \geq \alpha I_{(\nu+1)n}$ and $\tilde{D} \in \mathbb{R}_+^{(\nu+1)n\times l},\ \tilde{E}_{ij} \in \mathbb{R}_+^{l_{ij}\times(\nu+1)n}(i \in \overline{\nu}, j \in \underline{N})$. Hence, from Theorem 2.4, we get $r_{\mathbb{C}}(A_c, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i\in\overline{\nu},j\in\underline{N}};\ D(\alpha,r)) = r_{\mathbb{R}}(A_c, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i\in\overline{\nu},j\in\underline{N}};\ D(\alpha,r)) = \frac{1}{\max_{i\in\ \overline{\nu},\ j\in\underline{N}} \|(\tilde{E}_{ij}((\alpha+r)I_{(n+1)\nu} - A_c)^{-1}\tilde{D}_{ij}\|}$. On the other hand, it is easy to check that

$$(zI_{(\nu+1)n} - A_c)^{-1}\tilde{D} = \begin{pmatrix} P(z)^{-1} \\ zP(z)^{-1} \\ . \\ . \\ . \\ z^\nu P(z)^{-1} \end{pmatrix}.$$

Therefore, $r_{\mathbb{C}}^{m}(D(\alpha,r)) = r_{\mathbb{R}}^{m}(D(\alpha,r)) = \frac{1}{\max_{i\in\overline{\nu},\, j\in\underline{N}} \|(\alpha+r)^i E_{ij} P(\alpha+r)^{-1} D_{ij}\|}$. The proof of the case of $D_{ij} \in \mathbb{R}_+^{n\times l_{ij}}, E_{ij} := E \in \mathbb{R}_+^{q\times n} (i \in \overline{\nu}, j \in \underline{N})$, can be done by a similar way. This completes our proof. □

We now turn to the problem of computing of the complex, real $\mathcal{D}$-stability radius under affine perturbations (6). For every $i \in \overline{\nu}$, let us define

$$A_i^* := \frac{1}{r^{\nu+1-i}} \left(C_\nu^{\nu-i}\alpha^{\nu-i}A_\nu + C_{\nu-1}^{\nu-1-i}\alpha^{\nu-1-i}A_{\nu-1} + ... + A_i - C_{\nu+1}^{\nu+1-i}\alpha^{\nu+1-i}I_n \right), \quad (7)$$

where $C_u^v := \frac{u!}{v!(u-v)!}$, $u, v \in \mathbb{N}, u \geq v$. The following theorem is an extension of Theorem 3.2 to the general case of $\mathcal{D} = D(\alpha, r)$.

Theorem 3.6. *Let the linear discrete time-delay system (1) be $D(\alpha,r)$-stable. Suppose the system matrices $A_i, i \in \overline{\nu}$ are subjected to affine perturbations (6), where $B_{ij} \in \mathbb{R}_+^{n\times n} (i \in \overline{\nu}, j \in \underline{N})$. If either $\alpha \leq 0$ and $A_0, A_1, ..., A_{\nu-1}, (A_\nu - \alpha I_n) \in \mathbb{R}_+^{n\times n}$, or $\alpha > 0$ and $A_i^* \in \mathbb{R}_+^{n\times n}, i \in \overline{\nu}$, then $r_{\mathbb{C}}^{a}(D(\alpha,r)) = r_{\mathbb{R}}^{a}(D(\alpha,r)) = \frac{1}{\rho(P(\alpha+r)^{-1}B)}$, where $B := \sum_{i=0}^{\nu} \left(\sum_{j=1}^{N} B_{ij}\right)(\alpha+r)^i$.*

Proof. In the case of $\alpha \leq 0$ and $A_0, A_1, ..., A_{\nu-1}, (A_\nu - \alpha I_n) \in \mathbb{R}_+^{n\times n}$, the proof is similar to that of Theorem 3.5, based on the result of Theorem 2.4(i). Then, we have $r_{\mathbb{C}}^{a}(D(\alpha,r)) = r_{\mathbb{R}}^{a}(D(\alpha,r)) = \frac{1}{\rho(P(\alpha+r)^{-1}B)}$. We now assume that $\alpha > 0$ and $A_i^* \in \mathbb{R}_+^{n\times n}, i \in \overline{\nu}$. Denote by $P^*(z) := z^{\nu+1}I_n - A_\nu^* z^\nu - ... - A_0^*$. Let $s \in \mathbb{C}, |s-\alpha| \geq r$ satisfy $\det P(s) = 0$. Setting $z = \frac{s-\alpha}{r}, |z| \geq 1$, by a direct computation, we have $\det P(s) = 0$ if and only if $\det P^*(z) = 0$. So the discrete time-delay system (1) is $D(\alpha,r)$-stable if and only if the following discrete time-delay system

$$x(k+1) = A_\nu^* x(k) + A_{\nu-1}^* x(k-1) + ... + A_0^* x(k-\nu), \quad k \in \mathbb{N},\ k \geq \nu, \quad (8)$$

is $D(0,1)$-stable. Similarly, the perturbed system

$$x(k+1) = (A_\nu + \sum_{j=1}^{N} \delta_{\nu j} B_{\nu j}) x(k) + ... + (A_0 + \sum_{j=1}^{n} \delta_{0j} B_{0j}) x(k-\nu), \quad k \in \mathbb{N},\ k \geq \nu, \quad (9)$$

is $D(\alpha,r)$-stable if and only if the following discrete time-delay system is $D(0,1)$-stable

$$x(k+1) = (A_\nu^* + B_\nu^*) x(k) + ... + (A_0^* + B_0^*) x(k-\nu), \quad k \in \mathbb{N},\ k \geq \nu. \quad (10)$$

Here, $B_i^* := \left(\sum_{j=1}^{N} \delta_{\nu j} (\frac{1}{r^{\nu+1-i}} C_\nu^{\nu-i}\alpha^{\nu-i} B_{\nu j}) + \sum_{j=1}^{N} \delta_{(\nu-1)j} (\frac{1}{r^{\nu+1-i}} C_{\nu-1}^{\nu-1-i}\alpha^{\nu-1-i} B_{(\nu-1)j}) + ... + \sum_{j=1}^{N} \delta_{ij} (\frac{1}{r^{\nu+1-i}} B_{ij}) \right)$, $i \in \overline{\nu}$. Since $B_{ij} \in \mathbb{R}_+^{n\times n}$, $(i \in \overline{\nu}, j \in \underline{N})$, we have

$$\frac{1}{r^{\nu+1-i}} C_\nu^{\nu-i}\alpha^{\nu-i} B_{\nu j},\ \frac{1}{r^{\nu+1-i}} C_{\nu-1}^{\nu-1-i}\alpha^{\nu-1-i} B_{(\nu-1)j}, ..., \frac{1}{r^{\nu+1-i}} B_{ij} \in \mathbb{R}_+^{n\times n}, \quad i \in \overline{\nu}, j \in \underline{N}.$$

It follows from Theorem 3.2 that the system (10) is $D(0,1)$-stable for every δ satisfying $\max_{i\in \overline{\nu}, j\in \underline{N}} |\delta_{ij}| < \frac{1}{\rho(P^*(1)^{-1}G)}$, where

$$G := \sum_{i=0}^{\nu} \left(\sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} C_{\nu}^{\nu-i} \alpha^{\nu-i} B_{\nu j} + \sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} C_{\nu-1}^{\nu-1-i} \alpha^{\nu-1-i} B_{(\nu-1)j} + \right.$$

$$\left. \dots + \sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} B_{ij} \right). \tag{11}$$

Hence, the perturbed system (9) is $D(\alpha, r)$-stable for every complex perturbation δ such that $\max_{i\in \overline{\nu}, j\in \underline{N}} |\delta_{ij}| < \frac{1}{\rho(P^*(1)^{-1}G)}$. By the definition of the complex $D(\alpha, r)$-stability radius of the system (1) under affine perturbations of the form (6), we get $r_{\mathbb{C}}^{a}(D(\alpha, r)) \geq \frac{1}{\rho(P^*(1)^{-1}G)}$. On the other hand, taking Remark 3.3 into account, the system (10) is not $D(0,1)$-stable if $\delta := ((\delta_{01}, \dots, \delta_{0N}); \dots; (\delta_{\nu 1}, \dots, \delta_{\nu N})) \in \mathbb{R}^{(\nu+1)N}$; $\delta_{ij} = \frac{1}{\rho(P^*(1)^{-1}G)}$ $(i \in \overline{\nu}, j \in \underline{N})$. Then the perturbed system (9) is not $D(\alpha, r)$-stable if

$$\delta := ((\delta_{01}, \dots, \delta_{0N}); \dots; (\delta_{\nu 1}, \dots, \delta_{\nu N})) \in \mathbb{R}^{(\nu+1)N};\ \delta_{ij} = \frac{1}{\rho(P^*(1)^{-1}G)} \quad (i \in \overline{\nu}, j \in \underline{N}).$$

We derive that $r_{\mathbb{R}}^{a}(D(\alpha, r)) \leq \frac{1}{\rho(P^*(1)^{-1}G)}$. So we get the following inequalities

$$\frac{1}{\rho(P^*(1)^{-1}G)} \leq r_{\mathbb{C}}^{a}(D(\alpha, r)) \leq r_{\mathbb{R}}^{a}(D(\alpha, r)) \leq \frac{1}{\rho(P^*(1)^{-1}G)}.$$

Therefore $r_{\mathbb{C}}^{a}(D(\alpha, r)) = r_{\mathbb{R}}^{a}(D(\alpha, r)) = \frac{1}{\rho(P^*(1)^{-1}G)}$. Finally, by a direct computation, we get $P^*(1)^{-1}G = P(\alpha + r)^{-1}B$. This completes our proof. $\square$

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