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ON A PERTURBED SYSTEM OF CHEMOTAXIS

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1. Introduction

This is a joint work with Takasi Senba (Miyazaki University) and Takashi Suzuki (Osaka University). We consider the following parabolic-elliptic system describing chemotactic aggregation of the slime molds:

$$
\text{(CZ)} \quad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v), & (x,t) \in \Omega \times (0,T), \\ 0 = \Delta v - \gamma v - \beta |v|^{p-1} v + \alpha u, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$ and $T > 0$. Here $u(x,t)$ is the cell density of the cellular slime molds and $v(x,t)$ is the concentration of the chemical substance at place x and time t, respectively. χ, α, β, γ are positive constants and $1 < p < \infty$. ν denotes the unit outward normal vector to $\partial\Omega$. The term $F = \nabla u - \chi u \nabla v$ is the flux of u so that effect of diffusion $\nabla \cdot \nabla u$ and that of chemotaxis $\chi \nabla \cdot (u \nabla v)$ are competing for u to vary.

For the problem (CZ), in the case of $\beta = 0$, Nagai [6] has confirmed the conjectures of Childress and Percus [4], which is chemotaxis collaps can occur if a total cell number on $\Omega \subset \mathbb{R}^2$ is larger than a critical number, but can not occur for the total cell number on Ω less than it, and he find the critical number is $\frac{8\pi}{\alpha\chi}$. While Senba and Suzuki [11] have made clear the blowup mechanism.

Here we study the blowup solution to the simplest Keller-Segel model (CZ) in the case of $\beta \neq 0$. For the initial function u_0 we suppose

1. $u_0 \geq 0$ and u_0 is not identical to 0 on Ω,
2. u_0 is smooth on $\bar{\Omega}$.

H. Chen and X.H. Zhong [2] has shown that the system (CZ) has a unique classcal positive solution (u,v), $(x,t) \in \Omega \times [0, T_{\max})$. under the assumptions that $1 < p < +\infty$ for spatial dimension $N = 2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, where $T_{\max} = \sup\{T > 0; (u,v) \text{ exists for } x \in \Omega, t \in [0,T)\}$ denotes the maximal time of existence for the solution of (CZ). And they [3] obtained that if $\|u_0\|_{L^1} = \lambda < \frac{4\pi}{\alpha\chi}$ then $T_{\max} = +\infty$ and $\|u(t)\|_\infty < C$. Moreover they showed the critical number is $\frac{8\pi}{\alpha\chi}$, which determines occurence of blowup in case that u_0 is radially symmetric.

In this paper we show the blowup results of the problem. Henceforth we can assume that $\chi = \alpha = \beta = \gamma = 1$ without loss of generality. The main theorem is as follows. The first theorem justifies the teminology blowup.

Theorem 1. *If $T_{\max} < \infty$, then*

$$\lim_{t \nearrow T_{\max}} \|u(t)\|_\infty = \infty. \tag{1}$$

Regarding this, we define the blowup set $\mathcal{B}$ of u

$$\begin{aligned} \mathcal{B} = \{x_0 \in \bar{\Omega} : \text{ ther exist } t_k \nearrow T_{\max} \text{ and } x_k \to x_0 \\ \text{such that } u(x_k, t_k) \to \infty \text{ as } k \to \infty\} \end{aligned} \tag{2}$$

and call each $x_0 \in \mathcal{B}$ a blowup point. Condition $T_{\max} < +\infty$ implies $\mathcal{B} \neq \emptyset$, but more importantly, the finiteness of blowup point follows.

Theorem 2. *If $T_{\max} < \infty$, then $\sharp\mathcal{B}$ is finite.*

Remark 3. Keller and Segel(1970) discussed the iniation of cell aggregation as instability of the spatially homogeneus steady state. As concerned dynamics aspects of solutions, Nanjundiah [8] has posed a conjecture that cell density $u(x, t)$ will blow up in a finite time and form a δ-function singularity. Such a result is estanblished in [11].We expect a similar blowup mechanism for (CZ) system.

2. Preliminaries

First of all we recall the Gagliard-Nirenberg inequality in two dimensional cace;

$$\|w\|_{L^2}^2 \leq K^2(\|w\|_{L^2}^2 + \|w\|_{L^1}^2), \qquad w \in W^{1,1}(\Omega), \tag{3}$$

where K is a constant determined by Ω.

In this part we shall show some inequalities (3) for later use. Henceforth, we set $B_R(x_0) = \{x \in \mathbf{R}^2 : |x - x_0| < R\}$. We introduce the cut-off function φ satisfying

$$0 \leq \varphi \leq 1 \quad \text{in } \mathbf{R}^2, \qquad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{4}$$

Given $x_0 \in \Omega$, we have $0 < R' < R$ with $B_{2R}(x_0) \subset \Omega$. Then we take φ satisfying

$$\varphi_{x_0,R',R}(x) = \begin{cases} 1 & (x \in B_{R'(x_0)}) \\ 0 & (x \in \mathbf{R}^2 \backslash B_R(x_0)). \end{cases} \tag{5}$$

Given $x_0 \in \partial\Omega$, we take a smooth conformal mapping $X : B_{2R}(x_0) \cap \bar{\Omega} \mapsto \mathbf{R}^2$ satisfying $x_0 \to 0$ and

$$\begin{aligned} &X(B_{2R}(x_0) \cap \Omega) \subset \{(x_1, x_2) : x_2 > 0\} \\ &X(B_{2R}(x_0) \cap \partial\Omega) \subset \{(x_1, x_2) : x_2 = 0\} \\ &X(B_{R'}(x_0) \cap \Omega) \subset B_{\frac{1}{2}}(0), \quad X(\Omega \backslash B_R(x_0)) \subset \mathbf{R}^2 \backslash B_1(x_0) \end{aligned}$$

for $0 < R' \ll 1$. Then we have set $\varphi = \zeta(X(x))$. It holds that

$$\frac{\partial}{\partial \nu} \zeta \circ X = \frac{\partial X}{\partial \nu} \cdot (\nabla \zeta \circ X) = 0 \quad \text{on } \partial\Omega,$$

because $\frac{\partial X}{\partial \nu}$ is proportional to $(0,1)$ on $\partial\Omega$, and such a φ satisfies (4) and (5). Then, $\psi = \left(\varphi_{x_0,R',R}\right)^6$ satisfies

$$\psi(x) = \begin{cases} 1 & (x \in B_{R'(x_0)}) \\ 0 & (x \in \mathbf{R}^2 \backslash B_R(x_0)), \end{cases} \tag{6}$$

$$0 \le \psi \le 1 \quad \text{in } \mathbf{R}^2, \qquad \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega.$$

And it holds that

$$|\nabla \psi| \le A\psi^{\frac{5}{6}}, \qquad |\Delta \psi| \le B\psi^{\frac{2}{3}}, \tag{7}$$

where $A > 0$, $B > 0$ are constants determined by $0 < R' < 1 \ll 1$.

For the estimate of u, we have the following lemma from Gagliard-Nirenberg inequality and the characteristic of the cut-off function ψ.

Lemma 4. [11] *The follwing inequalities hold for any $s > 1$, where $C > 0$ is a constant:*

$$\int_\Omega u^2\psi dx \le 2K^2 \int_{B_R(x_0)\cap\Omega} udx \int_\Omega u^{-1}|\nabla u|^2 \psi dx + K^2(\frac{A^2}{2} + 1)\|u\|_{L^1}^2 \tag{8}$$

$$\int_\Omega u^2 dx \le \frac{2K^2}{\log s} \int_\Omega (u\log u + e^{-1}dx \int_\Omega u^{-1}|\nabla u|^2 dx + 2K^2\|u\|_{L^1}^2 + 3s^2|\Omega| \tag{9}$$

$$\begin{aligned} \int_\Omega u^3\psi dx \le & \frac{72K^2}{\log s} \int_{B_R(x_0)\cap\Omega} (u\log u + e^{-1})dx \int_\Omega |\nabla u|^2 \psi dx \\ & + C\|u\|_{L^1(B_R(x_0)\cap\Omega)}^3 + 10|\Omega|s^3. \end{aligned} \tag{10}$$

We can obtain the estimate of v. From $u > 0$, $v > 0$ and the sceond equation, we get

$$\|v\|_{L^1} + \|v\|_{L^p}^p = \|u\|_{L^1}. \tag{11}$$

We rewrite the second equation of v

$$-\Delta v + v = h \text{ in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{12}$$

by putting $h = u - |v|^{p-1}v$. From (11) we find

$$\|h\|_{L^1} \le \|u\|_{L^1} + \|v\|_{L^p}^p \le 2\|u\|_{L^1}. \tag{13}$$

Then the L^1 estimate (H.Brezis and W.Strauss [1]) to the second equation of (CZ) gives

$$\sup_{0\le t\le t_{\max}} \left\{\|v(t)\|_{W^{1,q}(\Omega)}\| + \|v\|_r\right\} < C(r)\|u\|_{L^1} \tag{14}$$

for $q \in [1,2)$ and $r \in [1,\infty)$.

3. Characterization of blowup point

Henceforth, we always assume that $T_{\max} < \infty$ and $\mathcal{B}$ denotes the blowup points. By using the estimate (14) and the first equation of (CZ), we lead to the following lemma.

Lemma 5. *$x_0 \in \Omega$ is a blowup point of u if and only if*

$$\limsup_{t\nearrow T_{\max}} \int_{B_R(x_0)\cap\Omega} u\log udx = +\infty \tag{15}$$

for $R > 0$ sufficiently small.

Proof. We start to prove the 'if' part. Let be $0 < R \ll 1$ and $\psi = (\varphi_{x_0,R',R})^6$. We assume that (15) holds. Multiplying the first equation of (CZ) by $u\psi$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi dx + \int_\Omega |\nabla u|^2\psi dx + \int_\Omega u\nabla u\cdot\nabla\psi dx = \int_\Omega u\psi\nabla v\cdot\nabla u dx + \int_\Omega u^2\nabla v\cdot\nabla\psi dx. \tag{16}$$

The first integral term of the right hand side in (16) is

$$\int_\Omega u\psi\nabla u\cdot\nabla v dx = -\frac{1}{2}\int_\Omega u^2\Delta v\cdot\psi dx - \frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi dx. \tag{17}$$

By the second equation of (CZ), the equation (17) is treated to get

$$\begin{aligned}\int_\Omega u\psi\nabla u\cdot\nabla v dx &= \frac{1}{2}\int_\Omega u^2 v\psi dx - \frac{1}{2}\int_\Omega u^2|v|^{p-1}v dx + \frac{1}{2}\int_\Omega u^3\psi dx - \frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi dx\\ &\le \frac{1}{2}\int_\Omega u^3\psi dx - \frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi dx\\ &= \frac{1}{2}\int_\Omega u^3\psi dx + \frac{1}{2}\int_\Omega v\nabla(u^2)\psi dx + \frac{1}{2}\int_\Omega u^2 v\Delta\psi dx. \end{aligned}\tag{18}$$

On the other hand, about the second integral term of right side in (16) we get

$$\int_\Omega u^2\nabla\psi\cdot\nabla v dx = -\int_\Omega v\nabla(u^2)\cdot\nabla\psi dx - \int_\Omega u^2 v\Delta\psi dx. \tag{19}$$

Therefore we have

(20)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi dx + \int_\Omega |\nabla u|^2\psi dx \int_\Omega u\nabla u\cdot\nabla\psi dx = \frac{1}{2}\int_\Omega u^3\psi dx - \frac{1}{2}\int_\Omega v\nabla(u^2)\nabla\psi dx - \int_\Omega u^2 v\Delta\psi dx.$$

By using Young's inequality and the estimate of ψ, we have

$$\frac{1}{2}\left|\int_\Omega u^2 v\Delta\psi dx\right| \le \frac{1}{3}\int_\Omega u^3\psi dx + \frac{B^3}{6}\|v\|_3^3, \tag{21}$$

$$\left|\int_\Omega u\nabla u\cdot\nabla\psi dx\right| \le \frac{1}{4}\int_\Omega |\nabla u|^2\psi dx + \frac{1}{3}\int_\Omega u^3\psi dx + \frac{4A^6}{3}|\Omega|, \tag{22}$$

and

$$\frac{1}{2}\left|\int_\Omega v\nabla u^2\cdot\nabla\psi dx\right| \le \frac{1}{4}\int_\Omega |\nabla u|^2\psi dx + \frac{1}{3}\int_\Omega u^3\psi dx + \frac{A^6}{48}\||v\|_6^6. \tag{23}$$

From (20)-(23) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi dx + \frac{1}{2}\int_\Omega |\nabla u|^2\psi dx \le \frac{3}{2}\int_\Omega u^3\psi dx + C_1. \tag{24}$$

From (10) of Lemma 4 with $s \ll 1$, we have

$$\frac{d}{dt}\int_\Omega u^2\psi dx + \frac{1}{2}\int_\Omega |\nabla u|^2\psi dx \le C_1. \tag{25}$$

This implies

$$\sup_{0\le t<T_{\max}}\int_\Omega u^2\psi dx \le +\infty. \tag{26}$$

In similar way, multiplying the first equation of (CZ) by $u^2\psi$ and integrating by parts, we have

$$\frac{d}{dt}\int_\Omega w^2\psi dx + \frac{5}{3}\int_\Omega |\nabla w|^2\psi dx \le 4\int_\Omega w^3\psi dx + C \tag{27}$$

for $w = u^{\frac{3}{2}}$. In particular, we have

$$\sup_{0\le t<T_{\max}} \int_{B_{R'}(x_0)\cap\Omega} w \log w dx \le +\infty,$$
$$\sup_{0\le t<T_{\max}} \|w\|_{L^1(B_{R'}(x_0)\cap\Omega)} \le +\infty.$$

Therefore, taking $R'' \in (0,R')$, we can apply the argument with u, R, and $\psi = (\varphi_{x_0,R',R})^6$, replacing by w, R', and $\psi_1 = (\varphi_{x_0,R'',R'})^6$, respectively. Similar to (26) it follows that

$$\sup_{0\le t<T_{\max}} \|w\|^{\frac{2}{3}}_{L^2(B_r(x_0)\cap\Omega)} = \sup_{0\le t<T_{\max}} \|u\|_{L^3(B_r(x_0)\cap\Omega)} < +\infty$$

for any $r \in (0,R)$, because $R' \in (0,R)$ and $R'' \in (0,R')$ are arbitrary. From second equation of (CZ) this implies $\sup_{0\le t<T_{\max}} \|v\|_{W^{2,3}(B_{r'}(x_0)\cap\Omega)} < +\infty$ for $r' \in (0,r)$. Therefore

$$\sup_{0\le t<T_{\max}} \|v\|_{C^1(B_r(x_0)\cap\Omega)} < +\infty \tag{28}$$

holds for any $r \in (0,R)$. Repeating the argument once more, we have

$$\sup_{0\le t<T_{\max}} \|u\|_{L^4(B_r(x_0)\cap\Omega)} < +\infty. \tag{29}$$

Next we take $r' \in (0,R)$ and put $\psi_1 = (\varphi_{x_0,r',r})^6$. For $p \ge 1$ we multiply the first equation of (CZ) by $u^p\psi_1^{p+1}$ and get

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega (u\psi_1)^{p+1}dx = -\int_\Omega \nabla(u^p\psi_1^{p+1})\cdot\nabla u dx + \int_\Omega u\nabla(u^p\psi_1^{p+1})\cdot\nabla v dx = -I + II.$$

Here we have

$$\begin{aligned} I &= \int_\Omega \left(pu^{p-1}\psi_1^{p+1}\nabla u + u^p\nabla\psi_1^{p+1}\right)\cdot\nabla u dx \\ &\ge \frac{2}{p+1}\int_\Omega \left|\nabla(u\psi_1)^{\frac{p+1}{2}}\right|^2 dx - \frac{A^2(p+1)}{2}\|u_0\|^{\frac{1}{3}}_{L^1(\Omega)}\left(\int_\Omega (u\psi_1)^{1+\frac{2}{3}p}\right)^{\frac{2}{3}}. \end{aligned}$$

On the other hand, estimate (28) means that

$$L \equiv \sup_{0\le t<T_{\max}} \|\nabla v\|_{L^\infty(B_r(x_0)\cap\Omega)} < +\infty.$$

We obtain

$$II \le \frac{1}{p+1}\int_\Omega \left|\nabla(u\psi_1)^{\frac{p+1}{2}}\right|^2 + 4L^2(p+1)\int_\Omega (u\psi_1)^{p+1}dx + LA(p+1)\|u_0\|^{\frac{1}{6}}_{L^1(\Omega)}\left(\int_\Omega (u\psi_1)^{1+\frac{6}{5}p}dx\right)^{\frac{5}{6}}.$$

It holds that

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u_1)^{p+1}dx &\le -\int_\Omega \left|\nabla(u_1)^{\frac{p+1}{2}}\right|^2 dx \\ &+ C_3(p+1)^2\int_\Omega u_1^{p+1}dx + C_3(p+1)^2\left(\left(\int_\Omega u_1^{1+\frac{3}{2}p}\right)^{\frac{2}{3}} + \left(\int_\Omega u_1^{1+\frac{6}{5}p}\right)^{\frac{5}{6}}\right), \end{aligned} \tag{30}$$

where $u_1 = u\psi_1$. Here, $C_3 > 0$ is independent of $p \ge 1$ and we can apply an iteration sheme of Moser's type. As a result, we have

$$\sup_{0\le t<T_{\max}} \|u_1\|_{L^\infty(\Omega)} \le C\max\left\{\left(\sup_{0\le t<T_{\max}} \|u_1\|^4_{L^4(\Omega)} + 1\right)^{\frac{1}{4}}, d\right\} \tag{31}$$

where $d = \|u_0\|_{L^\infty(\Omega)} + 1$. By using (29), we obtain

$$\sup_{0\le t<T_{\max}} \|u_1\|_{L^\infty(\Omega)} = \sup_{0\le t<T_{\max}} \|u\psi_1\|_{L^\infty(\Omega)} < +\infty, \tag{32}$$

or $\limsup_{t\nearrow T_{\max}} \|u\|_{L^\infty(B_{r'}(x_0)\cap\Omega)} < +\infty$. This means $x_0 \notin \mathcal{B}$.

On the other hand, the " only if "part is clear, because $x_0 \notin \mathcal{B}$ implies (15) for $0 < R \ll 1$ by the definition. The proof of Lemma 5 is complete.

□

4. Proof of Theorem 1

The global version of Lemma 5 is expressed as follows:

$$\limsup_{t\nearrow T} \int_\Omega u \log u dx < +\infty \tag{33}$$

implies

$$\limsup_{t\nearrow T} \|u\|_\infty < +\infty. \tag{34}$$

In fact, this is proven just by replacing the cut-off function φ with the constant function 1. If (34) follows, then general theory of parabolic equation yields that the solution u is continued after $t = T$. We shall show that (33) follows from

$$\liminf_{t\nearrow T} \int_\Omega u \log u dx < +\infty. \tag{35}$$

Then $t_{\max} < \infty$ holds only if

$$\liminf_{t\nearrow T} \int_\Omega u \log u dx = +\infty. \tag{36}$$

And in particular relation

$$\lim_{t\nearrow T} \|u\|_\infty = +\infty \tag{37}$$

follows.

Next we multiply $\log u$ by the first equation of (CZ). By using the second equation of (CZ), we have

$$\frac{d}{dt}\int_\Omega u \log u dx + \int_\Omega u^{-1}|\nabla u|^2 dx + \int_\Omega uv = \int_\Omega u^2 dx. \tag{38}$$

The right hand side is dominated by the second inequality (9) of Lemma 4. It follows that

$$\frac{d}{dt}\int_\Omega u \log u dx + \left(1 - \frac{2K^2}{\log s}\int_\Omega (u \log u + e^{-1}) dx\right)\int_\Omega u^{-1}|\nabla u|^2 dx \le C\|u_0\|_{L^1}^2 + 3s^2|\Omega|. \tag{39}$$

Taking $s = s(t) = \exp\left(2K^2 \int_\Omega u \log u + e^{-1}\right) dx > 1$, we obtain

$$\frac{dJ}{dt} \le C\|u_0\|_{L^1}^2 + 3|\Omega| \exp\left(4K^2 J\right), \tag{40}$$

where $J = \int_\Omega (u \log u + e^{-1}) dx$. Inequlity (40) and $\liminf_{t\nearrow T} J(t) < \infty$ implies $\limsup_{t\nearrow T} J(t) < \infty$ by the comparison theorem for ordinary differential equation. In particular, inequality (36) implies (33). The proof is complete.

5. Proof of Theorem 2

In this part we show the finiteness of blowup points. Given $x_0 \in \bar{\Omega}$, we take $0 < R' <\ll 1$ and set $\psi = \left(\psi_{x_0,R',R}\right)^6$. Let $G = G(x,y)$ be the Green's function of the operator $\mathcal{L}+1$, so that it solves

$$(-\Delta_y + 1)G = \delta(y-x) \qquad (y \in \Omega)$$

with $\frac{\partial}{\partial \nu_y} G = 0 \quad (y \in \partial\Omega)$ for $x \in \Omega$. From the elliptic regularity, it is extended to a smooth function on $\bar{\Omega}\times\bar{\Omega}\setminus\{(x,x) : x \in \bar{\Omega}\}$. Also the symmetry $G(x,y) = G(y,x)$ follows. Here we have the following.

Lemma 6. [11] *The function* $\rho(x,y) = \nabla\psi(x)\nabla_x G(x,y) + \nabla\psi(y)\nabla_y G(x,y)$ *belongs to* $L^\infty(\Omega \times \Omega)$.

And we have

Lemma 7. [11] *It holds that*

$$\frac{d}{dt}\int_\Omega (u \log u)\psi dx + \frac{1}{4}\int_\Omega u^{-1}|\nabla u|^2 \psi dx \le 2\int_\Omega u^2 \psi dx + C_6. \tag{41}$$

We start to prove Theorem 2. There is $\varepsilon_0 > 0$ such that any $x_0 \in \mathcal{B}$ and $0 < R \ll 1$ admit the estimate

$$\limsup_{t\nearrow T_{\max}} \int_{B_R(x_0)\cap\Omega} u dx \ge \varepsilon_0. \tag{42}$$

Take $R' \in (0,R)$ and set $\psi = \left(\varphi_{x_0,R',R}\right)^6$. From (41) and the first inequlity (8) of Lemma 4, we have

$$\frac{d}{dt}\int_\Omega (u\log u)\psi dx + \frac{1}{4}\left(1 - 16K^2 \int_{B_R(x_0)\cap\Omega} u dx\right)\int_\Omega u^{-1}|\nabla u|^2 \psi dx \le C_7. \tag{43}$$

Therefore, if $\limsup_{t\nearrow T_{\max}} \int_{B_R(x_0)\cap\Omega} udx < \varepsilon_0 \equiv \frac{1}{16K^2}$, then

$$\limsup_{t\nearrow T_{\max}} \int_{B_{R'}(x_0)\cap\Omega} u \log u dx < \limsup_{t\nearrow T_{\max}} \int_\Omega (u\log u)\psi dx < +\infty. \tag{44}$$

This implies $x_0 \notin \mathcal{B}$ by Lemma 5, that is a contradiction.

Next we show that

$$\left|\frac{d}{dt}\int_\Omega u\psi dx\right| < B\|u_0\|_{L^1} + \frac{1}{2}\|\rho\|_{L^\infty(\Omega\times\Omega)}\|u_0\|_{L^1}^2. \tag{45}$$

The first equation of (CZ) gives

$$\frac{d}{dt}\int_\Omega u\psi dx = \int_\Omega u\Delta\psi dx + \int_\Omega u\nabla v \cdot \nabla\psi dx. \tag{46}$$

The second integral term of the right side in (46) is

$$\begin{aligned} \int_\Omega u\nabla v\cdot\nabla\psi dx &= \int_\Omega\int_\Omega u(x,t)\nabla\psi(x)\cdot\nabla_x G(x,y)u(y,t)dydx \\ &= \frac{1}{2}\int_\Omega\int_\Omega \rho(x,y)u(x,t)u(y,t)dxdy. \end{aligned} \tag{47}$$

From Lemma 6 we have

$$\left|\int_\Omega\int_\Omega \rho(x,y)u(x,t)u(y,t)dxdy\right| \leq \|\rho\|_{\infty(\Omega\times\Omega)}\|u_0\|_{L^1}^2.$$

Since, it is obvious that $\int_\Omega u\Delta\psi dx \leq B\|u_0\|_{L^1}$, we get inequlity (45). This means that the value

$$\lim_{t\nearrow T_{\max}}\int_\Omega u\psi dx = \int_\Omega u_0(x)\psi dx + \int_0^{T_{\max}}\left(\frac{d}{dt}\int_\Omega u(\cdot,t)\psi dx\right)dt \tag{48}$$

exists. Because $0 < R \ll 1$ is arbitrary, (4) and inequality (42) are improved as

$$\liminf_{t\nearrow T_{\max}}\int_{B_R(x_0)\cap\Omega} udx \geq \lim_{t\nearrow T_{\max}}\int_\Omega u\psi dx \geq \limsup_{t\nearrow T_{\max}}\int_{B_{R'}(x_0)\cap\Omega} udx \geq \varepsilon_0. \tag{49}$$

Therefore, by using the L^1 norm preserving $\|u\|_{L^1} = \|u_0\|_{L^1}(\quad 0 \leq t \leq T_{\max})$, we conclude

$$\sharp\mathcal{B} \leq \frac{\|u_0\|_{L^1}}{\varepsilon_0} < \infty. \tag{50}$$

The proof of Theorem 2 is complete.

References

[1] H. Brezis and W. A. Strauss, Semilinear second-order elliptic equation in L^1, J. Math. Soc. Japan, **25** (1973), 565–590.

[2] Chen Hua and Zhong Xin-Hua, Existence and uniqueness of solutions to nonlinear parabolic-elliptic type chemotaxis system, Preprint.

[3] Chen Hua and Zhong Xin-Hua, Global existence and blow-up for the solutions to nonlinear parabolic-elliptic system modelling chemotaxis, IMA J. Appl. Math., **70** (2005), 221–240.

[4] S. Childress and J. K. Percus, Nonlinear aspects of chemotaxis, Math. Biosci., **56** (1981), 217–237.

[5] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., **26** (1970), 399–415.

[6] Toshitaka Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., **5** (1995), 581–601.

[7] Toshitaka Nagai, Blow-up of nonradially solutions to parabolic-elliptic systems modelling chemotaxis in two-dimensional domains, J. Inequal. Appl., **6** (2001), 37-55.

[8] V. Nanjundiah, Chemotaxis, single relaying, and aggregation morphology, J. Theor. Biol., **42** (1973), 63–105.

[9] Takasi Senba and Takashi Suzuki, *Applied analysis. Mathematical methods in natural science*, Imperial College Press, London, 2004.

[10] Takasi Senba and Takashi Suzuki, Time global solutions to a parabolic-elliptic system modelling chemotaxis, Asymptot. Anal. 32 (2002), no. 1, 63–89.

[11] Takasi Senba and Takashi Suzuki, Chemotactic collapse in a parabolic-elliptic system of mathematical biology, Adv. Differential Equations 6 (2001), no. 1, 21–50.

[12] Toshitaka Nagai and Takasi Senba, Global existence and blow-up of radial solutions to a parabolic-elliptic system of chemotaxis, Adv. Math. Sci. Appl. 8 (1998), no. 1, 145–156.