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Representations of p'-valenced schemes

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1 Introduction

In group representation theory, if a block has a cyclic defect group, then many things are well understood. The structure of such a block is described by a tree, so called a Brauer tree. In this talk, we try to generalize the theory to association schemes, but it seems to be very hard. So we show some results on this problem under many strong assumptions.

First of all, we note that we cannot define something like a defect group for a block of an association scheme. So we only consider the case of defect 1. Theory of a block of defect 1 in group representation theory was considered by Richard Brauer in [3]. In 2004, Professor Katsuhiro Uno said to me that the arguments in [3] might be generalized to the theory of association schemes, and we are trying to do it. A book by Goldschmidt [6] is also a good reference. Some modern articles and text books, for example [1], [2], [5], are not good for us, since they use deep results in group representation theory or group theory. A block of defect 0 is also in our interest. For the theory of blocks of defect 0, see [4].

Again we note that we do not have a good definition of "defect" for a block of an association scheme. So we want to consider the condition for a block such that the block is (similar to) a Brauer tree algebra. A Brauer tree algebra is a symmetric algebra, but the adjacency algebra of an association scheme need not be a symmetric algebra. Therefore we consider p'-valenced schemes. It is known that the adjacency algebra of a p'-valenced scheme over a field of characteristic p is a symmetric algebra.

2 Definitions and basic properties

We use the notations and terminologies in Zieschang's book [10]. Let X be a finite set, G a collection of non-empty subsets of $X \times X$. For $g \in G$, we define the *adjacency matrix* $\sigma_g \in Mat_X(\mathbb{Z})$ by $(\sigma_g)_{xy} = 1$ if $(x, y) \in g$, and 0 otherwise.

(X, G) is called an *association scheme* if

(1) $X \times X = \bigcup_{g \in G} g$ (disjoint),

(2) $1 := \{(x, x) \mid x \in X\} \in G$,

(3) if $g \in G$, then $g^* := \{(y, x) \mid (x, y) \in g\} \in G$,

(4) and $\sigma_f \sigma_g = \sum_{h \in G} p_{fg}^h \sigma_h$ for some $p_{fg}^h \in \mathbb{Z}$.

Then every row (column) of σ_g contains exactly $n_g := p_{gg^*}^1$ ones. We call n_g the *valency* of $g \in G$. An association scheme (X, G) is said to be *p'-valenced* if every valency is a p'-number.

Define

$$\mathbb{Z}G = \bigoplus_{g \in G} \mathbb{Z}\sigma_g \subset Mat_X(\mathbb{Z}),$$

then $\mathbb{Z}G$ is a $\mathbb{Z}$-algebra. For a commutative ring R with unity, we define

$$RG = R \otimes_{\mathbb{Z}} \mathbb{Z}G$$

and call this the *adjacency algebra* of (X, G) over R. We say that (X, G) is *commutative* if $\mathbb{Z}G$ is a commutative ring. The followings are known.

(1) [10, Theorem 4.1.3] If K is a field of characteristic zero, then KG is separable (semisimple).

(2) [8, Corollary 4.3] If F is a field of characteristic $p > 0$ and (X, G) is p'-valenced, then FG is a symmetric algebra.

We say that a field K is a *splitting field* of (X, G) if K is a splitting field of $\mathbb{Q}G$, namely $\mathrm{char}K = 0$ and KG is isomorphic to a direct sum of full matrix algebras over K. For an association scheme (X, G), there exists a finite Galois extension K of $\mathbb{Q}$ which is a splitting field of (X, G). We fix such K and denote the ring of integers in K by $\mathcal{O}$. Let p be a (rational) prime number, $\mathfrak{P}$ a prime ideal of $\mathcal{O}$ lying above $p\mathbb{Z}$. The *inertia group* T of $\mathfrak{P}$ is defined by

$$T = \{\tau \in Gal(K/\mathbb{Q}) \mid a - a^{\tau} \in \mathfrak{P} \quad \forall a \in \mathcal{O}\}.$$

We call the corresponding subfield of K the *inertia field* of $\mathfrak{P}$ and denote it by L. We denote $\mathcal{O}_L$ for the ring of integers in L, and $\mathfrak{p}$ for the unique prime ideal of $\mathcal{O}_L$ lying below $\mathfrak{P}$. It is known that $\mathfrak{p}$ is unramified in $L/\mathbb{Q}$, namely $p \notin \mathfrak{p}^2$. Let $\mathcal{O}_{\mathfrak{P}}$ be the localization of $\mathcal{O}$ by $\mathfrak{P}$. Put $F = \mathcal{O}_{\mathfrak{P}}/\mathfrak{P}\mathcal{O}_{\mathfrak{P}} \cong \mathcal{O}/\mathfrak{P}$, a field of characteristic p. We also suppose F is large enough. For $\alpha \in \mathcal{O}_{\mathfrak{P}}$, we denote $\alpha^* \in F$ for the image of the natural epimorphism $\mathcal{O}_{\mathfrak{P}} \to F$.

We denote the set of all irreducible characters of KG and FG by $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$, respectively. Note that $\mathrm{IBr}(G)$ denotes the set of irreducible modular characters, not Brauer characters. Brauer charcters are not defined for association schemes.

Let γ be the standard character, namely the character of the representation $\sigma_g \mapsto \sigma_g$. For $\chi \in \mathrm{Irr}(G)$, we denote m_χ for the multiplicity of χ in γ and call it the *multiplicity* of χ.

An indecomposable direct summand B of $\mathcal{O}_{\mathfrak{P}}G$ as a two-sided ideal is called a $\mathfrak{P}$-*block* of (X, G). Then there exists a central primitive idempotent e_B of $\mathcal{O}_{\mathfrak{P}}G$ such that $e_B\mathcal{O}_{\mathfrak{P}}G = B$. We say $\chi \in \mathrm{Irr}(G)$ belongs to a $\mathfrak{P}$-block B if $\chi(e_B) \neq 0$, and denote $\mathrm{Irr}(B)$ for the set of irreducible ordinary characters belonging to B. It is known that

$$e_B = \sum_{\chi \in \mathrm{Irr}(B)} e_\chi,$$

where $e_\chi = \frac{m_\chi}{n_G} \sum_{g \in G} \frac{1}{n_g} \chi(\sigma_{g^*})\sigma_g$. Also $\mathrm{Irr}(B)$ is a minimal subset S of $\mathrm{Irr}(G)$ such that $\sum_{\chi \in S} e_\chi \in \mathcal{O}_{\mathfrak{P}}G$.

Let Ψ be a matrix representation affording $\chi \in \mathrm{Irr}(G)$. We can suppose $\Psi(\sigma_g) \in Mat_{\chi(1)}(\mathcal{O}_{\mathfrak{P}})$ for every $g \in G$. Then we obtain a representation Ψ^* of FG. Consider the irreducible constituents of Ψ^* and denote the multiplicity of an irreducible modular character φ in Ψ^* by $d_{\chi\varphi}$. We call $d_{\chi\varphi}$ the *decomposition number* and the matrix $D = (d_{\chi\varphi})$ the *decomposition matrix.*

We say that $\varphi \in \mathrm{IBr}(G)$ belongs to a block B if there exists $\chi \in \mathrm{Irr}(B)$ such that $d_{\chi\varphi} \neq 0$. Then φ belongs to the only one block. We denote $\mathrm{IBr}(B)$ for the set of irreducible modular characters belonging to B. If $\chi \in \mathrm{Irr}(B)$, $\varphi \in \mathrm{IBr}(B')$, and $B \neq B'$, then $d_{\chi\varphi} = 0$. So we can consider the decomposition matrix D_B of a block B. Let Ψ be a matrix representation affording $\chi \in \mathrm{Irr}(G)$ such that $\Psi(\sigma_g) \in Mat_{\chi(1)}(\mathcal{O}_{\mathfrak{P}})$ for every $g \in G$ as before. For $\tau \in Gal(K/\mathbb{Q})$, we can define a representation Ψ^τ by $\Psi^\tau(\sigma_g) = \Psi(\sigma_g)^\tau$ (entry-wise action), and denote its character by χ^τ.

In general, χ and χ^τ may belong to different blocks. But if $\tau \in Gal(K/L)$, L is the inertia field of $\mathfrak{P}$, then they belong to the same block. We say that two irreducible ordinary characters are $\mathfrak{P}$-*conjugate* if they are conjugate by the action of the inertia group $Gal(K/L)$. Now $\mathrm{Irr}(B)$ is a disjoint union of some $\mathfrak{P}$-conjugate classes. We denote the size of the $\mathfrak{P}$-conjugate class containing χ by r_χ. We denote ν_p for the $\mathfrak{P}$-valuation on K such that $\nu_p(p) = 1$. Namely, if $p\mathcal{O}_{\mathfrak{P}} = \mathfrak{P}^e\mathcal{O}_{\mathfrak{P}}$ and $\alpha\mathcal{O}_{\mathfrak{P}} = \mathfrak{P}^f\mathcal{O}_{\mathfrak{P}}$, then $\nu_p(\alpha) = f/e$.

3 Questions

Let (X, G) be a p'-valenced scheme, B a $\mathfrak{P}$-block of (X, G) having an irreucible ordinary character χ such that $\nu_p(m_\chi)+1 = \nu_p(|X|)$. We think such a block is similar to that of defect 1 in group representation theory. We will consider the following questions, and give some partial results in the later section.

(1) For $\chi \in \mathrm{Irr}(B)$ and $\varphi \in \mathrm{IBr}(G)$, is it true that $d_{\chi\varphi} = 0$ or 1?

(2) For $\varphi \in \mathrm{IBr}(B)$, is it true that

$$\sharp\{\chi \in \mathrm{Irr}(B) \mid d_{\chi\varphi} \geq 1\}/(\mathfrak{P}\text{-conjugate}) = 2 \ \ ?$$

(3) If (2) is true, then we can define a graph by decomposition numbers. Is the graph a tree ?

(4) Is it true that there exists at most one exceptional vertex ? Namely, is there at most one $\mathfrak{P}$-conjugate class of irreducible characters in $\mathrm{Irr}(B)$ whose size is greater than one ?

(5) Does B^* have finite representation type ? Is it a Brauer tree algebra ?

4 Blocks of defect 0

In group representation theory, "defect 0" means the block over a field of characteristic p is a simple algebra. In the following, we suppose B is a block of an association scheme (X, G) and $\chi \in \mathrm{Irr}(B)$.

Proposition 4.1. *Let (X, G) be a p'-valenced scheme. If $\nu_p(m_\chi) \geq \nu_p(|X|)$, then $\nu_p(m_\chi) = \nu_p(|X|)$, $\mathrm{Irr}(B) = \{\chi\}$, χ^* is irreducible, and $\mathrm{IBr}(B) = \{\chi^*\}$.*

Proposition 4.2. *Let (X, G) be a p'-valenced scheme. Suppose $\nu_p(\chi(1)) = 0$. Then the following conditions are equivalent.*

(1) $\nu_p(m_\chi) \geq \nu_p(|X|)$.

(2) $\nu_p(m_\chi) = \nu_p(|X|)$.

(3) $\mathrm{Irr}(B) = \{\chi\}$.

Proposition 4.3. *Let (X, G) be a commutative scheme. If $\nu_p(m_\chi) < \nu_p(|X|)$, then $|\mathrm{Irr}(B)| \geq 2$.*

5 Blocks of defect 1

In group representation theory, the structure of a block of defect 1 is almost determined by the Brauer tree. For a p'-valenced scheme, we consider a block B with a character χ such that $\nu_p(m_\chi) + 1 = \nu_p(|X|)$.

Proposition 5.1. *Let (X, G) be a p'-valenced scheme. If $\nu_p(m_\chi) + 1 = \nu_p(|X|)$ and $\nu_p(r_\chi) > 0$, then $\mathrm{Irr}(B) = \{\chi^\tau \mid \tau \in Gal(K/L)\}$.*

For a block satisfying the property in the above proposition, we cannot define the Brauer tree, since it has only one vertex. But I do not know such an example.

We denote K^G for the set of K-valued functions on $\{\sigma_g \mid g \in G\}$. For $\alpha, \beta \in K^G$, we define

$$[\alpha, \beta] = \sum_{g \in G} \frac{1}{n_g} \alpha(\sigma_{g^*}) \beta(\sigma_g).$$

Let Φ be a matrix representatation of KG. We denote $\Phi_{ij} \in K^G$ for the (i, j)-entries of Φ, namely $\Phi_{ij}(\sigma_g) = \Phi(\sigma_g)_{ij}$.

Proposition 5.2 (Schur Relations [10, Theorem 4.2.4]). (1) *If Φ is an irreducible representation affording χ, then $[\Phi_{ij}, \Phi_{k\ell}] = \delta_{i\ell}\delta_{jk}|X|/m_\chi$. ($\delta$ is the Kronecker's delta.)*

(2) *If Φ and Ψ have no common irreducible constituent, then $[\Phi_{ij}, \Psi_{k\ell}] = 0$.*

Let Ψ_i, $i = 1, 2, 3$, be irreducible representations of KG affording ψ_i, respectively. We may assume that all $\Psi_i(\sigma_g)$, $g \in G$ are matrices over $\mathcal{O}_\mathfrak{P}$, and then, we can consider representations $\Psi_i{}^*$ of FG. Suppose $\Psi_i{}^*$, $i = 1, 2, 3$, have a common irreducible constituent S. We may assume

$$\Psi_i = \begin{pmatrix} S_i & * \\ * & * \end{pmatrix},$$

where $S_i{}^* = S$.

We define $u, v \in K^G$ by $u = (\Psi_1)_{11} - (\Psi_2)_{11}$ and $v = (\Psi_1)_{11} - (\Psi_3)_{11}$. Then $u(\sigma_g), v(\sigma_g) \in \mathfrak{P}\mathcal{O}_{\mathfrak{P}}$ for every $g \in G$. By Schur relation, we have

$$[(\Psi_1)_{11}, (\Psi_1)_{11}] = \frac{|X|}{m_{\psi_1}}.$$

Then

$$0 = [(\Psi_1)_{11}, (\Psi_2)_{11}] = [(\Psi_1)_{11}, (\Psi_1)_{11}] - [(\Psi_1)_{11}, u].$$

So we have

$$[(\Psi_1)_{11}, u] = [(\Psi_1)_{11}, (\Psi_1)_{11}],$$

and similarly

$$[(\Psi_1)_{11}, v] = [(\Psi_1)_{11}, (\Psi_1)_{11}].$$

Now

$$\begin{aligned} 0 &= [(\Psi_2)_{11}, (\Psi_3)_{11}] = [(\Psi_1)_{11}, (\Psi_1)_{11}] - [u, (\Psi_1)_{11}] - [(\Psi_1)_{11}, v] + [u, v] \\ &= -[(\Psi_1)_{11}, (\Psi_1)_{11}] + [u, v]. \end{aligned}$$

This means

$$\frac{|X|}{m_{\psi_1}} = [u, v].$$

Consider the traces over K/L of u and v, then we have

$$\frac{|X| \cdot |K : L|^2}{m_{\psi_1}} = \sum_{g \in G} \frac{1}{n_g} \mathrm{Tr}_{K/L}(u(\sigma_{g^*}))\mathrm{Tr}_{K/L}(v(\sigma_g)).$$

Suppose (X, G) is p'-valenced, $\nu_p(m_{\psi_1}) + 1 = \nu_p(|X|)$, and ψ_i, $i = 1, 2, 3$, are not $\mathfrak{P}$-conjugate to each other. Then we have $\nu_p(r_{\psi_i}) = 0$, $i = 1, 2, 3$.

Case 1. K is cyclotomic (abelian). In this case, we can prove that

$$\nu_p(\mathrm{Tr}_{K/L}(u(\sigma_{g^*}))) \geq \nu_p(|K : L|) + 1, \quad \nu_p(\mathrm{Tr}_{K/L}(v(\sigma_g))) \geq \nu_p(|K : L|) + 1.$$

This is a contradiction.

Case 2. $\nu_p(|K:L|) = 0$. In this case, we can prove that

$$\nu_p(\mathrm{Tr}_{K/L}(u(\sigma_{g^*}))) \geq 1$$

and this is a contradiction. (This condition is equivalent to that p is tamely ramified in $K/\mathbb{Q}$.)

Proposition 5.3. *Let (X, G) be a p'-valenced scheme, B a block of G, and $\varphi \in \mathrm{IBr}(B)$. Assume there exists $\chi \in \mathrm{Irr}(B)$ with $\nu_p(m_\chi) + 1 = \nu_p(|X|)$. Suppose that the minimal splitting field K of G is abelian or $\nu_p(|K:L|) = 0$ (p is tamely ramified in $K/\mathbb{Q}$). Then the number of $\mathfrak{P}$-conjugate classes of $\mathrm{Irr}(B)$ such that their modular characters contain φ is at most two.*

For $\psi \in \mathrm{Irr}(B)$ such that $d_{\psi\varphi} \geq 0$, we suppose $\nu_p(\psi(1)) = 0$. Then the number is exactly two.

Remark. If $\nu_p(\psi(1)) = 0$ for all $\psi \in \mathrm{Irr}(B)$, then we may assume $\nu_p(|K:L|) = 0$.

If all the numbers above are two, then we can draw a graph. Its vartex is a $\mathfrak{P}$-conjugate class, and its edge is an irreducible modular character. By a similar argument, we can show that the following.

Proposition 5.4. *Let (X, G) be a commutative p'-valenced scheme, B a block of G, and $\chi \in \mathrm{Irr}(B)$. Suppose $\nu_p(m_\chi) + 1 = \nu_p(|X|)$ and $\nu_p(r_\chi) = 0$. Then $\nu_p(m_\psi) + 1 = \nu_p(|X|)$ for all $\psi \in \mathrm{Irr}(B)$ and the number of $\mathfrak{P}$-conjugate classes of $\mathrm{Irr}(B)$ is exactly two.*

Corollary 5.5. *Let (X, G) be a commutative p'-valenced scheme with $\nu_p(|X|) = 1$. Then all non-trivial irreducible ordinary characters in the principal block are $\mathfrak{P}$-conjugate.*

Proposition 5.6. *If $|X| = p$, then all non-trivial irreducible ordinary characters are $\mathfrak{P}$-conjugate.*

Using this fact, we can prove that (X, G) is commutative, if $|X| = p$.

Proposition 5.7. *Let (X, G) be a commutative p'-valenced scheme, $\psi \in \mathrm{Irr}(G)$. Suppose $\nu_p(m_\chi) + 1 = \nu_p(|X|)$. If the Schur index $m_L(\chi) = 1$, $\nu_p(r_\chi) = 0$, and $p \neq 2$, then $d_{\chi\varphi} \leq 1$ for every $\varphi \in \mathrm{IBr}(G)$. (The assumption on Schur indices holds if there exists an L-representation of G affording χ.)*

Remark. (1) If $p \neq 2$, then the Schur index $m_L(\chi)$ equals to one for a group character χ. (Note that the base field is not $\mathbb{Q}$.)

(2) If $L(\chi(\sigma_g) \mid g \in G)$ is a Galois extension of L, then the condition $\nu_p(r_\chi) = 0$ holds.

(3) If we can define a graph, $d_{\chi\varphi} \leq 1$ holds for $\chi \in \mathrm{Irr}(B)$ and $\varphi \in \mathrm{IBr}(B)$, and $p \neq 2$, then the graph is bipartite. Of cource, a tree is bipartite. The original proof to show that the graph is a tree uses the fact that the Cartan matrix is invertible. But this is not true for association schemes. I do not know whether it is true or not for p'-valenced schemes.

Concerning the above remark, we have one more question. Let (X, G) be a p'-valenced scheme. Suppose $\nu_p(m_\chi) + 1 = \nu_p(|X|)$, $d_{\chi\varphi} \leq 1$ for all $\chi \in \mathrm{Irr}(B)$ and all $\varphi \in \mathrm{IBr}(B)$, and a graph is defined. Then the graph is a tree if and only if $\mathrm{rank} D_B = |\mathrm{IBr}(B)|$. Especially, if the Cartan matrix C_B is invertible, then the graph is a tree.

Question 5.8. For a p'-valenced scheme, is the Cartan matrix invertible ?

Remark. (1) Almost all results in this talk are not true for non p'-valenced schemes.

(2) For commutative p'-valenced scheme, it is reasonable to define the "defect" of a block by $\max\{\nu_p(|X|) - \nu_p(m_\chi) \mid \chi \in \mathrm{Irr}(B)\}$. But, in general, it is still difficult.

(3) After my talk, Yoshimasa Hieda pointed out the following facts. Let G be a finite group, and H a p'-subgroup of G. Consider the Schurian scheme $G//H$. Then $G//H$ is p'-valenced and the decomposition matrix of $G//H$ is a submatrix of the decomposition matrix of the group G by [7, §6.2] or [9]. So if G has a cyclic Sylow p-subgroup, then many things on our problem are well understood.

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