A Classification of Semiregular RDS's with $k = 12$

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1. Introduction

Definition 1.1. An incidence structure $(\mathcal{P}, \mathcal{B})$ is called a square $(m, u, k, \lambda)$-divisible design if the following conditions (i)-(iii) are satisfied.

(i) $|\mathcal{P}| = |\mathcal{B}| = mu$.

(ii) There exists a partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_m$ of $\mathcal{P}$ satisfying $|\mathcal{P}_1| = \cdots = |\mathcal{P}_m| = u$ and

$[p, q] = \begin{cases} 0 & \text{if } p, q \in \mathcal{P}_i, \exists i, \\ \lambda & \text{otherwise.} \end{cases} (p \neq q \in \mathcal{P}).$

(iii) $|\mathcal{B}| = k$ $(\forall B \in \mathcal{B})$.

The following hold.

$$k(k-1) = (m-1)u\lambda, \quad |p| = k \quad (\forall p \in \mathcal{P})$$  

$$k \geq u\lambda \quad (\text{Bose - Connor[1]})$$

Let $p \in \mathcal{P}_1$ and $B \in \mathcal{B}$ and assume that an automorphism group $G$ of $(\mathcal{P}, \mathcal{B})$ acts regularly on both $\mathcal{P}$ and $\mathcal{B}$. Set $D = \{ x \in G \mid px \in B \}$ and $U = \{ x \in G \mid px \in \mathcal{P}_1 \}$. Then $|D| = k$ and $U$ is a subgroup of $G$ of order $u$ satisfying

$$DD^{(-1)} = k + \lambda(G - U).$$

The equation (3) is equivalent to the following.

$$|aD \cap bD| = \begin{cases} 0 & \text{if } aU = bU, \\ \lambda & \text{otherwise.} \end{cases} (a \neq b \in G)$$

Definition 1.2. Let $G$ be a group of order $mu$ and $U$ a subgroup of $G$ of order $u$. A $k$-subset $D$ is called a $(m, u, k, \lambda)$-difference set relative to $U$ if $D$ satisfies (3). $D$ is also called a relative difference set (RDS) relative to $U$. 
Conversely, given a \((m, u, k, \lambda)\)-difference set \(D\) in \(G\) relative to \(U\). Then we can show that \(\text{dev}(D)\) is a square \((m, u, k, \lambda)\)-divisible design, where

\[
\text{dev}(D) := (G, \{Dx \mid x \in G\}).
\]

**Definition 1.3.** A square \((m, u, k, \lambda)\)-divisible design is said to be **symmetric** if its dual is also a square \((m, u, k, \lambda)\)-divisible design. In other words, there is a partition \(B = B_1 \cup \cdots \cup B_m\) of \(B\) satisfying

\[
|B \cap C| = \begin{cases} 0 & \text{if } B, C \in B_i, \exists i, \quad (B \neq C \in B) \\ \lambda & \text{otherwise.} \end{cases}
\]

**Result 1.4.** (W. S. Connor [3]) Let \((P, B)\) be a square \((m, u, k, \lambda)\)-divisible design such that \(k > u\lambda\). If \((k, \lambda) = 1\), then \((P, B)\) is symmetric.

**Remark 1.5.** Let \(D\) be a \((m, u, k, \lambda)\)-difference set in \(G\) relative to a subgroup \(U\). If \(D D^{(-1)} = D^{(-1)} D\), then \(\text{dev}(D)\) is symmetric.

**Result 1.6.** (D. Jungnickel [9]) If \(G \supset U\), then \(D D^{(-1)} = D^{(-1)} D\).

Concerning this, we have the following results.

**Proposition 1.7.** \(\text{dev}(D)\) is symmetric if and only if \(D^{(-1)} D = u \lambda + \lambda (G - V)\) for a subgroup \(V \) of \(G\).

**Proof.** Set \((P, B) = \text{dev}(D)\) and assume \((P, B)\) is symmetric. Then, there exists a partition \(B = B_1 \cup \cdots \cup B_u\) of \(B\) such that

\[
|B \cap C| = \begin{cases} 0 & \text{if } B, C \in B_i, \exists i, \quad (B \neq C \in B) \\ \lambda & \text{otherwise.} \end{cases}
\]

Set \(B_1 = \{Dg_1, Dg_2, \cdots , Dg_u\}\), where \(g_1 = 1\). As \(Dd_i \cap Dg_j = \emptyset\) for any distinct \(i, j \in \{1,2, \cdots, u\}\), for each \(B_k\) there is an element \(g \in G\) so that \(B_k = \{Dg_1g, Dg_2g, \cdots , Dg_ug\}\).

We note that

\[
Dg_i \cap Dg_j = \emptyset \quad \Leftrightarrow \quad \{(d_1, d_2) \mid d_1, d_2 \in D, d_1g_i = d_2g_j = \emptyset\}
\]

\[
\Leftrightarrow \quad \{(d_1, d_2) \mid d_1, d_2 \in D, \quad d_1^{-1} d_2 = g_i g_j^{-1} = \emptyset\} = \emptyset \quad (\ast)
\]

Set \(V = \{g_1 = 1, g_2, \cdots , g_u\}\). Let \(g_i, g_j \in V\). Then, by (\ast), \(Dg_i g_j^{-1} \cap D = \emptyset\).

Hence \(Dg_i g_j^{-1} = Dg_k\) for some \(g_k \in V\). Thus \(g_i g_j^{-1} = g_k \in V\) and so \(V\) is a subgroup of \(G\) of order \(u\). By \((\ast)\), we have the lemma.

**Corollary 1.8.** Let \(D\) be an RDS. Then \(D^{(-1)}\) is also an RDS if and only if \(\text{dev}(D)\) is symmetric.

**Definition 1.9.** An RDS \(D\) is called **symmetric** if \(\text{dev}(D)\) is symmetric, otherwise **non-symmetric**.

If the equality in (2) holds, then \(k = m = u \lambda\) and so \((m, u, k, \lambda) = (u \lambda, u, u \lambda, \lambda)\).
**Definition 1.10.** A square $(m, u, k, \lambda)$-divisible design $(\mathcal{P}, \mathcal{B})$ is called a transversal design and denoted by $\text{TD}_\lambda(k; u)$ if $|B \cap \mathcal{P}_i| = 1$ for $\forall B \in \mathcal{B}$ and $\forall i \in \{1, 2, \ldots, m\}$.

Therefore, a square $(m, u, k, \lambda)$-divisible design is a transversal design if

$$k = m(= u\lambda).$$

**Definition 1.11.** If $k = m = u\lambda$, then a $(m, u, k, \lambda)$-difference set $D$ in a group $G$ is said to be semiregular. Clearly $|G| = u^2\lambda$.

**Remark 1.12.** Under the above assumption, $DD^{-1} \neq D^{-1}D$ in general. However, every known transversal design obtained from semiregular RDS is symmetric.

In this talk we give examples of semiregular RDS's which do not satisfy the condition of Proposition 1.7. Then it gives us examples so that dev($D$)'s are non-symmetric, and consequently non-symmetric transversal designs.

2. **Known non-normal semiregular RDS's**

Let $D$ be a $(u\lambda, u, u\lambda, \lambda)$-difference set (i.e. semiregular RDS) in $G$ relative to $U$. Then $D$ is called normal and non-normal according as $U \triangleleft G$ and $U \not\triangleleft G$, respectively. We note that dev($D$) is symmetric for every normal RDS applying Jungnickel’s result.

**Example 2.1.** ([6], [7]) The following are all known examples of non-normal semiregular RDS's.

(i) $(u, \lambda) = (2, 2), G = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1} \rangle(\simeq D_8), U = \langle y \rangle : (4, 2, 4, 2)\text{-DS}$

(ii) $(u, \lambda) = (4, 4), G = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1} \rangle(\simeq Q_{16}), U = \langle y \rangle : (4, 4, 4, 2)\text{-DS}$

(iii) $(u, \lambda) = (2, 8), G = \langle x, y \mid x^{16} = y^2 = 1, y^{-1}xy = x^7 \rangle(\simeq SD_{32}), U = \langle y \rangle : (16, 2, 16, 8)\text{-DS}$

(iv) $(u, \lambda) = (2, 8), G = \langle x, y \mid x^{16} = y^2 = 1, y^{-1}xy = x^9 \rangle(\simeq M_5(2)), U = \langle y \rangle : (16, 2, 16, 8)\text{-DS}$

(v) Let $A$ be a $(4m^2, 2m^2 - m, m^2 - m)$-difference set in a group $N$. Assume $t$ is an automorphism of $N$ of order 2. Then $D = A \cup (N \setminus A^t)$ is a non-normal $(4m^2, 2, 4m^2, 2m^2)\text{-DS}$ in a group $N(t)$ relative to $t$.
3. Semiregular RDS’s with $|D| = 12$

From now on we assume that $D$ is a semiregular RDS in a group $G$ relative to a subgroup $U$ with $|D| = 12$. Set $u = |U|$. Then $G = 12u$ and $D$ is a $(12, u, 12, \lambda)$-DS, where $u\lambda = 12$. Thus $|D| = 12$ and one of the following holds.

(i) $(u, \lambda) = (2, 6), \ |G| = 24, \ |U| = 2.$
(ii) $(u, \lambda) = (3, 4), \ |G| = 36, \ |U| = 3.$
(iii) $(u, \lambda) = (4, 3), \ |G| = 48, \ |U| = 4.$
(iv) $(u, \lambda) = (6, 2), \ |G| = 72, \ |U| = 6.$
(v) $(u, \lambda) = (12, 1), \ |G| = 144, \ |U| = 12.$

Remark 3.1. Let $D$ be a semiregular RDS in a group $G$ relative to $U$ and let $s$ be an automorphism of $G$. Then $D^s$ is also a semiregular RDS (with the same parameters as $D$) in $G$ relative to $U^s$.

CASE $(u, \lambda) = (2, 6), \ |G| = 24, \ |U| = 2$

The following lemma holds.

Lemma 3.2. ([7]) If there exists a $(2n, 2, 2n, n)$-difference set in $G$ relative to $U$ such that $G = NU$ for a subgroup $N$ of $G$ of index 2, then $n^* = 2$.

By Lemma 3.2 we have the following.

Lemma 3.3. If $(u, \lambda) = (2, 6)$, then $[G, G] \geq U$.

Lemma 3.4. N. Ito ([8]) If a group $G$ of order $4n (> 4)$ contains a normal $(2n, 2, 2n, n)$-DS relative to $U$, then a Sylow 2-subgroup of $G$ is neither cyclic nor dihedral.

By Remark 3.1, Lemmas 3.3, 3.4, there are five possibilities.

1. $G = Q_8 \times Z_3, \ U = Z(Q_8) \times 1.$
2. $G = Q_{24}, \ U = Z(Q_{24}).$
3. $G = Z_2 \times A_4$ and there are three possibilities for $U(\simeq Z_2)$.
4. $G = SL(2, 3), \ U = Z(SL(2, 3)).$
5. $G = S_4$ and there are two possibilities for $U(\simeq Z_2)$.

By a computer search, we have the following.

Lemma 3.5. Assume $(u, \lambda) = (2, 6)$. Then, a group $G$ of order $24$ contains a $(12, 2, 12, 6)$-DS if and only if $G \simeq Q_8 \times Z_3, \ Q_{24}$ or $SL(2, 3)$. 
CASE \((u, \lambda) = (3, 4), \ |G| = 36, \ |U| = 3.\)

In this case \((m, u, k, \lambda) = (12, 3, 12, 4).\)

Lemma 3.6. Let \(G\) be a nonabelian group of order 36. Then there are eleven possibilities.

(i) \(G \cong D_{36},\)

(ii) \(G \cong Q_{36},\)

(iii) \(G \cong D_{18} \times \mathbb{Z}_{2},\)

(iv) \(G \cong (\mathbb{Z}_{2} \times (\mathbb{Z}_{3} \times \mathbb{Z}_{3})) \times \mathbb{Z}_{2},\)

(v) \(G \cong S_{3} \times \mathbb{Z}_{6},\)

(vi) \(G \cong A_{4} \times \mathbb{Z}_{2},\)

(vii) \(G \cong (\mathbb{Z}_{4} \ltimes \mathbb{Z}_{3}) \times \mathbb{Z}_{3},\)

(viii) \(|Z(G)| = 2,\)

(ix) \(G = \langle d \rangle \langle a, b \rangle \simeq \mathbb{Z}_{4} \times D_{12},\)

(x) \(G \cong A_{4} \times \mathbb{Z}_{3},\)

(xi) \(G \cong \mathbb{Z}_{9} \times (\mathbb{Z}_{2} \times \mathbb{Z}_{2}).\)

By a computer search we have the following.

Lemma 3.7. Assume \((u, \lambda) = (3, 4).\) Then, a nonabelian group \(G\) of order 36 contains a \((12, 3, 12, 4)\)-DS if and only if

\(|G| = (\mathbb{Z}_{4} \times \mathbb{Z}_{3}) \times \mathbb{Z}_{3},\)

where \(|Z(G)| = 6,\)

\(|U| = \text{O}_{3}(Z(G)).\)

The first and the second cases have been previously known. But the third is a new one and has unusual properties.

Example 3.8. Let \(G = \langle a, b, c \mid a^{3} = b^{2} = c^{6} = 1,\)

\(b^{-1}ab = a^{-1}, \ ac = ca, \ bc = cb\) and set \(D = \{1, c, c^{2}, a, ac, b, a^{2}bc^{5}, abc^{4}, a^{2}bc, bc^{4}, abc\}.\)

Then \(D\) is a non-symmetric \((12, 3, 12, 4)\)-DS relative to \(U = \text{O}_{3}(Z(G)).\)

Let \((P, B) (= \text{dev}(D))\) be the corresponding transversal design and let \(A\) be an incidence matrix of \((P, B).\) Then

\[\begin{bmatrix}
\end{bmatrix}\]

where, \(I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\) and \(J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\).
However,

\[
\begin{bmatrix}
12 & 4 & 4 & 4 & \cdots & \cdots & 5 & 4 & 5 & 4 & 2 \\
4 & 12 & 4 & 4 & \cdots & \cdots & 4 & 5 & 4 & 5 & 4 \\
4 & 4 & 12 & 4 & \cdots & \cdots & 2 & 4 & 5 & 4 & 5 \\
4 & 4 & 4 & 12 & \cdots & \cdots & 4 & 2 & 4 & 5 & 4 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
4 & 5 & 4 & 2 & \cdots & \cdots & 4 & 12 & 4 & 4 & 4 \\
5 & 4 & 5 & 4 & \cdots & \cdots & 4 & 12 & 4 & 4 & 4 \\
4 & 5 & 4 & 5 & \cdots & \cdots & 4 & 4 & 4 & 12 & 4 \\
2 & 4 & 5 & 4 & \cdots & \cdots & 4 & 4 & 4 & 12 & 4 \\
\end{bmatrix}
\]

$A^TA = \box{	t \small REJECT}$

An RDS $D$ is called symmetric if $D^{(-1)}$ is also an RDS. Since $AA^T$ has entries $2, 5 \not\in \{0, 4, 12\}$, the dual of $\text{dev}(D)$ is not a transversal design. Applying Proposition 1.7, $D^{(-1)}$ is a non-symmetric RDS. As far as I know, this is the only known non-symmetric RDS.

**CASE** $(u, \lambda) = (4, 3), \quad |G| = 48, \quad |U| = 4$

In this case $(m, u, k, \lambda) = (12, 4, 12, 3)$.

We use the following two lemmas to settle the present case.

**Lemma 3.9.** If $2 \mid u$ and $2 \nmid \lambda$, then $U$ contains every involution of $G$.

**Lemma 3.10.** (Elliott-Butson) Let $D$ be a $(u\lambda, u, u\lambda, \lambda)$-DS in $G$ relative to $U$. If $U$ contains a normal subgroup $N$ of $G$ of order $v$, then $DN/N$ is a $(u\lambda, v, u\lambda, v\lambda)$-DS in $G/N$ relative to $U/N$.

Applying Lemmas 3.5 and 3.10, it suffices to check the following case.

\[G = \langle a, b, c \mid a^4 = b^4 = c^3 = 1, c^{-1}ac = ab, c^{-1}bc = ab^2 \rangle, \quad U = \langle a^2, b^2 \rangle\]

We have the following by a computer search.

**Lemma 3.11.** There is no $(12, 4, 12, 3)$-DS.

**CASE** $(u, \lambda) = (6, 2), \quad |G| = 72, \quad |U| = 6$

In this case $(m, u, k, \lambda) = (12, 6, 12, 2)$.

**Observation.** For every known semiregular RDS, $|U|$ is a power of a prime.

The smallest undecided case is $u = 6$. 
Lemma 3.12. Let $G$ be a nonabelian group of order 72. Then there are five possibilities.

(i) $G \cong SL(2,3) \times \mathbb{Z}_3$,
(ii) $G \cong A_4 \times \mathbb{Z}_6$,
(iii) $G \cong A_4 \times S_3$,
(iv) $G \cong \langle t \rangle (M \times T)$, $t^2 = 1$, $M \cong A_4$, $T \cong \mathbb{Z}_3$,
$v$ $G \triangleright Q$, $|Q| = 9$.

Applying Lemmas 3.5, 3.7 and 3.10, we have the following by a computer search.

Lemma 3.13. There is no (12,6,12,2)-DS.

CASE $(u, \lambda) = (12, 1)$, $|G| = 144$, $|U| = 12$.

In this case $(m, u, k, \lambda) = (12, 12, 12, 1)$.

By Lemma 1 of [2], the following holds.

Theorem 3.14. Every transversal design with $\lambda = 1$ is symmetric.

The above theorem implies that if there is a $(u, u, 1)$-DS in a group $G$, then the corresponding transversal design can be extended to a projective plane of order $u$ which admits $G$ as a collineation group of order $u^2$. Thus $u \neq 12$ by Baumert-Hall [4] and the following holds.

Lemma 3.15. There is no (12,12,12,1)-DS.

By Lemmas 3.5, 3.7, 3.11, 3.13 and 3.15, we have the following.

Theorem 3.16. A group $G$ contains a $(u\lambda, u, u\lambda, 1)$-DS $D$ with $|D| = 12$ if and only if $G$ is isomorphic to one of the following.

(i) $(u, \lambda) = (2, 6)$, $G = Q_8 \times \mathbb{Z}_3$, $U = Z(Q_8) \times 1 \cong \mathbb{Z}_2$.
(ii) $(u, \lambda) = (2, 6)$, $G = Q_{24}$, $U = Z(Q_{24}) \cong \mathbb{Z}_2$.
(iii) $(u, \lambda) = (2, 6)$, $G = SL(2,3)$, $U = Z(SL(2,3)) \cong \mathbb{Z}_2$.
(iv) $(u, \lambda) = (3, 4)$, $G = S_3 \times \mathbb{Z}_6$, $U \cong \mathbb{Z}_3$ (a non-symmetric RDS).
(v) $(u, \lambda) = (3, 4)$, $G = (Z_4 \ltimes Z_3) \times Z_3$, $|Z(G)| = 6$, $U = O_3(Z(G))$.
(vi) $(u, \lambda) = (3, 4)$, $G = A_4 \times \mathbb{Z}_3$, $U = 1 \times \mathbb{Z}_3$.
(vii) $(u, \lambda) = (3, 4)$, $G = \mathbb{Z}_6 \times \mathbb{Z}_6$, $U \cong \mathbb{Z}_3$. 
4. Construction of non-symmetric RDS's

In this section we show the following.

**Theorem 4.1.** There exists a non-symmetric \((2^{2m}3^{m}, 2^{2m}3^{m}, 2^{2m}3^{m-1})\) difference set in \((S_{3} \times Z_{6}) \times (Z_{6} \times Z_{2}) \times \cdots \times (Z_{6} \times Z_{2})\) relative to \(U \times 1 \times \cdots \times 1\), where \(D\) is a non-symmetric \((12, 3, 12, 4)\)-difference set in \(S_{3} \times Z_{6}\) relative to its non-normal subgroup \(U\) of order 3 (see Example 3.8).

**Corollary 4.2.** There exists a non-symmetric \(TD_{2m}3^{m-1}[2^{2m}3^{m}; 3]\) for every \(m \in \mathbb{N}\).

**Example 4.3.**

In order to prove Theorem 4.1, we need the following lemma.

**Lemma 4.4.** Let \(L = G \times H\), where \(G\) be a group of order \(u^2\lambda\) and \(H\) is a group of order \(u\mu\). Let \(D\) be a \((u\mu, u, u\lambda, \lambda)DS\) in \(G\) relative to a subgroup \(U\) of \(G\) of order \(u\) and let \(C\) be a \((u\mu, u, u\mu, \mu)DS\) in \(U \times H\) relative to \(U\). Then

(i) \(CD\) is a \((u^2\lambda\mu, u, u^2\lambda\mu, u\lambda\mu)DS\) in \(L\) relative to \(U\).

(ii) \(CD\) is symmetric if and only if \(D\) is symmetric.

**Proof.** Let \(c_{1}, c_{2} \in C\) and \(d_{1}, d_{2} \in D\) and assume \(c_{1}d_{1} = c_{2}d_{2}\). Then \(c_{1}^{-1}c_{2} = d_{1}d_{2}^{-1} \in UH \cap G = U\). Thus \(d_{1} = d_{2}\) and so \(c_{1} = c_{2}\). Therefore \(CD\) is a subset of \(L\).

By assumption, the following hold.

\[
DD^{(-1)} = u\lambda + \lambda(G - U) \tag{5}
\]
\[
CC^{(-1)} = u\mu + \mu(UH - U) \tag{6}
\]
\[
G = UD, \quad UC = UH \tag{7}
\]

Hence \((CD)(CD)^{(-1)} = CD\) \((DD^{(-1)})C^{(-1)} = C(u\lambda + \lambda(G - U))C^{(-1)} = u\lambda CC^{(-1)} + \lambda CUC^{(-1)} - \lambda CUC^{(-1)}\) as \(C, U \subset UH \triangleright U\), \(CU = UC\). Similarly \(GC = CG\). It follows that \((CD)(CD)^{(-1)} = u\lambda(u\mu + \mu(UH - U)) + \lambda GCC^{(-1)} - \lambda UCC^{(-1)} = u^2\mu\lambda + u\mu\lambda UH - u\mu\lambda U + AG(u\mu + \mu UH - \mu U) - \lambda UM(u\mu + \mu UH - \mu U) = u^2\mu\lambda + u\mu(L - U)\). Thus we have (i).

Since \(UH \triangleright U\), \(C^{(-1)}C = CC^{(-1)}\). Hence \((CD)^{(-1)}CD = D^{(-1)}(CC^{(-1)})D = D^{(-1)}(u\mu + \mu UH - \mu U)D\). By (7), the following holds.

\[
(CD)^{(-1)}CD = u\mu D^{(-1)}D + u\mu\lambda L - u\mu\lambda G \tag{8}
\]

Assume \(CD\) is symmetric. Then \((CD)^{(-1)}CD = u^2\mu\lambda + u\mu\lambda(L - V)\) for a subgroup \(V\) of \(L\) of order \(u\). By (8), \(u\mu D^{(-1)}D - u\mu\lambda G = u^2\mu\lambda - u\mu\lambda V\). Thus \(D^{(-1)}D = u\lambda + \lambda(G - V)\). In particular, \(V\) is a subgroup of \(G\) of order \(u\) and so \(D\) is symmetric. Conversely, assume \(D\) is symmetric. Then \(D^{(-1)}D = u\lambda + \lambda(G - V)\) for a subgroup \(V\) of \(G\) of order \(u\). Then, by (8), \((CD)^{(-1)}CD = u\mu(u\lambda + \lambda(G - V)) + u\mu\lambda L - u\mu\lambda G = u^2\mu\lambda + u\mu\lambda(L - V)\). Therefore \(CD\) is symmetric. Thus we have (ii).
We note that Lemma 4.4(i) is a modification of Result 2.4 of [11], where $N$ is assumed to be normal in $G$.

**Proof of Theorem**

Let $D$ be a non-symmetric $(12, 3, 12, 4)$DS in $M = S_3 \times \mathbb{Z}_6$ relative to a non-normal subgroup $U$ of $M$ (see Example 3.8). Let $H = \langle a \rangle \times \langle b \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_3$. We note that an abelian group $H \times \langle c \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Let $H = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Set $G = H \times M$ and choose $\langle c \rangle$ as a non-normal subgroup of $M$. Then, applying Lemma 4.4, $H \times M$ contains a non-symmetric $(2^3 3^2 \cdot 4, 3, 2^2 3^2 2^2 3 \cdot 4)$DS in $G$ relative to $1 \times U(\simeq \mathbb{Z}_3)$ as $D$ is non-symmetric. Repeating this procedure again and again we have the theorem.

**References**


