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Finite Groups of Local Characteristic 2

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Abstract

A finite group X has local characteristic 2 if $C_L(O_2(L)) \leq O_2(L)$ for every 2-local subgroup L of X. Ulrich Meierfrankenfeld, Bernd Stellmacher and Gernot Stroth have started a project on a revision of the classification of finite simple groups of local characteristic 2. In this paper we report on work on the global analysis part of this project contained in the author's doctoral thesis.

1 Introduction

Let p be a fixed prime (in this section we do not assume $p = 2$), X a finite group whose order is divisible by p, and $P \in \mathrm{Syl}_p(X)$.

A ***p*-local subgroup** of X is the normalizer in X of a nontrivial p-subgroup of X. The group X has **characteristic *p*** if $C_X(O_p(X)) \leq O_p(X)$. We say that X has **local characteristic *p*** if all p-local subgroups of X have characteristic p. Finally, we say that X has **parabolic characteristic *p*** if every p-local subgroup L of X with $P \leq L$ has characteristic p.

Examples.

1. The generic examples of groups of local characteristic p are finite groups of Lie type defined over a field of characteristic p.

2. M_{12}, A_{10}, $P\Omega_8^+(3)$ are examples of groups which have parabolic characteristic 2 but not local characteristic 2.

3. Finite groups of Lie type defined over a field of characteristic $r \neq p$ usually do not have parabolic characteristic p, but there are exceptions such as $L_2(2^n \pm 1)$ or $G_2(3)$ which have local characteristic 2.

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X is a $\mathcal{K}$-group if every simple group involved in X is a known simple group. If every p-local subgroup of X is a $\mathcal{K}$-group, then X is called a $\mathcal{K}_p$-group.

In the original proof of CFSG[1] the finite simple groups of local characteristic 2 caused major difficulties. It is the goal of the project "Finite groups of local characteristic p" originated by Ulrich Meierfrankenfeld, Bernd Stellmacher and Gernot Stroth to give a revised proof of that part of CFSG. For an overview and more information on the project see [4] and [5].

2 The Setup

The bulk of the project consists of the investigation of the p-local structure of a finite group G of local characteristic p. This information is then used to identify G up to isomorphism. In the H-structure theorem [6], a large nonlocal subgroup H of G with $F^*(H)$ simple is constructed. If G is a simple group, in most cases we have $G = H$, but there are exceptions like $P\Omega_8^+(2)$ in $P\Omega_8^+(3)$ for $p = 2$, or $P\Omega_7(3)$ in $M(22)$ for $p = 3$.

In his doctoral thesis the author investigates the following

Setup.

Let G be a finite $\mathcal{K}_2$-group of parabolic characteristic 2 with $O_2(G) = 1$.

We assume there exists a subgroup H of G of odd index.

Further we assume $K = F^*(H)$ is a simple group of Lie type defined over a field of even characteristic with Lie rank greater than 1.

Fix a Sylow 2-subgroup S of H (so $S \in \mathrm{Syl}_2(G)$).

Finally, we assume that $R = Z(K \cap S)$ is a (long) root subgroup.

We also need some 2-local subgroups of G. Set $C = N_G(Z(S))$ and let $\widetilde{C}$ be a fixed maximal 2-local subgroup of G such that $C \leq \widetilde{C}$.

We also assume that there is a maximal 2-local subgroup M of G with $S \leq M$, but M is not contained in $\widetilde{C}$.

Remarks.

1. The group H is a group constructed in the H-structure theorem [6]. Therefore we have some information on the group H and its embedding in G.

2. The condition on R can be weakened to $R = \Omega_1(X_\alpha)$ for some root subgroup X_α. This would allow K to be a unitary group in odd dimension or a group of type ${}^2F_4$, too.

[1] The classification of the finite simple groups.

For further comments on this setup see [3].
The aim is to show that usually the following holds:

$$\text{Every 2-local subgroup } L \text{ of } G \text{ with } S \leq L \text{ is contained in } H. \qquad (*)$$

This is not true in every case, for instance $H \cong L_4(2) : 2 \cong S_8$, $G \cong A_{10}$ and $\widetilde{C} \cong 2^4 : S_5$, or $H \cong P\Omega_8^+(2)$, $G \cong P\Omega_8^+(3)$ and $\widetilde{C} \cong 2((A_4 \wr 2) \wr 2)$; in both cases $\widetilde{C} \not\leq H$. So we also want to characterize the pairs (G, H) where $(*)$ fails.

The first and most difficult step to reach this aim is to determine when $\widetilde{C}$ is contained in H. In this paper we outline the proof of the following

Theorem 1. *Assume the Setup. If K is isomorphic to one of the following groups $P\Omega_{2m}^{\pm}(q)$ ($\neq P\Omega_8^+(2)$), $U_n(q)$ (n even), $G_2(q)$ ($q > 2$), $E_6(q)$, $E_7(q)$, $E_8(q)$, ${}^2E_6(q)$, ${}^3D_4(q)$, where $q = 2^f$ for some positive integer f, then we have*

$$\widetilde{C} \leq H.$$

Remarks.

1. In the exceptional cases of Theorem 1 we try to prove: If $\widetilde{C} \not\leq H$ and G is simple, then either $H \cong P\Omega_8^+(2)$ and $G \cong P\Omega_8^+(3)$, or $H \cong G_2(2)$ and $G \cong G_2(3)$.

2. We are working on a corresponding theorem for K isomorphic to $L_n(q)$. Here the exception $K \cong L_4(2)$ occurs. We have proven: If G is simple, $H \cong L_4(2) : 2$, and $\widetilde{C} \not\leq H$, then $G \cong A_{10}$.

3. In all exceptional cases the main difficulty is the determination of $\widetilde{C}$. If this is done, we have enough information about G to quote existing characterization results from the literature.

4. If we allow $R = \Omega_1(X_\alpha)$ for some nonabelian root subgroup X_α, then the proof of Theorem 1 shows $\widetilde{C} \leq H$ if $K \cong U_{2m+1}(q)$. In case $K \cong {}^2F_4(q)$, we try to prove that $\widetilde{C}$ is contained in H.

Conjecture. Assume the Setup. If $\widetilde{C} \leq H$, then $(*)$ holds.

3 Preliminary Observations

From now on assume the Setup. We frequently use facts from the theory of finite groups of Lie type without explicit notification. These are often well known, a standard reference for them is [2].

Because $R = Z(K \cap S)$ is a root subgroup, we have

Lemma 1. *K is isomorphic to one of the following groups: $L_n(q)$, $P\Omega_{2m}^{\pm}(q)$, $U_n(q)$ (n even), $G_2(q)$, $G_2(2)'$, $E_6(q)$, $E_7(q)$, $E_8(q)$, ${}^2E_6(q)$, ${}^3D_4(q)$, for a suitable $q = 2^f$.*

It is well known that $N_K(R)$ is a parabolic subgroup of K. We have $N_K(R) = Q_0K_0H_0$, $C_K(R) = Q_0K_0$, where $Q_0 = O_2(N_K(R))$, K_0 is a central product of groups of Lie type defined over some extension field of $\mathbb{F}_q$, $H_0 \cong \mathbb{Z}_{q-1}$ acts regularly on $R^\sharp$, $[K_0, H_0] = 1$.

From [2, 2.6.5e] we get $Z(S) \leq R$. Because all nonidentity elements in the root subgroup R have the same centralizer in K, it follows $C_K(Z(S)) = C_K(R)$, in particular $C_K(R) \leq \widetilde{C} \cap K$.

We set $Q = O_2(\widetilde{C})$.

Lemma 2. *Except for $K \cong L_4(q)$ or $G_2(2)'$, we have $Q \leq K$.*

Proof. Set $Q_1 = O_2(\widetilde{C} \cap H)$ and $Q_2 = O_2(\widetilde{C} \cap K)$. Obviously, $Q \leq Q_1$. We show $Q_1 \leq K$.

Suppose Q_1 is not contained in K. Since $Q_2 = Q_1 \cap K$ and $K = F^*(H)$, the quotient group Q_1/Q_2 is isomorphic to a subgroup of $\mathrm{Out}(K)$. We have $[Q_1/Q_2, K_0] = 1$, so there is an involution in $\mathrm{Out}(K)$ which centralizes K_0. Then [1, §19] yields $K \cong L_4(q)$ or $G_2(2)'$, a contradiction. □

The following identification of Q is a key ingredient for the proof of Theorem 1.

Lemma 3. *If $K \not\cong L_n(q)$, ${}^2F_4(q)$, or $G_2(2)'$, then $Q = Q_0$. In particular $\widetilde{C} = N_G(R)$.*

Proof. Note that Q_0/R is irreducible as K_0-module. By Lemma 2, Q is contained in K, so it is easy to see that $Q \leq Q_0$. We have $K_0 \leq C_K(R) \leq \widetilde{C}$, hence Q/R is a K_0-invariant subspace of Q_0/R. Now the assertion follows, for $C_G(Q) \leq Q$. □

Remark. If $K \cong L_n(q)$, then K_0 does not act irreducibly on Q_0/R. This is the reason why the linear groups need a separate treatment and are not included in Theorem 1.

A very easy but extremely useful observation is

Lemma 4. *$\widetilde{C}/Q$ acts faithfully on Q/R.*

Proof. Observe that $Q = Q_0$ is a nonabelian special group with Frattini subgroup R. So $C_{\widetilde{C}/Q}(Q/R)$ is a 2-group, for G has parabolic characteristic 2. Now the assertion follows immediately. □

Assume K is one of the groups from Theorem 1. From Lemma 3 we have $Q = Q_0$ and thus $\widetilde{C} = N_G(Q_0) = N_G(R)$. The argument to show $\widetilde{C} \leq H$ is divided into several steps. First of all, we have to determine $F(\widetilde{C}/Q)$. After this is done, it will be evident that $\widetilde{C}/Q$ contains at least one component. We must identify the components of $\widetilde{C}/Q$ up to isomorphism. Here the following result from the author's doctoral thesis is of particular importance.

Lemma 5. *Let X be an almost simple group with $F^*(X)$ of Lie type defined over a field of characteristic 2 and Y be a known quasisimple group such that X is a subgroup of Y of odd index. Then $X = Y$, or one of the following holds:*

1. $X \cong L_2(4)$, *and* $Y \cong L_2(p^a)$ *with* $p^a \equiv \pm 3 \pmod 8$ *and* $p^a \equiv \pm 1 \pmod 5$ *or* $L_2(5^a)$ *with* $a > 1$ *odd.*
2. $X \cong L_2(4).2$, *and* $Y \cong A_7$ *or* $L_2(5^a)$ *with* $a \equiv 2 \pmod 4$.
3. $X \cong L_2(8)$, *and* $Y \cong {}^2G_2(3^{2a+1})$.
4. $X \cong L_2(8).3$, *and* $Y \cong {}^2G_2(3^{2a+1})$.
5. $X \cong L_3(2)$, *and* $Y \cong L_2(7^a)$ *with* $a > 1$ *odd.*
6. $X \cong L_3(2).2$, *and* $Y \cong L_2(7^a)$ *with* $a \equiv 2 \pmod 4$.
7. $X \cong L_3(4).2_2$, *and* $Y \cong M_{23}$ *or* McL.
8. $X \cong L_3(4).2^2$, *and* $Y \cong Ly$ *or* $\mathrm{Aut}(McL)$.
9. $X \cong L_4(2)$, *and* $Y \cong A_9$.
10. $X \cong L_4(2).2$, *and* $Y \cong A_{10}$ *or* A_{11}.
11. $X \cong U_4(2)$, *and* $Y \cong PSp_4(q)$ *with* $q \equiv \pm 3 \pmod 8$.
12. $X \cong U_4(2).2$, *and* $Y \cong L_4(q)$ *or* $U_4(q)$ *with* $q \equiv \pm 3 \pmod 8$.
13. $X \cong Sp_4(2)'$, *and* $Y \cong A_7$ *or* $L_2(9^a)$ *with* $a > 1$ *odd.*
14. $X \cong Sp_4(2)'.2 \cong PGL_2(9)$, *and* $Y \cong L_2(9^a)$ *with* $a \equiv 2 \pmod 4$.
15. $X \cong Sp_4(2)'.2_3 \cong M_{10}$, *and* $Y \cong M_{11}$ *or* $L_3(q)$ *with* $q \equiv 3 \pmod 8$ *or* $U_3(q)$ *with* $q \equiv -3 \pmod 8$.
16. $X \cong Sp_6(2)$, *and* $Y \cong P\Omega_7(q)$ *with* $q \equiv \pm 3 \pmod 8$.
17. $X \cong P\Omega_8^+(2)$, *and* $Y \cong P\Omega_8^+(q)$ *with* $q \equiv \pm 3 \pmod 8$.

18. $X \cong G_2(2)'$, *and* $Y \cong U_3(q)$ *with* $q = 3^{2a+1}$.

19. $X \cong G_2(2)$, *and* $Y \cong G_2(q)$ *with* $q \equiv \pm 3 \pmod 8$.

In the last step of the proof of Theorem 1, we use the information on $E(\widetilde{C}/Q)$ together with some information on H and M from [6] to show that every element $u \in \widetilde{C} \setminus H$ acts nontrivially on K. By replacing H with $\langle H, u \rangle$, we may therefore assume H maximal with $F^*(H) = K$ from the outset.

We will demonstrate this method by the example $K \cong E_8(q)$.

4 The proof of Theorem 1 in case $K \cong E_8(q)$

In this section we assume that $K \cong E_8(q)$, where $q = 2^f$ for some integer $f > 0$.

Then $N_K(R)$ is the $E_7(q)$-parabolic, i.e., $K_0 \cong E_7(q)$, $Q \cong q^{1+56}$, $V = Q/R$ is the $V(\lambda_7)$-module for K_0. In particular, V is an absolutely irreducible $\mathbb{F}_q K_0$-module. As in any case $C_K(R) = QK_0$, $H_0 \cong \mathbb{Z}_{q-1}$ acts regularly on the set of nonidentity elements of R.

We first prove an auxiliary result.

Lemma 6. *Every element of* $\widetilde{C}/Q$ *which is centralized by* K_0Q/Q *is contained in* H_0Q/Q.

Proof. Let $xQ \in \widetilde{C}/Q$ with $[K_0Q/Q, xQ] = 1$. Consider the group A generated by K_0Q/Q and the element xQ. As V is an absolutely irreducible $\mathbb{F}_q A$-module, it follows from Schur's Lemma that xQ acts by scalar multiplication on V. But we have got all the $q-1$ possible scalars in H_0Q/Q already. Now Lemma 4 yields the assertion. □

Lemma 7. *The Fitting subgroup* $F(\widetilde{C}/Q)$ *is contained in* H_0Q/Q.

Proof. Fix a prime $p > 2$. It is enough to show $O_p(\widetilde{C}/Q) \leq H_0Q/Q$. As K_0 is contained in $\widetilde{C}$, the group K_0Q/Q acts on $O_p(K_0Q/Q)$. By Lemma 6 we can assume that this action is nontrivial, i.e., faithful as K_0Q/Q is simple.

Suppose first $f > 1$. Consider a root subgroup $\widehat{R}$ of K_0Q/Q. By Thompson's Dihedral Lemma there is a subgroup $P \leq O_p(\widetilde{C}/Q)$ with $P\widehat{R} = P_1\langle r_1 \rangle \times \cdots \times P_f \langle r_f \rangle$, where $P_i \langle r_i \rangle$ is a dihedral group of order $2p$ for $i = 1, \ldots, f$. It is well known that $C_V(R) = C_V(r)$ for all $r \in R^\sharp$. This implies $[C_V(r_1), P_2 \times \cdots \times P_f] = 1$. Then Thompson's $A \times B$ Lemma gives $[V, P_2 \times \cdots \times P_f] = 1$. This is a contradiction to Lemma 4.

Let now $q = 2$. Set $P = \Omega_1(Z(O_p(\widetilde{C}/Q)))$. The group K_0Q/Q acts faithfully on P by Lemma 6. Coprime action gives $V = [V, P] \oplus C_V(P)$, both summands are $\widetilde{C}/Q$-invariant. Hence Lemma 4 and the irreducible action of K_0Q/Q yield $V = [V, P]$. Therefore V is the direct sum of $C_V(E)$ with $E \in \Gamma$, where Γ is the set of hyperplanes E of P with $C_V(E) \neq 0$. Let $E \in \Gamma$ and $\mathcal{O}$ the orbit of E under K_0Q/Q. Then we have

$$V = \bigoplus_{F \in \mathcal{O}} C_V(F).$$

It follows $\dim V \geq 2|\mathcal{O}|$. Therefore $28 \geq |\mathcal{O}| > 1$, but certainly $K_0Q/Q \cong E_7(q)$ does not have a nontrivial permutation representation of that degree. So we get again a contradiction and have thereby completed the proof of the lemma. □

Lemma 8. *K_0Q/Q is the only component of $\widetilde{C}/Q$.*

Proof. Recall $[H_0, K_0] = 1$. Hence $E(\widetilde{C}/Q) \neq 1$. Let $T \in \mathrm{Syl}_2(E(\widetilde{C}/Q))$, without loss $T \leq S/Q$. Note S/Q, thus also T, normalizes K_0Q/Q. Let L be a component of $\widetilde{C}/Q$. As K_0Q/Q is simple, we have $K_0Q/Q \leq L$ or $K_0Q/Q \cap L = 1$.

In the first case, it follows from Lemma 6 that L is the only component of $\widetilde{C}/Q$. Now $(K_0Q/Q)T$ is an almost simple subgroup of L of odd index. From Lemma 5 we get $K_0Q/Q = L$, hence $K_0Q/Q = E(\widetilde{C}/Q)$.

So we can assume $K_0Q/Q \cap E(\widetilde{C}/Q) = 1$. Then $[K_0Q/Q, T] = 1$. But $T \neq 1$; this contradicts Lemma 6. □

We need some definitions from the project. Recall that there is a maximal 2-local subgroup M of G with $S \leq M$ but $M \not\leq \widetilde{C}$. Define $M^0 = \langle Q^M \rangle$. Then M^0 dominates the structure of M, for we have $M = M^0(M \cap \widetilde{C})$, cf. [5, 2.4.2]. Let Y_M be the unique maximal elementary abelian 2-subgroup of M with $O_2(M/C_M(Y_M)) = 1$, cf. [5, 2.0.1].

From the H-structure theorem [6, 3.2], we know that we can choose M with M^0 the $P\Omega_{14}^+(q)$-parabolic of K, i.e., $F^*(M^0/O_2(M^0)) \cong P\Omega_{14}^+(q)$, $Y_M \cong q^{14}$ is the natural module, and $O_2(M^0)/Y_M \cong q^{64}$ is a halfspin module for $F^*(M^0/O_2(M^0))$. Observe that R is a singular point in Y_M with $R^\perp = [Y_M, Q] = Y_M \cap Q$.

We assume that $H \leq G$ with $|G : H|$ odd is chosen maximal with respect to $F^*(H) = K \cong E_8(q)$. This choice does not affect the use of the results from [6].

Lemma 9. *$\widetilde{C}$ is contained in H.*

Proof. Suppose there is an element $u \in \widetilde{C} \backslash H$. Since H contains $S \in \mathrm{Syl}_2(G)$, we can assume that u has odd order. Obviously, u acts on Q and on R, by Lemma 8 also on K_0Q.

By a Frattini argument we may assume that u normalizes $K \cap S \in \mathrm{Syl}_2(K_0Q)$. Since $|Y_MQ/Q| = q$ and $Y_MQ/Q \trianglelefteq (K \cap S)Q/Q$, we have $Y_MQ/Q = Z((K \cap S)Q/Q)$. Therefore u acts on Y_MQ/Q, thus on Y_MQ. Then $(Y_MQ)' = Y_M \cap Q$ and as a result $C_{K\cap S}(Y_M \cap Q)$ are $\langle u \rangle$-invariant, too.

Because $O_2(M/C_M(Y_M))$ is trivial, $O_2(M^0)$ is contained in $C_{K\cap S}(Y_M \cap Q)$. In fact, we have equality, since otherwise $P\Omega_{14}^+(q) \cong K_1 \leq M^0$ would contain an involution t such that $C_{Y_M}(t) = Y_M \cap Q$ is a hyperplane. But in orthogonal groups there are no transvections with a singular point as center.

As M is a maximal 2-local subgroup of G, we have $u \in M = N_G(O_2(M^0))$. That is, u acts on $\langle M^0, K_0Q \rangle = K$. Of course this action is nontrivial, therefore $F^*(\langle H, u \rangle) = K$ contradicting the maximal choice of H. □

Remarks.

1. Note that for all K from Theorem 1 with the exceptions $U_4(2)$ and $P\Omega_{2m}^{\pm}(q)$ the group K_0 always is quasisimple. This allows us to give unified proofs of Lemmas 6 to 8.

2. If $K \cong U_4(2)$, then $Q \cong 2_+^{1+4}$ and $N_K(R)/Q \cong S_3 \times \mathbb{Z}_3$. We get that $\widetilde{C}/Q$ is isomorphic to a subgroup of $\mathrm{Out}(Q) \cong O_4^+(2)$, from which $\widetilde{C} \leq H$ is clear.

3. If $K \cong P\Omega_{2m}^{\pm}(q)$, then $K_0 \cong L_2(q) \times P\Omega_{2\cdot(m-2)}^{\pm}(q)$. In case $q > 2$, we can modify the argument given above for $E_8(q)$ to handle these groups, too. If $q = 2$, then Lemma 7 does not hold and we show that $F(\widetilde{C}/Q) \leq O_3(K_0Q/Q)$ instead.

4. Note that for the general case of Theorem 1 in the proof of Lemma 8 there are examples where $(K_0Q/Q)T$ is possibly contained in a larger component L. But this group L has to act faithfully on Q/R, and in each case it is easy to see that this is not possible.

5. The H-structure theorem [6] gives us a connection between the maximal 2-local subgroup M of G and some parabolic subgroup of K. This enables us to continue our investigations in the well known finite simple group of Lie type and to derive from these all the information needed to finish the proof of Theorem 1.

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