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A NOTE ON THE BEST-CHOICE PROBLEM RELATED TO THE WEIGHTED RANDOM PERMUTATION

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概 要

A best-choice problem related to the weighted random permutation is considered. The optimal stopping rule will be derived in some simple cases.

1. Introduction

A weighted random permutation of $1, 2, \ldots, n$ with weight $\lambda_1, \lambda_2, \ldots, \lambda_n$ is one whose first element is j with probability $\lambda_j / \sum_i \lambda_i, j = 1, \ldots, n$. If the first element in the permutation is j, then the next element is $i, i \neq j$, with probability $\lambda_i / \sum_{k \neq j} \lambda_k$. In general, each subsequent element of the permutation will equal any value not yet appearing with a probability that is equal to the weight of that value divided by the sums of the weights of all those values that have not yet appeared in the permutation (see Ross[4]).

Imagine a situation where a known number n of rankable applicants appear one at a time. The arrival order of these applicants constitutes a weighted random permutation, if the ith best applicant has her weight $\lambda_i, 1 \leq i \leq n$. The optimal stopping problem we consider here is a best-choice problem related to this weighted random permutation based on the relative ranks of the applicants observed. That is, if we denote by $Y_j, 1 \leq j \leq n$, the relative rank of the jth applicant, the decision of whether we should accept (select) or reject the kth applicant is made based on the observed values of $\{Y_j\}_{j=1}^k$. The objective of the best-choice problem is to find a stopping rule which maximizes the probability of selecting the very best.

In a special case where all the weights are the same, i.e., $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, this problem is greatly simplified into the celebrated classical secretary problem (see, e.g., Gilbert and Mosteller[2]), because, in this case, $Y_1, Y_2, \ldots, Y_n$ turn out to be independent random variables with $P\{Y_j = i\} = 1/j$ for $1 \leq i \leq j$ and $1 \leq j \leq n$. However, when the weights are rank dependent, the problem becomes very complicated because this will result in the sequence $Y_1, Y_2, \ldots, Y_n$ being dependent with the consequent complication of the form of the optimal stopping rule.

In Section 2, we consider the case of $n = 3$ in detail. In section 3, we treat a continuous version of the problem in a special case where $\lambda_1 = 1$, and $\lambda_i \equiv \lambda, 2 \leq i \leq n$.

2. Best-choice problem for $n = 3$

When $n = 2$, it is obvious that the optimal rule selects the first applicant iff $\lambda_1 \geq \lambda_2$. Thus we consider here the case of $n = 3$ as a prelude. Let X_i and Y_i denote respectively the absolute and relative ranks of the *ith* applicant, $i = 1, 2, 3$. Then the joint distribution of (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are given as follows:

$$\begin{aligned}
P\{X_1 = 1, X_2 = 2, X_3 = 3\} &= P\{Y_1 = 1, Y_2 = 2, Y_3 = 3\} \\
&= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 + \lambda_3}\right), \\
P\{X_1 = 1, X_2 = 3, X_3 = 2\} &= P\{Y_1 = 1, Y_2 = 2, Y_3 = 2\} \\
&= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_3}{\lambda_2 + \lambda_3}\right), \\
P\{X_1 = 2, X_2 = 1, X_3 = 3\} &= P\{Y_1 = 1, Y_2 = 1, Y_3 = 3\} \\
&= \left(\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_1}{\lambda_1 + \lambda_3}\right), \\
P\{X_1 = 2, X_2 = 3, X_3 = 1\} &= P\{Y_1 = 1, Y_2 = 2, Y_3 = 1\} \\
&= \left(\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_3}{\lambda_1 + \lambda_3}\right), \\
P\{X_1 = 3, X_2 = 1, X_3 = 2\} &= P\{Y_1 = 1, Y_2 = 1, Y_3 = 2\} \\
&= \left(\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right),
\end{aligned}$$

$$\begin{aligned} P\{X_1 = 3, X_2 = 2, X_3 = 1\} &= P\{Y_1 = 1, Y_2 = 1, Y_3 = 1\} \\ &= \left(\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right). \end{aligned}$$

From this joint distribution, we can obtain the marginal and conditional distributions of X_i's and Y_i's. Some quantities of interest are as follows:

$$P\{Y_2 = 1\} = \frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3^2}{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_3)},$$

$$P\{Y_2 = 2\} = \frac{\lambda_1^2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_3)},$$

$$P\{Y_3 = 1\} = \left(\frac{\lambda_2\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_1 + \lambda_3}\right),$$

$$P\{Y_3 = 2\} = \left(\frac{\lambda_1\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2 + \lambda_3}\right),$$

$$P\{Y_3 = 3\} = \left(\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}\right)\left(\frac{1}{\lambda_1 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3}\right),$$

$$P\{X_2 = 1 | Y_2 = 1\} = \frac{\lambda_1\lambda_2(\lambda_1 + \lambda_2) + \lambda_1\lambda_3(\lambda_1 + \lambda_3)}{(\lambda_1 + \lambda_2)(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3^2)},$$

$$P\{X_3 = 1 | Y_2 = 1\} = \frac{(\lambda_1 + \lambda_3)\lambda_2\lambda_3}{(\lambda_1 + \lambda_2)(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3^2)},$$

$$P\{X_3 = 1 | Y_2 = 2\} = \frac{\lambda_2\lambda_3}{\lambda_1^2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}.$$

Let v_i be the optimal value when the *ith* applicant is a *candidate*, i.e., relatively best applicant. Let also s_i and c_i be the corresponding stopping value and the continuation value respectively, $i = 1, 2$. If we pass over the first two applicants, we select the last applicant irrespective of her quality. Then,

$$v_i = \max\{s_i, c_i\}, \quad i = 1, 2,$$

where

$$s_1 = P\{X_1 = 1\} = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3},$$

$$\begin{aligned} c_1 &= P\{Y_2 = 1\} v_2 + P\{Y_2 = 2\} P\{X_3 = 1 | Y_2 = 2\} \\ &= \frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3^2}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_3)} v_2 + \frac{\lambda_2\lambda_3}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_3)}, \end{aligned}$$

$$s_2 = P\{X_2 = 1 | Y_2 = 1\} = \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2) + \lambda_1\lambda_3(\lambda_1+\lambda_3)}{(\lambda_1+\lambda_2)(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3^2)},$$

$$c_2 = P\{X_3 = 1 | Y_2 = 1\} = \frac{\lambda_2\lambda_3(\lambda_1+\lambda_3)}{(\lambda_1+\lambda_2)(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_3^2)}.$$

Let T_i denote the threshold rule with critical number i, i.e., a stopping rule which starts to select a candidate from time i onward. Then the optmal stopping rule can be summarized as follows.

Theorem 1
Let α and β, functions of λ_1 and λ_2, be defined by

$$\begin{aligned} \alpha(\lambda_1, \lambda_2) &= \frac{(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)}{\lambda_1}, \quad \lambda_1 \geq \lambda_2, \\ \beta(\lambda_1, \lambda_2) &= -\frac{\lambda_1}{2} + \sqrt{\frac{\lambda_1^2}{4} + \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2)}{\lambda_2 - \lambda_1}}, \quad \lambda_1 < \lambda_2. \end{aligned}$$

Then the optimal rule can be described as follows:
(i)When $\lambda_1 \geq \lambda_2$,
T_1 is optimal if $\lambda_3 \leq \alpha(\lambda_1, \lambda_2)$.
T_2 is optimal if $\lambda_3 > \alpha(\lambda_1, \lambda_2)$.
(ii)When $\lambda_1 < \lambda_2$,
T_2 is optimal if $\lambda_3 \leq \beta(\lambda_1, \lambda_2)$.
T_3 is optimal if $\lambda_3 > \beta(\lambda_1, \lambda_2)$.

Proof By straightforward calculations.

Remark: For some given pair λ_1 and λ_3, there exist three numbers

$(\lambda_1 <)\lambda_2^* < \tilde{\lambda_2} < \lambda_2^{**}$ such that T_2 is optimal for the $(\lambda_1, \lambda_2^*, \lambda_3)$- and $(\lambda_1, \lambda_2^{**}, \lambda_3)$-problems, whereas T_3 is optimal for the $(\lambda_1, \tilde{\lambda_2}, \lambda_3)$-problem.

3. Continuous arrival time model

Consider a model in which the *ith* best applicant appear at time U_i, where $U_1, U_2, \ldots, U_n$ are independent exponential random variables with respective rates $\lambda_1, \lambda_2, \ldots, \lambda_n$. It follows from the well known memoryless property of exponential distribution that the order in which the applicants appear is probabilistically the same as in the original model.

We consider the best-choice problem related to this continuous model in a special case where $\lambda_1 = 1$ and $\lambda_i \equiv \lambda, 2 \le i \le n$. Denote by (t,k) the state in which the *kth* applicant, a candidate, has just arrived at time t (note that the state is not path-dependent in this case). Let $p_k(t)$ be the success probability when we choose the current applicant in state (t,k). Two cases are distinguished in state (t,k) depending on whether the relatively best applicant observed by time t is the very best or not. Let $p_k^1(t)$ be the probability that the relatively best is the best overall and $p_k^2(t)$ the probability that the relatively best is not the best overall. Then

$$p_k^1(t) = \binom{n-1}{k-1}\left(1-e^{-\lambda t}\right)^{k-1}\left(e^{-\lambda t}\right)^{n-k}(1-e^{-t}),$$

$$p_k^2(t) = \binom{n-1}{k}\left(1-e^{-\lambda t}\right)^{k}\left(e^{-\lambda t}\right)^{n-1-k}(e^{-t}),$$

and so we obtain

$$\begin{aligned} p_k(t) &= \frac{p_k^1(t)}{p_k^1(t)+p_k^2(t)} \\ &= \frac{e^{-\lambda t}(1-e^{-t})}{\left(\frac{n-k}{k}\right)(1-e^{-\lambda t})e^{-t} + e^{-\lambda t}(1-e^{-t})}. \end{aligned}$$

On the other hand, we can show that the success probability when we choose the next candidate to appear after leaving state (t,k) is given by

$$q_k(t) = \frac{p_k^3(t)}{p_k^1(t)+p_k^2(t)}, \tag{1}$$

where

$$p_k^3(t) = \int_t^{\infty} e^{-s} ds [\sum_{j=0}^{n-k-1} \left(\frac{k}{k+j}\right) \frac{(n-1)!}{k!j!(n-1-k-j)!}$$
$$\times \left(1-e^{-\lambda t}\right)^k \left(e^{-\lambda t} - e^{-\lambda s}\right)^j \left(e^{-\lambda s}\right)^{n-1-k-j}].$$

Given that the best applicant appears at time $s(> t)$ after leaving state (t, k), the conditional probability that we can choose the best applicant becomes

$$\sum_{j=0}^{n-k-1} \left(\frac{k}{k+j}\right) \frac{(n-1)!}{k!j!(n-1-k-j)!} \left(1-e^{-\lambda t}\right)^k \left(e^{-\lambda t} - e^{-\lambda s}\right)^j \left(e^{-\lambda s}\right)^{n-1-k-j},$$

because $(k/k+j)$ is just the probability that no candidate appears in $[t, s)$, provided that $k + j$ applicants appear before time s. Thus Equation (1) follows.

Using the beta function

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt,$$

we can write $q(t)$ as

$$q_k(t) = \frac{\frac{1}{\lambda} e^{-t}(1-e^{-\lambda t}) \sum_{j=0}^{n-k-1} (\frac{n-k}{j+k}) \binom{n-k-1}{j} B(n-k-j-1+\frac{1}{\lambda}, j+1)}{\left(\frac{n-k}{k}\right)(1-e^{-\lambda t})e^{-t} + e^{-\lambda t}(1-e^{-t})}.$$

Let G be the OLA stopping region, i.e., the set of states in which stopping immediately is at least as good as continuing and then stopping with the next candidate. Then

$$\begin{aligned} G &= \{(t, k) : p_k(t) \geq q_k(t)\} \\ &= \{(t, k) : g(t) \geq G_k\}, \end{aligned}$$

where

$$g(t) = \lambda \left(\frac{e^t - 1}{e^{\lambda t} - 1} \right) \quad 0 < t,$$

$$G_k = \sum_{j=0}^{n-k-1} (\frac{n-k}{j+k}) \binom{n-k-1}{j} B(n-k-j-1+\frac{1}{\lambda}, j+1), \quad 1 \leq k < n.$$

Remark: When $\lambda = 1$,

$$g(t) \equiv 1,$$

$$G_k = \sum_{j=0}^{n-k-1} \frac{1}{j+k} = \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1}.$$

Thus G becomes

$$G = \left\{(t,k) : 1 \geq \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1}\right\},$$

which gives the optimal stopping region of the classical secretary problem.

Theorem 2
When $\lambda \leq 1$, G gives the optimal stopping region. This implies that the process stops as soon as it enters a state in G. More specifically, there exists a decreasing sequence of critical numbers $\{t_k\}_{k=1}^{n}$ such that the optimal rule stops in state (t,k) iff $t \geq t_k$, where t_k is a unique root of the equation $g(t) = G_k$ for $G_k \geq 1$. Possibly, $t_k = 0, k \geq r^*$ for some positive integer r^*.

Proof It is well known that the OLA stopping region G gives the optimal stopping region if G is closed in a sense that once the process enters G, then it stays in G for additional time (see Ross[3] or Chow, Robbins and Siegmund [1]). To show that G is closed, it is sufficient to show that $g(t)$ is increasing in t and G_k is decreasing in k. $g(t)$ is obviously increasing. Hereafter we show the monotonicity of G_k. Let $\Gamma(a)$ be the gamma function defined by

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx.$$

This function satisfies the following properties

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

$$\Gamma(a+1) = a\Gamma(a).$$

Using these properties, we have

$$G_{k-1} - G_k = \sum_{j=0}^{n-k} \left(\frac{n-k+1}{j+k-1}\right) \frac{(n-k)!}{j!(n-k-j)!} \frac{\Gamma(n-k-j+\frac{1}{\lambda})\Gamma(j+1)}{(n-k+\frac{1}{\lambda})\Gamma(n-k+\frac{1}{\lambda})}$$
$$- \sum_{j=0}^{n-k-1} \left(\frac{n-k}{j+k}\right) \frac{(n-k-1)!}{j!(n-k-j-1)!} \frac{\Gamma(n-k-j-1+\frac{1}{\lambda})\Gamma(j+1)}{\Gamma(n-k+\frac{1}{\lambda})}$$

$$= \frac{(n-k+1)!\Gamma(\frac{1}{\lambda})}{(n-1)\Gamma(n-k+1+\frac{1}{\lambda})} + \sum_{j=0}^{n-k-1} (n-k)!\Gamma(n-k-j-1+\frac{1}{\lambda})$$
$$\times \frac{\{(n-k-j)(n-k+\frac{1}{\lambda}) + (\frac{1}{\lambda}-1)(j+1)(j+k)\}}{(n-k-j)!\Gamma(n-k+1+\frac{1}{\lambda})(j+k)(j+k-1)}$$

implying that $G_{k-1} - G_k \geq 0$, because $\lambda \leq 1$.

参考文献

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