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American-Style Fractional Lookback Options*

北海道大学・経済学研究科 木村 俊一 (Toshikazu Kimura)
菊地 一哲 (Kazuaki Kikuchi)
Graduate School of Economics and Business Administration
Hokkaido University

1 Introduction

Lookback options are *path-dependent* options whose payoff at (or prior to) expiry depends on the realized extremum of the underlying asset price attained over the options' lifetimes. Lookback options can be classified into two types: *fixed strike* and *floating strike*. Let $(S_t)_{t\geq 0}$ be the price process of the underlying asset, and let $m_t = \min_{0\leq u\leq t} S_u$ and $M_t = \max_{0\leq u\leq t} S_u$. Assume that the price process is monitored continuously; see Heynen and Kat [14] for discrete monitoring. Then, a fixed-strike lookback call (put) is defined as an ordinary option written on the process $(M_t)_{t\geq 0}$ $((m_t)_{t\geq 0})$ instead of $(S_t)_{t\geq 0}$. For European-style lookback options with maturity date T and strike price K, payoffs at the maturity for fixed-strike lookback call and put are respectively given by

$$(M_T - K)^+ \quad \text{and} \quad (K - m_T)^+,$$

where $(x)^+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. These payoffs mean that a fixed-strike lookback call (put) option entitles the holder to the difference between the highest (lowest) realized price of the underlying asset over the trading period and the strike price. Closed-form pricing formulas for European fixed-strike lookback options have been derived by Conze and Viswanathan [7]. Russian options [12, 22] can be considered as a perpetual (*i.e.*, $T = \infty$) American fixed-strike lookback call option with $K = 0$. On the other hand, a floating-strike lookback call (put) depends on the processes $(S_t)_{t\geq 0}$ and $(m_t)_{t\geq 0}$ $((M_t)_{t\geq 0})$, and it always gives the option holder the right to buy (sell) at the lowest (highest) realized price. For European floating-strike lookback call and put with maturity date T, their *standard* terminal payoffs are given by

$$(S_T - m_T)^+ = S_T - m_T \quad \text{and} \quad (M_T - S_T)^+ = M_T - S_T,$$

respectively. Goldman et al. [13] provided closed-form pricing formulas for European floating-strike lookback options and analyzed their properties for some particular cases.

Clearly, standard floating-strike lookback options are *not* genuine option contracts since they are always exercised until the maturity, finishing in-the-money. This means that high premiums are charged for the standard floating-strike lookback options, being less attractive to investors. Conze and Viswanathan [7] introduced a more general and less expensive variant called a *fractional* or *partial* lookback option, where the strike is fixed at some fraction over (for a call) or below (for a put) the extreme value. Specifically, the payoffs for European lookback call and put with fractional floating strikes and maturity date T are respectively given by

$$(S_T - \alpha m_T)^+ \quad \text{and} \quad (\beta M_T - S_T)^+,$$

where α and β are positive constants, allowing flexible adjustment of option premiums. To reduce option premiums, we assume that $\alpha \geq 1$ and $0 < \beta \leq 1$. When $\alpha = \beta = 1$, the fractional floating-strike lookback option agrees with the standard one as a special case. Closed-form valuation formulas for European fractional floating-strike lookback options can be found

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in Conze and Viswanathan [7]. For American fractional floating-strike case, Lai and Lim [20] obtained an integral representation of an early exercise premium. As with vanilla options, there are no pricing formulas for American lookback options, except for the perpetual case; see Dai [8] for the standard floating-strike case and Lai and Lim [20] for the fractional floating-strike case. The purpose of this paper is to develop a fast and accurate numerical method for valuing American fractional floating-strike lookback options.

A number of approximations and/or numerical methods have been developed for numerical valuation of American options, most of which can be also applied to lookback options. For American fractional floating-strike lookback puts, Conze and Viswanathan [7] proposed an explicit upper bound using a technique based on Snell envelopes, which was later shown to be quite loose for short maturities by Barraquand and Pudet [4]. Hull and White [15], Kat [16], Barraquand and Pudet [4], Cheuck and Vorst [6], Babbs [3], Dai [9], and Lai and Lim [20] developed binomial or lattice methods. Among them, a forward shooting grid method in Barraquand and Pudet [4] has a superior performance for American path-dependent options. Yu *et al.* [24] adopted the partial differential equation (PDE) approach together with the finite difference method. It is, however, well known that both the lattice and finite difference methods are quite time consuming if we need solutions with high-precision. In addition, Dai [9] showed that a simple binomial tree is not necessarily consistent with its continuous model, resulting the low speed of convergence. To achieve quick and accurate pricing for practical purposes, this paper adopts a Laplace transform (LT) approach to valuing American floating-strike lookback options; see other related LT approaches of Carr [5] and Kimura [17] for American vanilla options and of Petrella and Kou [21] for a European standard floating-strike lookback option with discrete monitoring.

2 PDE Formulation

Assume that $(S_t)_{t\geq 0}$ is a risk-neutralized diffusion process described by the linear stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \geq 0, \tag{2.1}$$

where $r > 0$ is the risk-free rate of interest, $\delta \geq 0$ is the continuous dividend rate, and $\sigma > 0$ is the volatility coefficient of the asset price. Also, $W \equiv (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration corresponding to W and the probability measure $\mathbb{P}$ is chosen so that the stock has mean rate of return r. Let $C \equiv C(t, S_t, m_t)$ denote the value of the American floating-strike lookback call option at time $t \in [0, T]$. Note that the values of American and European call options are equal if the underlying asset pays no dividends, *i.e.*, $\delta = 0$; see Conze and Viswanathan [7]. In the absence of arbitrage opportunities, the value $C(t, S_t, m_t)$ is a solution of an *optimal stopping problem*

$$C(t, S_t, m_t) = \operatorname*{ess\,sup}_{\mathcal{T}_t \in [t,T]} \mathbb{E}\left[e^{-r(\mathcal{T}_t - t)}(S_{\mathcal{T}_t} - \alpha m_{\mathcal{T}_t})^+ \,|\, \mathcal{F}_t\right] \tag{2.2}$$

for $t \in [0, T]$, where $\mathcal{T}_t$ is a stopping time of the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$. Solving the optimal stopping problem (2.2) is equivalent to finding the points (t, S_t) for which early exercise is optimal. Let $\mathcal{S}$ and $\mathcal{C}$ denote the *stopping region* and *continuation region*, respectively. In terms of the value function $C(t, S, m)$ ($S \equiv S_t$ and $m \equiv m_t$ for abbreviation), the stopping region $\mathcal{S}$ is defined by

$$\mathcal{S} = \left\{(t, S) \in [0, T] \times [m, +\infty) \,|\, C(t, S, m) = (S - \alpha m)^+\right\},$$

for which the stopping time T_t satisfies

$$T_t = \inf\{u \in [t,T] \mid (u, S_u) \in \mathcal{S}\}.$$

The continuation region $\mathcal{C}$ is the complement of $\mathcal{S}$ in $[0,T] \times [m, +\infty)$, *i.e.*,

$$\mathcal{C} = \left\{(t,S) \in [0,T] \times [m,+\infty) \mid C(t,S,m) > (S - \alpha m)^+\right\}.$$

The boundary that separates $\mathcal{S}$ from $\mathcal{C}$ is referred as the *early exercise boundary* (or *critical asset price*), which is defined by

$$\bar{S}_t = \sup\left\{S \in [m,+\infty) \mid C(t,S,m) > (S - \alpha m)^+\right\}, \quad t \in [0,T]. \tag{2.3}$$

At the early exercise boundary $(\bar{S}_t)_{t\in[0,T]}$, the American fractional floating-strike lookback call option would be optimally exercised.

It has been known that $C(t,S,m)$ is obtained by solving a *free boundary problem*; see, *e.g.*, Kwok [19] and Wilmott *et al.* [23, pp. 207–209]. Define the differential operator $\mathcal{A}$ by

$$\mathcal{A} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - \delta) S \frac{\partial}{\partial S} - r.$$

Then, the free boundary problem can be written in linear complimentary form as

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mathcal{A}C \le 0, \qquad C - (S - \alpha m)^+ \ge 0, \\ \left(\dfrac{\partial C}{\partial t} + \mathcal{A}C\right) \cdot \left(C - (S - \alpha m)^+\right) = 0, \quad (t,S) \in \mathcal{C} \end{cases} \tag{2.4}$$

together with auxiliary conditions

$$\left| \begin{array}{l} C(T,S,m) - (S - \alpha m)^+ = 0, \\ \lim\limits_{S \downarrow m} \dfrac{\partial C}{\partial m} = 0. \end{array} \right. \tag{2.5}$$

For the free boundary $(\bar{S}_t)_{t\in[0,T]}$, this problem is equivalent to solving the Black-Scholes-Merton PDE

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - \delta) S \frac{\partial C}{\partial S} - rC = 0, \quad m < S < \bar{S}_t \tag{2.6}$$

with the terminal condition

$$C(T,S,m) = (S - \alpha m)^+ \tag{2.7}$$

and the boundary conditions

$$\left| \begin{array}{l} \lim\limits_{S \uparrow \bar{S}_t} C(t,S,m) = \bar{S}_t - \alpha m, \\ \lim\limits_{S \uparrow \bar{S}_t} \dfrac{\partial C}{\partial S} = 1, \\ \lim\limits_{S \downarrow m} \dfrac{\partial C}{\partial m} = 0, \end{array} \right. \tag{2.8}$$

which are called the *value matching*, *smooth pasting* and *Neumann* conditions in order.

Similarly, if we denote the value of the American floating-strike lookback put option by $P \equiv P(t, S_t, M_t)$, then $P(t,S,M)$ satisfies the same PDE as (2.6), *i.e.*,

$$\frac{\partial P}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - \delta) S \frac{\partial P}{\partial S} - rP = 0, \quad \underline{S}_t < S < M, \tag{2.9}$$

where $(\underline{S}_t)_{t\in[0,T]}$ is the early exercise boundary for put. The boundary conditions for put are

$$\left|\begin{array}{l}\lim_{S\downarrow\underline{S}_t} P(t,S,M) = \beta M - \underline{S}_t,\\ \lim_{S\downarrow\underline{S}_t} \dfrac{\partial P}{\partial S} = -1,\\ \lim_{S\uparrow M} \dfrac{\partial P}{\partial m} = 0,\end{array}\right. \tag{2.10}$$

and the terminal condition is given by

$$P(T,S,M) = (\beta M - S)^+. \tag{2.11}$$

3 Laplace-Carlson Transforms

3.1 Option Values

For the remaining time to maturity $\tau = T - t$, define the time-reversed value $\widetilde{C}(\tau,S,m) = C(T-\tau,S,m)$ $(\tau \geq 0)$ and its Laplace-Carlson transform (LCT) as

$$C^*(\lambda,S,m) = \mathcal{LC}[\widetilde{C}(\tau,S,m)] \equiv \int_0^\infty \lambda e^{-\lambda\tau}\widetilde{C}(\tau,S,m)\mathrm{d}\tau, \tag{3.1}$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. No doubt, there is no essential difference between the LCT (3.1) and the Laplace transform (LT)

$$\widehat{C}(\lambda,S,m) = \mathcal{L}[\widetilde{C}(\tau,S,m)] \equiv \int_0^\infty e^{-\lambda\tau}\widetilde{C}(\tau,S,m)\mathrm{d}\tau.$$

Clearly, we have $C^*(\lambda,S,m) = \lambda\widehat{C}(\lambda,S,m)$ for $\mathrm{Re}(\lambda) > 0$. The principal reason why we prefer LCT to LT is that LCT generates relatively simpler formulas than LT for option pricing problems because constant values are invariant after taking transformation. In the context of option pricing, LCTs have been first adopted in the *randomization* of Carr [5] for valuing an American vanilla put option with a random maturity. Mathematically, Carr's randomization is equivalent to the Post-Widder LT inversion method [1].

For the LCT $C^*(\lambda,S,m)$, we obtain

Theorem 1

$$C^*(\lambda,S,m) = \begin{cases} S - \alpha m, & S \geq \bar{S}^*,\\ A_1 S\left(\dfrac{\alpha m}{S}\right)^{\theta_1} + A_2 S\left(\dfrac{\alpha m}{S}\right)^{\theta_2} + \dfrac{\lambda}{\lambda+\delta}S - \dfrac{\lambda}{\lambda+r}\alpha m, & \alpha m < S < \bar{S}^*,\\ A_3 S\left(\dfrac{\alpha m}{S}\right)^{\theta_1} + A_4 S\left(\dfrac{\alpha m}{S}\right)^{\theta_2}, & m < S \leq \alpha m \end{cases} \tag{3.2}$$

where

$$\begin{aligned} A_1 &\equiv A_1(\theta_1,\theta_2) = \frac{\theta_2}{\theta_2-\theta_1}\left(\frac{\delta}{\lambda+\delta} + \frac{1-\theta_2}{\theta_2}\frac{r}{\lambda+r}\frac{\alpha m}{\bar{S}^*}\right)\left(\frac{\bar{S}^*}{\alpha m}\right)^{\theta_1},\\ A_2 &= A_1(\theta_2,\theta_1),\\ A_3 &\equiv A_3(\theta_1,\theta_2) = A_1(\theta_1,\theta_2) + \frac{\theta_2}{\theta_2-\theta_1}\left(\frac{\delta}{\lambda+\delta} + \frac{1-\theta_2}{\theta_2}\frac{r}{\lambda+r}\right),\\ A_4 &= A_3(\theta_2,\theta_1), \end{aligned} \tag{3.3}$$

and the parameters $\theta_1 = \theta_+ > 0$ and $\theta_2 = \theta_- < 0$ are given by

$$\theta_\pm \equiv \theta\pm(\lambda) = \frac{1}{\sigma^2}\left\{-(\delta - r - \tfrac{1}{2}\sigma^2) \pm \sqrt{(\delta - r - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+\delta)}\right\},$$

which are two real roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\theta^2 + (\delta - r - \tfrac{1}{2}\sigma^2)\theta - (\lambda+\delta) = 0. \tag{3.4}$$

In addition, $\bar{S}^* \equiv \bar{S}^*(\lambda)$ is defined by

$$\bar{S}^* = \frac{m}{\xi^*},$$

of which $\xi^* \equiv \xi^*(\lambda)$ $(0 < \xi^* < \alpha^{-1} \leq 1)$ satisfies the equation

$$\begin{aligned}
&\lambda\left\{\frac{\alpha^{\theta_2}}{\theta_1} - \frac{\alpha^{\theta_1}}{\theta_2} + (\alpha^{\theta_2} - \alpha^{\theta_1})\frac{r-\delta}{\lambda+\delta}\right\}(\xi^*)^{\theta_1+\theta_2} \\
&= \delta\frac{\lambda+r}{\lambda+\delta}\left\{(\xi^*)^{\theta_2} - (\xi^*)^{\theta_1}\right\} + \alpha r\left\{\frac{1-\theta_2}{\theta_2}(\xi^*)^{\theta_2+1} - \frac{1-\theta_1}{\theta_1}(\xi^*)^{\theta_1+1}\right\}.
\end{aligned} \tag{3.5}$$

Let $P \equiv P(t,S,M)$ denote the value of the American floating-strike lookback put option at time $t \in [0,T]$, and let $P^*(\lambda,S,M)$ be the LCT of $\widetilde{P}(\tau,S,M) = P(T-\tau,S,M)$ for $\mathrm{Re}(\lambda) > 0$. For the LCT of $\widetilde{P}(\tau,S,M)$, we can obtain an analogous result to Thorem 1 in much the same way.

Theorem 2

$$P^*(\lambda,S,M) = \begin{cases}
\beta M - S, & S \leq \underline{S}^*, \\
B_1 S\left(\dfrac{\beta M}{S}\right)^{\theta_1} + B_2 S\left(\dfrac{\beta M}{S}\right)^{\theta_2} - \dfrac{\lambda}{\lambda+\delta}S + \dfrac{\lambda}{\lambda+r}\beta M, & \underline{S}^* < S < \beta M, \\
B_3 S\left(\dfrac{\beta M}{S}\right)^{\theta_1} + B_4 S\left(\dfrac{\beta M}{S}\right)^{\theta_2}, & \beta M \leq S < M
\end{cases} \tag{3.6}$$

where

$$\begin{aligned}
B_1 &\equiv B_1(\theta_1,\theta_2) = \frac{\theta_2}{\theta_1-\theta_2}\left(\frac{\delta}{\lambda+\delta} + \frac{1-\theta_2}{\theta_2}\frac{r}{\lambda+r}\frac{\beta M}{\underline{S}^*}\right)\left(\frac{\underline{S}^*}{\beta M}\right)^{\theta_1}, \\
B_2 &= B_1(\theta_2,\theta_1), \\
B_3 &\equiv B_3(\theta_1,\theta_2) = B_1(\theta_1,\theta_2) + \frac{\theta_2}{\theta_1-\theta_2}\left(\frac{\delta}{\lambda+\delta} + \frac{1-\theta_2}{\theta_2}\frac{r}{\lambda+r}\right), \\
B_4 &= B_3(\theta_2,\theta_1),
\end{aligned} \tag{3.7}$$

and $\underline{S}^* \equiv \underline{S}^*(\lambda)$ is defined by

$$\underline{S}^* = \frac{M}{\eta^*},$$

of which $\eta^* \equiv \eta^*(\lambda)$ $(\eta^* > \beta^{-1} \geq 1)$ satisfies the equation

$$\begin{aligned}
&\lambda\left\{\frac{\beta^{\theta_2}}{\theta_1} - \frac{\beta^{\theta_1}}{\theta_2} + (\beta^{\theta_2} - \beta^{\theta_1})\frac{r-\delta}{\lambda+\delta}\right\}(\eta^*)^{\theta_1+\theta_2} \\
&= \delta\frac{\lambda+r}{\lambda+\delta}\left\{(\eta^*)^{\theta_2} - (\eta^*)^{\theta_1}\right\} + \beta r\left\{\frac{1-\theta_2}{\theta_2}(\eta^*)^{\theta_2+1} - \frac{1-\theta_1}{\theta_1}(\eta^*)^{\theta_1+1}\right\}.
\end{aligned} \tag{3.8}$$

From (3.5) and (3.8), $\xi^*(\lambda)$ ($\in (0, \alpha^{-1})$) and $\eta^*(\lambda)$ ($\in (\beta^{-1}, \infty)$) can be obtained by solving a functional equation of the form

$$x = f_\lambda(x), \tag{3.9}$$

where f_λ is an operator mapping defined by

$$f_\lambda(x) \equiv f_\lambda(x;\nu) = \frac{\lambda g(\lambda) x^{\theta_1+\theta_2} - \delta \dfrac{\lambda + r}{\lambda + \delta}\left(x^{\theta_2} - x^{\theta_1}\right)}{\nu r \left(\dfrac{1-\theta_2}{\theta_2} x^{\theta_2} - \dfrac{1-\theta_1}{\theta_1} x^{\theta_1}\right)}, \quad \nu = \alpha, \beta \tag{3.10}$$

with

$$g(\lambda) \equiv g(\lambda;\nu) = \frac{\nu^{\theta_2}}{\theta_1} - \frac{\nu^{\theta_1}}{\theta_2} + (\nu^{\theta_2} - \nu^{\theta_1})\frac{r-\delta}{\lambda+\delta}, \quad \nu = \alpha, \beta. \tag{3.11}$$

Note that $f_\lambda(x)$ is symmmetric with respect to θ_1 and θ_2. From the functional equation (3.9), we can show some asymptotic properties of the early exercise boundaries.

Theorem 3 For the early exercise boundaries $(\bar{S}_t)_{t\in[0,T]}$ and $(\underline{S}_t)_{t\in[0,T]}$ of the perpetual fractional lookback options with $T = \infty$, we have

$$\bar{S}_t = \bar{S}_\infty \equiv \frac{m}{\xi_\infty} \quad \text{and} \quad \underline{S}_t = \underline{S}_\infty \equiv \frac{M}{\eta_\infty} \tag{3.12}$$

for all $t \geq 0$. If $\delta > 0$, the constants $\xi_\infty \in (0, \alpha^{-1})$ and $\eta_\infty \in (\beta^{-1}, \infty)$ exist uniquely and they are solutions of the common equation

$$x = f_0(x), \tag{3.13}$$

where

$$f_0(x) \equiv f_0(x;\nu) = -\frac{\theta_1^\circ \theta_2^\circ}{\nu} \frac{1 - x^{\theta_1^\circ - \theta_2^\circ}}{\theta_1^\circ(1-\theta_2^\circ) - \theta_2^\circ(1-\theta_1^\circ)x^{\theta_1^\circ - \theta_2^\circ}}, \quad \nu = \alpha, \beta,$$

and $\theta_i^\circ = \lim_{\lambda\to 0}\theta_i(\lambda)$ $(i = 1, 2)$. If $\delta = 0$, then

$$\bar{S}_\infty = \infty \quad \text{and} \quad \underline{S}_\infty = 0.$$

Theorem 4 For the early exercise boundaries $(\bar{S}_t)_{t\in[0,T]}$ and $(\underline{S}_t)_{t\in[0,T]}$ of the fractional lookback options with $T < \infty$, we have

$$\lim_{t\to T} \bar{S}_t = \max\left(\frac{r}{\delta}, 1\right)\alpha m \quad \text{and} \quad \lim_{t\to T} \underline{S}_t = \min\left(\frac{r}{\delta}, 1\right)\beta M. \tag{3.14}$$

3.2 Early Exercise Premiums

For American vanilla options, it has been well known that the value of an American option can be represented as the sum of the value of the corresponding European option and the so-called *early exercise premium*. For American fractional lookback options, Lai and Lim [20] proved that the value has such a decomposition and that the premium has an integral representation; see Proposition 2 in Lai and Lim [20]. Here, we will derive closed-form LCTs of early exercise premiums for the American fractional lookback call and put options.

First, we will derive the LCT of the European call value: Let $c(t, S, m)$ denote the value of the European fractional floating-strike lookback call option at time $t \in [0, T]$. As in the American counterpart, $c(t, S, m)$ satisfies the Black-Scholes-Merton PDE

$$\frac{\partial c}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + (r-\delta)S\frac{\partial c}{\partial S} - rc = 0, \quad S > m \tag{3.15}$$

together with the boundary conditions

$$\begin{aligned} &\lim_{S\uparrow\infty}\frac{\partial c}{\partial S}<\infty, \\ &\lim_{S\downarrow m}\frac{\partial c}{\partial m}=0, \end{aligned} \tag{3.16}$$

and the terminal condition

$$c(T,S,m)=(S-\alpha m)^+. \tag{3.17}$$

The solution can be find in Zhu *et al.* [25, p. 152] for $\delta\neq r$ (that includes a typo) or in Lai and Lim [20, Proposition 2]. Since the notation and assumptions used in these results are fairly different from those in this paper, we rewrite it to obtain

$$\begin{aligned} c(t,S,m) &= Se^{-\delta(T-t)}\Phi(d_1^+)-\alpha me^{-r(T-t)}\Phi(d_1^-) \\ &+\begin{cases} \dfrac{\alpha S}{\gamma}\left\{e^{-r(T-t)}\left(\dfrac{m}{S}\right)^{\gamma}\Phi(d_2^+)-e^{-\delta(T-t)}\alpha^{\gamma}\Phi(d_2^-)\right\}, & \delta\neq r \\ \alpha Se^{-r(T-t)}\sigma\sqrt{T-t}\left(d_2^+\Phi(d_2^+)+\phi(d_2^+)\right), & \delta=r, \end{cases} \end{aligned} \tag{3.18}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ respectively denote the cdf and pdf of the standard normal distribution,

$$\begin{aligned} d_1^{\pm} &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{S}{\alpha m}\right)+(r-\delta\pm\tfrac{1}{2}\sigma^2)(T-t)\right\}, \\ d_2^{\pm} &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{m}{\alpha S}\right)\pm(r-\delta\mp\tfrac{1}{2}\sigma^2)(T-t)\right\}, \end{aligned}$$

and

$$\gamma=\frac{2(r-\delta)}{\sigma^2}.$$

For the time-reversed value $\widetilde{c}(\tau,S,m)=c(T-\tau,S,m)$ ($\tau\geq 0$), define its LCT as $c^*(\lambda,S,m)=\mathcal{LC}[\widetilde{c}(\tau,S,m)]$. Then, in much the same way as in the American case, we have

Proposition 1

$$c^*(\lambda,S,m)=\begin{cases} a_1S\left(\dfrac{\alpha m}{S}\right)^{\theta_1}+\dfrac{\lambda}{\lambda+\delta}S-\dfrac{\lambda}{\lambda+r}\alpha m, & S>\alpha m, \\ a_3S\left(\dfrac{\alpha m}{S}\right)^{\theta_1}+a_4S\left(\dfrac{\alpha m}{S}\right)^{\theta_2}, & m<S\leq\alpha m \end{cases} \tag{3.19}$$

where

$$\begin{aligned} a_1 &= \frac{\theta_2}{\theta_1-\theta_2}\left\{(\alpha^{\theta_1}-\alpha^{\theta_2})\frac{\lambda}{\lambda+\delta}+\left(\frac{1-\theta_2}{\theta_2}\alpha^{\theta_1}-\frac{1-\theta_1}{\theta_1}\alpha^{\theta_2}\right)\frac{\lambda}{\lambda+r}\right\}\alpha^{-\theta_1}, \\ a_3 &= \frac{\theta_2}{\theta_2-\theta_1}\left(\frac{\lambda}{\lambda+\delta}+\frac{1-\theta_1}{\theta_1}\frac{\lambda}{\lambda+r}\right)\alpha^{\theta_2-\theta_1}, \\ a_4 &= \frac{\theta_1}{\theta_1-\theta_2}\left(\frac{\lambda}{\lambda+\delta}+\frac{1-\theta_1}{\theta_1}\frac{\lambda}{\lambda+r}\right). \end{aligned} \tag{3.20}$$

Theorem 5

$$C^*(\lambda, S, m) = \begin{cases} c^*(\lambda, S, m) + e_c^*(\lambda, S, m), & m < S < \bar{S}^*, \\ S - \alpha m, & S \geq \bar{S}^* \end{cases} \tag{3.21}$$

where $e_c^*(\lambda, S, m)$ is the early exercise premium

$$e_c^*(\lambda, S, m) = \begin{cases} S(A_1 - a_1)\left(\frac{\alpha m}{S}\right)^{\theta_1} + SA_2\left(\frac{\alpha m}{S}\right)^{\theta_2}, & \alpha m < S < \bar{S}^*, \\ S(A_3 - a_3)\left(\frac{\alpha m}{S}\right)^{\theta_1} + S(A_4 - a_4)\left(\frac{\alpha m}{S}\right)^{\theta_2}, & m < S \leq \alpha m \end{cases} \tag{3.22}$$

with A_i and a_i $(i = 1, \ldots, 4)$ defined in (3.3) and (3.20), respectively.

Let $p(t, S, m)$ denote the value of the European fractional floating-strike lookback put option at time $t \in [0, T]$. As in the American counterpart, $p(t, S, m)$ satisfies the Black-Scholes-Merton PDE

$$\frac{\partial p}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + (r - \delta)S\frac{\partial p}{\partial S} - rp = 0, \quad S < M \tag{3.23}$$

together with the boundary conditions

$$\begin{aligned} &\lim_{S \downarrow 0} p(t, S, M) = \beta M e^{-r(T-t)}, \\ &\lim_{S \uparrow M} \frac{\partial p}{\partial M} = 0, \end{aligned} \tag{3.24}$$

and the terminal condition

$$p(T, S, m) = (\beta M - S)^+. \tag{3.25}$$

The European put value $p(t, S, M)$ can be written as

$$\begin{aligned} p(t, S, m) &= \beta M e^{-r(T-t)}\Phi(-h_1^-) - Se^{-\delta(T-t)}\Phi(-h_1^+) \\ &+ \begin{cases} \frac{\beta S}{\gamma}\left\{e^{-\delta(T-t)}\beta^\gamma\Phi(-h_2^-) - e^{-r(T-t)}\left(\frac{M}{S}\right)^\gamma \Phi(-h_2^+)\right\}, & \delta \neq r \\ \beta S e^{-r(T-t)}\sigma\sqrt{T-t}\left(\phi(-h_2^+) - h_2^+\Phi(-h_2^+)\right), & \delta = r, \end{cases} \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} h_1^\pm &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{S}{\beta M}\right) + (r - \delta \pm \tfrac{1}{2}\sigma^2)(T-t)\right\}, \\ h_2^\pm &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{M}{\beta S}\right) \pm (r - \delta \mp \tfrac{1}{2}\sigma^2)(T-t)\right\}. \end{aligned}$$

For the time-reversed value $\widetilde{p}(\tau, S, m) = p(T-\tau, S, m)$ $(\tau \geq 0)$, define its LCT as $p^*(\lambda, S, m) = \mathcal{LC}[\widetilde{p}(\tau, S, m)]$. Then, as with the call case, we can obtain the following proposition and theorem, which proofs are omitted.

Proposition 2

$$p^*(\lambda, S, M) = \begin{cases} b_2 S\left(\frac{\beta M}{S}\right)^{\theta_2} - \frac{\lambda}{\lambda + \delta}S + \frac{\lambda}{\lambda + r}\beta M, & S < \beta M, \\ b_3 S\left(\frac{\beta M}{S}\right)^{\theta_1} + b_4 S\left(\frac{\beta M}{S}\right)^{\theta_2}, & \beta M \leq S < M \end{cases} \tag{3.27}$$

where

$$b_2 = \frac{\theta_1}{\theta_2 - \theta_1}\left\{(\beta^{\theta_1} - \beta^{\theta_2})\frac{\lambda}{\lambda+\delta} + \left(\frac{1-\theta_2}{\theta_2}\beta^{\theta_1} - \frac{1-\theta_1}{\theta_1}\beta^{\theta_2}\right)\frac{\lambda}{\lambda+r}\right\}\beta^{-\theta_2},$$

$$b_3 = \frac{\theta_2}{\theta_1 - \theta_2}\left(\frac{\lambda}{\lambda+\delta} + \frac{1-\theta_2}{\theta_2}\frac{\lambda}{\lambda+r}\right), \tag{3.28}$$

$$b_4 = \frac{\theta_1}{\theta_2 - \theta_1}\left(\frac{\lambda}{\lambda+\delta} + \frac{1-\theta_2}{\theta_2}\frac{\lambda}{\lambda+r}\right)\beta^{\theta_1-\theta_2}.$$

Theorem 6

$$P^*(\lambda, S, M) = \begin{cases} p^*(\lambda, S, M) + e_p^*(\lambda, S, M), & \underline{S}^* < S < M, \\ \beta M - S, & S \le \underline{S}^* \end{cases} \tag{3.29}$$

where $e_p^*(\lambda, S, M)$ is the early exercise premium

$$e_p^*(\lambda, S, M) = \begin{cases} SB_1\left(\frac{\beta M}{S}\right)^{\theta_1} + S(B_2 - b_2)\left(\frac{\beta M}{S}\right)^{\theta_2}, & \underline{S}^* < S < \beta M, \\ S(B_3 - b_3)\left(\frac{\beta M}{S}\right)^{\theta_1} + S(B_4 - b_4)\left(\frac{\beta M}{S}\right)^{\theta_2}, & \beta M \le S < M \end{cases} \tag{3.30}$$

with B_i and b_i $(i = 1, \ldots, 4)$ defined in (3.7) and (3.28), respectively.

4 Conclusion

In this paper, we analyzed American fractional floating-strike lookback options via a Laplace transform approach to obtain the transforms of the early exercise boundaries, option values, hedging parameters and early exercise premiums, all of which can be expressed in terms of a root of a functional equation. Applying Abelian theorems to this equation, we characterized asymptotic behaviors of the early exercise boundaries at time to close to expiration and at infinite time to expiration.

The Laplace transform approach is so general that it could be applied to other American path-dependent options whose payoff functions are continuous with respect to the state variables. For options with discontinuous payoff functions such as digital options, however, there remains some numerical issues in Laplace transform inversion. In addition, the approach could be extended to the cases that the underlying asset price has jumps [18], and it is discretely monitored [21].

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