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On the continuity of positive definite functions on conelike semigroups

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Dedicated to the memory of Knud Maack Bisgaard

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Abstract

Let S be a conelike semigroup in $\mathbb{Q}^k$. In [5], P. Ressel showed an integral representaion of bounded positive definite functions on S which is continuous at 0. In this paper, we will analyze some integral representations of unbounded positive definite functions on S which is continuous at 0.

1 Introduction

Let S be an abelian semigroup with the identity 0. A function $\varphi : S \to \mathbb{R}$ is called *positive definite* if

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j + s_k) \geq 0$$

for all $n \in \mathbb{N}$, $s_1, \cdots, s_n \in S$, $c_1, \cdots, c_n \in \mathbb{R}$.
A function $\sigma : S \to \mathbb{R}$ is called a *character* if it is multiplicative and not identically zero. In particular, if $0 \notin \sigma(S)$, σ is called *zerofree.* The set of characters on S is denoted by S^*. Denote by $\mathcal{A}(S^*)$ the least σ-ring of subsets of S^* rendering the mapping $S^* \ni \sigma \mapsto \sigma(s) \in \mathbb{R}$ measurable for each $s \in S$. A function $\varphi : S \to \mathbb{R}$ is called a *moment function* if there is a measure μ defined on $\mathcal{A}(S^*)$ such that

$$\varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma)$$

for all $s \in S$. Note that every moment function is positive definite and every bounded positive definite function on S is a moment function whose representing

measure is unique (see [1], Theorem 4.2.8). But a positive definite function is not necessarily a moment function (see [1], Theorem 6.3.5), and a representing measure is not necessarily unique if any (see [1], Example 6.4.3).
An abelian $*$-semigroup S is called *determinate* if whenever μ and ν are measures on $\mathcal{A}(S^*)$ such that

$$\int_{S^*} \sigma(s) d\mu(\sigma) = \int_{S^*} \sigma(s) d\nu(\sigma), \quad s \in S$$

then $\mu = \nu$. The semigroup S is called *semiperfect* if every positive definite function $\varphi : S \to \mathbb{R}$ is a moment function, and *perfect* if S is semiperfect and determinate.

A subset M of a vector space over the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$) is called *conelike* if for each $s \in M$ there is some $a \in \mathbb{K}$ such that $\alpha s \in M$ for all $\alpha \in \mathbb{K}$ satisfying $\alpha \geq a$.

P. Ressel has proved the following theorem (see [5], Theorem 2):

Ressel's Theorem *Let S be a conelike semigroup in the real vector space $\mathbb{R}^k$, $k \geq 1$, with $\overset{\circ}{S} \neq \emptyset$ and $0 \in \overline{\widetilde{S}}$, where $\widetilde{S} := \{s \in S \mid (\mathbb{R}_+ s) \cap \overset{\circ}{S} \neq \emptyset\}$. For a bounded positive definite function $\varphi : S \to \mathbb{R}$ the following properties are equivalent:*

(i) *φ is uniformly continuous.*

(ii) *φ is continuous at 0.*

(iii) *$\exists \{s_n\} \subset \widetilde{S}$ with $s_n \to 0$ and $\varphi(s_n) \to \varphi(0)$.*

(iv) *There is a bounded nonnegative measure μ on $S^{\square}$ such that $\varphi(s) = \int_{S^{\square}} e^{-\langle v,s \rangle} d\mu(v)$, $s \in S$, where $S^{\square} := \{v \in \mathbb{R}^k \mid \langle v, s \rangle \geq 0$ for all $s \in S\}$.*

It is natural to consider this theorem for unbounded positive definite functions. In general, every unbounded positive definite function is not a moment function. But every conelike semigroup in the rational vector space $\mathbb{Q}^k$, $k \geq 1$, is perfect (see [4], Theorem 3.3, [2], Theorem 6). In section 3, we will prove a Ressel-type theorem for unbounded positive definite functions on conelike semigroups in $\mathbb{Q}$. In section 4, we will show that such a Ressel-type theorem in $(\mathbb{Q}_+ \setminus \{0\})^2 \cup \{(0,0)\}$ does not hold. In section 5, for some conelike semigroups in $\mathbb{Q}^k$, we will prove that the implication (ii) $\Rightarrow$ (iv) holds.

Throughout this paper, an abelian semigoup S in $\mathbb{Q}^k$ (or $\mathbb{R}^k$) is conelike, and the composition on S is the ordinary addition. See [1] for other details on positive definite and moment functions, and see [3] on positive definite functions on conelike semigroups.

2 Pleliminaries

In this section, we will determine explicitly the zerofree characters on S with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$, where $\overset{\circ}{S}_{\mathbb{Q}}$ is the interior of S in the rational vector space $\mathbb{Q}^k$ with the relative topology. This argument is similar to P. Ressel's (cf. [5]).

Proposition 1 *Let S be a conelike subsemigroup of $\mathbb{Q}^k$ with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$. Then every zerofree character $\sigma \in S^*$ is of the form*

$$\sigma(s) = \exp\langle v, s\rangle$$

for some $v \in \mathbb{R}^k$.

Put $\widetilde{S_{\mathbb{Q}}} := \{s \in S \mid (\mathbb{Q}_{+}s) \cap \overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset\}$. The set $\widetilde{S_{\mathbb{Q}}}$ contains $\overset{\circ}{S}_{\mathbb{Q}}$. By the similar proof of [5], Lemma 3, we have the following.

Lemma 2 *Let S be a conelike subsemigroup of $\mathbb{Q}^k$ with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$, and $\sigma \in S^*$ is not zerofree. Then $\sigma \equiv 0$ on $\widetilde{S_{\mathbb{Q}}}$, in particular on $\overset{\circ}{S}_{\mathbb{Q}}$.*

Define the sets

$$W := \{\sigma \in S^* \mid \sigma : \text{ zerofree}\},$$
$$N := \{\sigma \in S^* \mid \sigma : \text{ not zerofree}\}.$$

If $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$, by Propositon 1, W is topological semigroup isomorphic to $\mathbb{R}^k$ by the correspondence

$$f : (s \mapsto \exp\langle v, s\rangle) \mapsto v.$$

Since S is perfect, every positive definite function φ on S has the following integral representation with the unique measure μ on S^*:

$$\varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma), \quad s \in S.$$

Since every character $\sigma \in N$ is identically zero on $\overset{\circ}{S}_{\mathbb{Q}}$ by Lemma 2, then

$$\varphi(s) = \int_{\mathbb{R}^k} \exp\langle v, s\rangle d\nu(v), \quad s \in \overset{\circ}{S}_{\mathbb{Q}},$$

where ν is the image measure defined by $\nu := \mu^f$.

3 In the Case of S in $\mathbb{Q}$

In the case of $S \subsetneq \mathbb{Q}$ with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$, it is easily obtained that $S^* = W \cup N = W \cup \{\mathbf{1}_{\{0\}}\}$, where $\mathbf{1}_{\{0\}}$ is the indicator function of $\{0\}$. We have the following:

Theorem 3 *Let S be a conelike semigroup in the rational vector space $\mathbb{Q}$ with $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$ and $0 \in \overline{\widetilde{S_{\mathbb{Q}}}}$. For a positive definite function $\varphi : S \to \mathbb{R}$ the following properties are equivalent :*

(i) *φ is continuous.*

(ii) *φ is continuous at 0.*

(iii) *$\exists \{s_n\} \subset \widetilde{S_{\mathbb{Q}}}$ with $s_n \to 0$ and $\varphi(s_n) \to \varphi(0)$.*

(iv) *There is a nonnegative measure* ν *on* $\mathbb{R}$ *such that* $\varphi(s) = \int_{\mathbb{R}} e^{vs} d\nu(v)$, $s \in S$.

Corollary 4 *Let* S *be a conelike semigroup in the real vector space* $\mathbb{R}$ *and define* $S_{\mathbb{Q}} := S \cap \mathbb{Q}$. *Suppose that* $\overset{\circ}{S} \neq \emptyset$, $0 \in \overline{\overset{\circ}{S_{\mathbb{Q}}}}$ *and* $S = \overline{S_{\mathbb{Q}}}$. *Then a function* $\varphi : S \to \mathbb{R}$ *is continuous and positive definite if and only if there exists a nonnegative measure* ν *on* $\mathbb{R}$ *such that*

$$\varphi(s) = \int_{\mathbb{R}} e^{vs} d\nu(v), \quad s \in S.$$

4 In the Case of S in $\mathbb{Q}^2$

In the case of S in $\mathbb{Q}$, we proved a Ressel-type theorem for unbounded positive definite functions. But, in the case of S in $\mathbb{Q}^2$, a Ressel-type theorem such as Theorem 3 does not hold. In this section, we will show some counterexamples. Throughout this section, let S be the abelian semigroup $(\mathbb{Q}_+ \setminus \{0\})^2 \cup \{(0,0)\}$.

Example 1 (Counterexample of (iv) $\Rightarrow$ (ii)) For each $k \in \mathbb{N}$, define $v_k \in \mathbb{R}^2$ by $v_k = (k, -k^2)$. Let m be the measure $\sum_{k=1}^{\infty} \frac{1}{k^2} \varepsilon_{v_k}$ on $\mathbb{R}^2$, where ε_{v_k} is the Dirac measure supported by $\{v_k\}$. Define

$$\varphi(x,y) := \int_{\mathbb{R}^2} e^{\langle v,(x,y)\rangle} dm(v) = \sum_{k=1}^{\infty} k^{-2} e^{kx-k^2 y} < \infty, \quad (x,y) \in S.$$

Now φ is not continuous at $(0,0)$. In fact, let $\{x_n\}$ be any sequence of positive numbers tending to 0. For each n, since $\varphi(x_n, y) \to \infty$ as $y \to 0$, we can choose y_n such that $0 < y_n < \frac{1}{n}$ and $\varphi(x_n, y_n) > n$. Then $(x_n, y_n) \to (0,0)$ but $\varphi(x_n, y_n) \to \infty$.

Example 2 (Counterexample of (iii) + (iv) $\Rightarrow$ (ii)) Let φ be the function as above. We only have to show that there is a sequence $\{s_n\}$ in $\overset{\circ}{S_{\mathbb{Q}}}$ such that $s_n \to 0$ and $\varphi(s_n) \to \varphi(0)$ as $n \to \infty$. For each $n \in \mathbb{N}$, define a continuous mapping γ_n on $(-1,1)$ by $\gamma_n(-t) = \left(\frac{1-t}{n}, \frac{1}{n}\right)$ and $\gamma_n(t) = \left(\frac{1}{n}, \frac{1-t}{n}\right)$ for $0 \le t < 1$. We can easily prove that $\varphi(\gamma_n(-t)) \downarrow \sum_{k=1}^{\infty} \frac{1}{k^2} e^{\frac{k-k^2}{n}} < \varphi(0)$ and $\varphi(\gamma_n(t)) \to \infty$ as $0 \le t \uparrow 1$. By continuity, we can choose $t_n \in (-1,1) \cap \mathbb{Q}$ such that $\varphi(\gamma_n(t_n)) = \varphi(0)$. Putting $s_n = \gamma_n(t_n) \in \overset{\circ}{S_{\mathbb{Q}}}$, we have that $s_n = \gamma_n(t_n) \to 0$ and $\varphi(s_n) = \varphi(\gamma_n(t_n)) = \varphi(0)$. Then we can obtain the result.

Example 3 (Counterexample of (iii) $\Rightarrow$ (iv)) Let φ and $\{s_n\}$ be as above, and let μ be the representing measure of φ on S^*. Choose a number α such that $\sum_{k=1}^{\infty} \frac{1}{k^2} e^{\frac{k-k^2}{n}} < \alpha < \varphi(0)$. Define the function ψ as follows:

$$\psi(x,y) = \begin{cases} \varphi(x,y) & ((x,y) \in S \setminus \{(0,0)\}) \\ \alpha & ((x,y) = (0,0)) \end{cases}$$

Then ψ is positive definite on S. By the similar argument to take $\{t_n\}$, we can choose $\widetilde{t}_n \in (-1,1) \cap \mathbb{Q}$ such that $\psi(\gamma_n(\widetilde{t}_n)) = \alpha$. Putting $\widetilde{s}_n = \gamma_n(\widetilde{t}_n) \in \overset{\circ}{S}_{\mathbb{Q}}$, we have that $\widetilde{s}_n = \gamma_n(\widetilde{t}_n) \to 0$ and $\psi(\widetilde{s}_n) = \psi(\gamma_n(\widetilde{t}_n)) = \psi(0)$. But the support of the representing measure of ψ contains $\{\mathbf{1}_{\{0\}}\}$. In fact, Since ψ is a moment function on S, there exists the measure μ_0 on S^* such that

$$\psi(s) = \int_{S^*} \sigma(s) d\mu_0(\rho), \quad s \in S.$$

Put $H := S \setminus \{(0,0)\}$. By [6], Lemma 2.2, the mapping $f : \sigma \mapsto \sigma|_H$ is a one-to-one correspondence between $S^* \setminus \{\mathbf{1}_{\{0\}}\}$ and H^*. Let $\widetilde{\mu}$ and $\widetilde{\mu_0}$ be the images of μ and μ_0, respectively, i.e., $\widetilde{\mu} = \mu^f$ and $\widetilde{\mu_0} = {\mu_0}^f$. For $s \in H$,

$$\begin{aligned}\int_{H^*} \sigma(s) d\widetilde{\mu_0}(\sigma) = \int_{S^*} \sigma(s) d\mu_0(\sigma) = \psi(s) \\ = \varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma) = \int_{H^*} \sigma(s) d\widetilde{\mu}(\sigma).\end{aligned}$$

By [6], Theorem 3.2, H is perfect (see [6] for the definition of pefectness of H). By [6], Proposition 3.1, $\widetilde{\mu} = \widetilde{\mu_0}$ on H^*. Suppose $\mu_0(\{\mathbf{1}_{\{0\}}\}) = 0$, then $\mu = \mu_0$ on S^*, hence $\varphi = \psi$ on S. This contradicts to $\varphi \neq \psi$. Therefore $\mu_0(\{\mathbf{1}_{\{0\}}\}) \neq 0$.

5 In the case of S in $\mathbb{Q}^k$

In the case of S in $\mathbb{Q}^2$, a Ressel-type theorem such as Theorem 3 does not hold. But, under an assumption of S, we will show the implication (ii) $\Rightarrow$ (iv).

Proposition 5 *Let S be a conelike semigroup in the rational vector space $\mathbb{Q}^k$, $k \geq 2$, such that $\overset{\circ}{S}_{\mathbb{Q}} \neq \emptyset$ and there exists a sequence $\{s_n\}$ of $\widetilde{S_{\mathbb{Q}}}$ satisfying $\lim_{n\to\infty} s_n = 0$ and $\dim(\text{linspan}\{s_n\}) = 1$. For a continuous and positive definite function φ on S there exists the nonnegative measure ν on $\mathbb{R}^k$ such that*

$$\varphi(s) = \int_{\mathbb{R}^k} e^{\langle v,s \rangle} d\nu(v), \quad s \in S.$$

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