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A generalization of the Sobolev-Lieb-Thirring inequality

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1 Introduction

In this article we explain about a generalization of the Sobolev-Lieb-Thirring inequality and its application. In the proof of our theorem we use the φ-transform of Frazier and Jawerth.

In 1976 Lieb and Thirring proved the following inequality([8]).

Theorem 1.1 (The Lieb-Thirring inequality) *Let V be a non-negative measurable function on $\mathbb{R}^n$ and*

$$\gamma \geq \frac{1}{2} \quad \text{for} \quad n=1,$$

$$\gamma > 0 \quad \text{for} \quad n=2,$$

$$\gamma \geq 0 \quad \text{for} \quad n \geq 3.$$

Then we have

$$\sum_i |\lambda_i|^\gamma \leq c_{n,\gamma} \int_{\mathbb{R}^n} V^{n/2+\gamma} dx,$$

where $\lambda_1 \leq \lambda_2 \leq \cdots$ are the negative eigenvalues of the Schrödinger operator $-\Delta - V$ on $L^2(\mathbb{R}^n)$.

The case $\gamma > 1/2, n=1$ or $\gamma > 0, n \geq 2$ was proved by Lieb and Thirring([8]). The case $\gamma = 1/2, n=1$ was proved by Weidl([15]). The case $\gamma = 0, n \geq 3$ was established by Cwikel([1]), Lieb([7]) and Rozenbljum([9],[10]).

Furthermore Lieb and Thirring proved the following inequality as an application of Theorem 1.1.

Theorem 1.2 (The Sobolev-Lieb-Thirring inequality) *Suppose that* $n \in \mathbb{N}$, ψ_i, $|\nabla\psi_i| \in L^2(\mathbb{R}^n)$ $(i = 1, \ldots, N)$, *and that* $\{\psi_i\}_{i=1}^N$ *is orthonormal in* $L^2(\mathbb{R}^n)$. *Then we have*

$$\int_{\mathbb{R}^n} \rho^{1+2/n} dx \leq c_n \sum_{i=1}^{N} \int_{\mathbb{R}^n} |\nabla\psi_i|^2 dx,$$

where

$$\rho(x) = \sum_{i=1}^{N} |\psi_i(x)|^2.$$

The Sobolev-Lieb-Thirring inequality has important applications such as the stability of matter or the estimates of the dimension of attractors of nonlinear equations(c.f.[5],[8],[14]).

2 Proof of Theorem 1.2

In this section we explain about the outline of an alternative proof of Theorem 1.2.

First we recall the definition of A_p-weights. By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. A locally integrable function $w > 0$ *a.e.* on $\mathbb{R}^n$ is an A_p-weight for some $p \in (1, \infty)$ if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(x)\, dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. We say that w is an A_1-weight if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(y)\, dy \leq Cw(x) \qquad a.e.\ x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. We write A_p for the class of A_p-weights. It is easy to show $A_1 \subset A_p$ for $1 < p < \infty$. An example of A_p-weight for $1 < p < \infty$ is given by $w(x) = |x|^\alpha \in A_p$ where $x \in \mathbb{R}^n$ and $-n < \alpha < n(p-1)$. Let Ω be a bounded C^1-domain in $\mathbb{R}^n, n \geq 2$. Then $w(x) = \mathrm{dist}(x, \partial\Omega)^\alpha$, $(-1 < \alpha < p-1)$, is another example of A_p-weight.

For $f \in L^1_{loc}(\mathbb{R}^n)$, we define the Hardy-Littlewood maximal operator as

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ such that $x \in Q$. For a nonnegative, locally integrable function w on $\mathbb{R}^n$ and $p \in [1, \infty)$ we set

$$L^p(w) = \{f\ :\ \text{measurable},\ \|f\|_{L^p(w)} = \left\{ \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right\}^{1/p} < \infty\}.$$

The proof of the following proposition is in [6].

Proposition 2.1 *(i) Let* $1 < p < \infty$ *and* w *be a nonnegative, locally integrable function on* $\mathbb{R}^n$*. Then* M *is bounded on* $L^p(w)$ *if and only if* $w \in A_p$*.*

(ii) Let $0 < \tau < 1$*,* $f \in L^1_{loc}(\mathbb{R}^n)$*, and* $M(f)(x) < \infty$ *a.e.. Then* $M(f)(x)^\tau \in A_1$*.*

(iii) Let $1 < p < \infty$ *and* $w_1, w_2 \in A_1$*. Then* $w_1 w_2^{1-p} \in A_p$*.*

We consider a function φ which satisfies the following properties.

(A1) $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(A2) $\operatorname{supp} \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$.

(A3) $|\hat{\varphi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$.

(A4) $\displaystyle\sum_{\nu \in \mathbf{Z}} |\hat{\varphi}(2^{-\nu}\xi)|^2 = 1$ for all $\xi \neq 0$.

For $\nu \in \mathbb{Z}, k \in \mathbb{Z}^n$, $Q = \{(x_1, \ldots, x_n) : k_i \leq 2^\nu x_i < k_i + 1,\ i = 1, \ldots, n\}$, and $x \in \mathbb{R}^n$, we set $\varphi_Q(x) = 2^{\nu n/2}\varphi(2^\nu x - k)$. The cube Q described above is called a dyadic cube. Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^n$.

Now we explain about the outline of a proof of Theorem 1.2. We may assume $\psi_i \in C_0^\infty(\mathbb{R}^n)$ for $i = 1, \ldots, N$. Let $V(x) = \delta\rho(x)^{2/n}$ where δ is a positive constant. Then we get $\int_{\mathbb{R}^n} V^{1+n/2}\, dx < \infty$. Set $v(x) = M(V^\kappa)(x)^{1/\kappa}$. Then (i) of Proposition 2.1 leads to

$$\int_{\mathbb{R}^n} v^{1+n/2}\, dx = \int_{\mathbb{R}^n} M(V^\kappa)^{(1+n/2)/\kappa}\, dx \leq c_1 \int_{\mathbb{R}^n} V^{1+n/2}\, dx < \infty.$$

Furthermore we have $v \in A_1$ and $V \leq v$ a.e..

The following two lemmas are essentially proved by Frazier and Jawerth , where (f, g) denotes the inner product in $L^2(\mathbb{R}^n)$(c.f.[11]).

Lemma 2.1 *There exists an* $\alpha > 0$ *such that*

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2/n} |(f, \varphi_Q)|^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2\, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$*.*

Lemma 2.2 *Let $v \in A_2$. Then there exists a $\beta > 0$ such that*

$$\int_{\mathbb{R}^n} |f|^2 v\,dx \le \beta \sum_{Q\in\mathcal{Q}} |(f,\varphi_Q)|^2 \frac{1}{|Q|}\int_Q v\,dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

By Lemmas 2.1 and 2.2 we have for $f \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla f|^2\,dx - \int_{\mathbb{R}^n} V|f|^2\,dx \\ &\ge \alpha \sum_{Q\in\mathcal{Q}} |Q|^{-2/n}|(f,\varphi_Q)|^2 - \beta \sum_{Q\in\mathcal{Q}} |(f,\varphi_Q)|^2 \frac{1}{|Q|}\int_Q v\,dx.\end{aligned}$$

Let

$$\mathcal{I} = \{Q \in \mathcal{Q} : \beta \int_Q v\,dx > \alpha |Q|^{-2/n}\}$$

and $\{\mu_k\}_{1\le k}$ be the non-decreasing rearrangement of

$$\left\{\alpha|Q|^{-2/n} - \beta|Q|^{-1}\int_Q v\,dx\right\}_{Q\in\mathcal{I}}.$$

When

$$\mu_k = \alpha|Q|^{-2/n} - \beta|Q|^{-1}\int_Q v\,dx,$$

we define $\varphi_k = \varphi_Q$. Then we get

$$\begin{aligned}&\sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla\psi_i|^2\,dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V|\psi_i|^2\,dx \\ &\ge \sum_{i=1}^N \sum_{Q\in\mathcal{Q}} |(\psi_i,\varphi_Q)|^2 \left\{\alpha|Q|^{-2/n} - \beta|Q|^{-1}\int_Q v\,dx\right\} \\ &\ge \sum_{i=1}^N \sum_k \mu_k |(\psi_i,\varphi_k)|^2 = \sum_k \mu_k \sum_{i=1}^N |(\psi_i,\varphi_k)|^2 \\ &= -c\sum_k |\mu_k|.\end{aligned}$$

Now we use the following lemma in [13].

Lemma 2.3 *There exists a positive constant c such that*

$$\sum_k |\mu_k| \le c\int_{\mathbb{R}^n} v^{1+n/2}\,dx,$$

where c depends only on n.

Hence by Lemma 2.3 we have

$$\sum_{i=1}^{N}\int_{\mathbb{R}^n}|\nabla\psi_i|^2\,dx-\sum_{i=1}^{N}\int_{\mathbb{R}^n}V|\psi_i|^2\,dx$$
$$\geq -c\int_{\mathbb{R}^n}V^{1+n/2}\,dx=-c\delta^{1+n/2}\int_{\mathbb{R}^n}\rho^{1+2/n}\,dx.$$

Therefore

$$\sum_{i=1}^{N}\int_{\mathbb{R}^n}|\nabla\psi_i|^2\,dx\geq\delta\int_{\mathbb{R}^n}\rho^{1+2/n}\,dx-c\delta^{1+n/2}\int_{\mathbb{R}^n}\rho^{1+2/n}\,dx$$
$$=\{\delta-c\delta^{1+n/2}\}\int_{\mathbb{R}^n}\rho^{1+2/n}\,dx.$$

If we take δ small enough, then we get the inequality in Theorem 1.2.

3 Some generalizations

We have the following generalization of the Sobolev-Lieb-Thirring inequality for $n\geq 3$(c.f.[11, Lemma 3.2], [13]).

Theorem 3.1 *Let $n\in\mathbb{N}$, $n\geq 3$, $w\in A_2$ and $w^{-n/2}\in A_{n/2}$. Suppose that $\psi_i\in L^2(\mathbb{R}^n)$, $|\nabla\psi_i|\in L^2(w)$ $(i=1,\ldots,N)$, and $\{\psi_i\}_{i=1}^N$ is orthonormal in $L^2(\mathbb{R}^n)$. Then we have*

$$\int_{\mathbb{R}^n}\rho(x)^{1+2/n}w(x)\,dx\leq c\sum_{i=1}^{N}\int_{\mathbb{R}^n}|\nabla\psi_i(x)|^2w(x)dx,$$

where

$$\rho(x)=\sum_{i=1}^{N}|\psi_i(x)|^2$$

and c is a positive constant depending only on n and w.

An example of w which satisfies the conditions in Theorem 3.1 is given by $w(x)=|x|^\alpha$ for $-n+2<\alpha<2$.

In the proof of Theorem 3.1 we use the following lemma.

Lemma 3.1 *Let $w\in A_2$. Then there exists an $\alpha>0$ such that*

$$\alpha\sum_{Q\in\mathcal{Q}}|Q|^{-2/n}|(f,\varphi_Q)|^2\frac{1}{|Q|}\int_Q w\,dx\leq\int_{\mathbb{R}^n}|\nabla f|^2w\,dx$$

for all $f\in C_0^\infty(\mathbb{R}^n)$.

We omit the detail of the proof of Theorem 3.1.

By Theorem 3.1 we can prove the following L^p version of the Sobolev-Lieb-Thirring inequality.

Theorem 3.2 *Let $n \in \mathbb{N}$, $n \geq 3$ and $2n/(n+2) < p < n$. Then there exists a positive constant c such that for every family $\{\psi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ which is orthonormal and $|\nabla\psi_i(x)| \in L^p(\mathbb{R}^n)$, $(i = 1, \dots, N)$, we have*

$$\int_{\mathbb{R}^n} \rho(x)^{(1+2/n)p/2}\, dx \leq c \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\nabla\psi_i(x)|^2 \right)^{p/2} dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\psi_i(x)|^2$$

and c depends only on n and p.

Proof

Our proof is very similar to that of the extrapolation theorem in harmonic analysis(c.f.[2, Theorem 7.8]).

Let $2 < p < n$ and $2/p + 1/q = 1$. Let $u \in L^q$, $u \geq 0$ and $\|u\|_{L^q} = 1$. We take a γ such that $n/(n-2) < \gamma < q$. Then we have $u \leq M(u^\gamma)^{1/\gamma}$ a.e and $M(u^\gamma)^{1/\gamma} \in A_1$. Furthermore let $\alpha = \frac{n}{(n-2)\gamma}$. Then $0 < \alpha < 1$ and

$$M(u^\gamma)^{-n/(2\gamma)} = \{M(u^\gamma)^\alpha\}^{1-n/2} \in A_{n/2},$$

where we used $M(u^\gamma)^\alpha \in A_1$ and (iii) of Proposition 2.1. Therefore we have

$$\begin{aligned}\int \rho^{1+2/n} u\, dx &\leq \int \rho^{1+2/n} M(u^\gamma)^{1/\gamma}\, dx \leq c \int \left(\sum_{i=1}^N |\nabla\psi_i|^2 \right) M(u^\gamma)^{1/\gamma}\, dx \\ &\leq c \left(\int \left(\sum_{i=1}^N |\nabla\psi_i|^2 \right)^{p/2} dx \right)^{2/p} \left(\int M(u^\gamma)^{q/\gamma} dx \right)^{1/q} \\ &\leq c \left(\int \left(\sum_{i=1}^N |\nabla\psi_i|^2 \right)^{p/2} dx \right)^{2/p},\end{aligned}$$

where we used Theorem 3.1 and the inequality

$$\int M(u^\gamma)^{q/\gamma}\, dx \leq c \int u^q dx = c.$$

If we take the supremum for all $u \in L^q$, $u \geq 0$ and $\|u\|_{L^q} = 1$, then we get

$$\left(\int \rho^{(1+2/n)p/2}\, dx\right)^{2/p} \leq c \left(\int \left(\sum_{i=1}^{N} |\nabla \psi_i|^2\right)^{p/2} dx\right)^{2/p}.$$

Next we consider the case $2n/(n+2) < p < 2$. Let

$$f = \left(\sum_{i=1}^{N} |\nabla \psi_i|^2\right)^{1/2}.$$

We can take γ such that $(2-p)n/2 < \gamma < p$. Then we have

$$M(f^\gamma)^{-(2-p)/\gamma} \in A_2$$

because

$$M(f^\gamma)^{(2-p)/\gamma} \in A_1$$

by (ii) of Proposition 2.1. Furthermore we have

$$\{M(f^\gamma)^{-(2-p)/\gamma}\}^{-n/2} = M(f^\gamma)^{(2-p)n/(2\gamma)} \in A_1 \subset A_{n/2}.$$

Therefore

$$\begin{aligned}
\int \rho^{(1+2/n)p/2}\, dx &= \int \rho^{(1+2/n)p/2} M(f^\gamma)^{-(2-p)p/(2\gamma)} M(f^\gamma)^{(2-p)p/(2\gamma)}\, dx \\
&\leq \left(\int \rho^{1+2/n} M(f^\gamma)^{-(2-p)/\gamma}\, dx\right)^{p/2} \left(\int M(f^\gamma)^{p/\gamma}\, dx\right)^{1-p/2} \\
&\leq c \left(\int f^2 M(f^\gamma)^{-(2-p)/\gamma}\, dx\right)^{p/2} \left(\int f^p\, dx\right)^{1-p/2} \\
&\leq c \left(\int M(f^\gamma)^{2/\gamma} M(f^\gamma)^{-(2-p)/\gamma}\, dx\right)^{p/2} \left(\int f^p\, dx\right)^{1-p/2} \\
&\leq c \left(\int M(f^\gamma)^{p/\gamma}\, dx\right)^{p/2} \left(\int f^p\, dx\right)^{1-p/2} \leq c \int f^p\, dx,
\end{aligned}$$

where we used Theorem 3.1 in the second inequality.

We shall give a generalization of Theorem 3.1. We say a family $\{\psi_i\}_{i=1}^N \subset L^2(\mathbb{R}^n)$ is suborthonormal if

$$\sum_{i,j=1}^{N} \xi_i \overline{\xi_j} (\psi_i, \psi_j) \leq \sum_{i=1}^{N} |\xi_i|^2$$

for all $\xi_i \in \mathbb{C}, i = 1, \ldots, N$ (c.f.[5]).

For $w \in A_2$ and $s > 0$ let $\mathcal{H}^s(w)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x)\, dx + \|f\|^2 \right\}^{1/2}.$$

For any $Q \in \mathcal{Q}$ there exists a unique $Q' \in \mathcal{Q}$ such that $Q \subset Q'$ and the side-length of Q' is double of that of Q. We call Q' the parent of Q.

We have the following generalization of Theorem 3.1([13]).

Theorem 3.3 *Let $n \in \mathbb{N}$, $s > 0$, $\max(1, \frac{n}{2s}) < p \leq 1 + \frac{n}{2s}$, and $w \in A_2$. If $2s < n$, then we assume that $w^{-n/(2s)} \in A_{n/(2s)}$. If $2s \geq n$, then we assume that $w^{-n/(2s)} \in A_p$ and*

$$\int_{Q'} w\, dx \leq 2^{2s} \int_Q w\, dx$$

for all dyadic cubes Q and its parent Q'.

Then for $\{\psi_i\}_{i=1}^N \subset \mathcal{H}^s(w)$ which is suborthonormal in $L^2(\mathbb{R}^n)$ we have

$$\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))}\, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \psi_i(x)|^2 w(x)\, dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\psi_i(x)|^2$$

and c is a positive constant depending only on $n, p, s,$ and w.

Remarks

(1) The case $s \in \mathbb{N}$ adn $w \equiv 1$ is studied by Ghidaglia, Marion and Temam([5]).

(2) The case $w \equiv 1$ is studied by Edmunds and Ilyin([3]) for $\{\psi_i\}_{i=1}^N$ which is orthonormal in $L^2(\mathbb{R}^n)$.

(3) When $2s < n$, an example of w is given by $w(x) = |x|^\alpha$ for $-n + 2s < \alpha < 2s$.

(4) When $2s > n$, an example of w is given by $w(x) = |x|^\alpha$ for $0 \leq \alpha < \min\{2s-n, n\}$.

(5) When $2s = n$, our condition means $w \approx 1$.

4 Estimate of the Hausdorff dimension of the attractor of a nonlinear equation

In this section we apply Theorem 3.1 to a nonlinear equation. In [14] the following result is proved.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let*

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad \text{where} \quad b_j \in \mathbb{R},\ b_{2p-1} > 0,$$

and

$$\kappa_1 \geq 0, \quad g'(s) \geq -\kappa_1, \quad \forall s \in \mathbb{R}.$$

Let $d > 0$ and $u_0 \in L^2(\Omega)$. Then the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u + g(u) &= 0 & \text{in } \Omega \times \mathbb{R}_+ \\ u(x,t) &= 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u(x,0) &= u_0(x) & x \in \Omega \end{cases}$$

has a unique solution $u = u(x,t)$ such that

$$u \in L^2(0,T; H_0^1(\Omega)) \cap L^{2p}(0,T; L^{2p}(\Omega)), \quad \forall T > 0$$

and

$$u \in C(\mathbb{R}_+; L^2(\Omega)).$$

Furthermore there exists a maximal attractor $\mathcal{A}$ which is bounded in $H_0^1(\Omega)$, compact and connected in $L^2(\Omega)$. Let m be the integer such that

$$m - 1 < c\left(\frac{\kappa_1}{d}\right)^{n/2} |\Omega| \leq m,$$

where c is a constant depending only on n. Then the Hausdorff dimension of $\mathcal{A}$ is less than or equal to m.

We have the following result as an application of Theorem 3.1.

Theorem 4.2 *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain. Let*

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_j \in \mathbb{R}, \; b_{2p-1} > 0$$

and

$$\kappa_1 \geq 0, \quad g'(s) \geq -\kappa_1, \quad \forall s \in \mathbb{R}.$$

Let

$$d(x) = \mathrm{dist}(x, \partial\Omega),$$
$$-1 + \frac{2}{n} < a < \frac{2}{n}, \; w(x) = d(x)^a,$$

and $H_0^1(\Omega, w)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|_{H_0^1(\Omega,w)} = \left\{ \int_\Omega (|\nabla f|^2 + |f|^2) w \, dx \right\}^{1/2}.$$

Let $d > 0$ and $u_0 \in L^2(\Omega)$. Then the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - d \displaystyle\sum_{i=1}^{n} \partial_{x_i}(w(x)\partial_{x_i} u) + g(u) & = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ u(x,t) & = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u(x,0) & = u_0(x) & x \in \Omega \end{cases}$$

has a a unique solution $u = u(x,t)$ such that

$$u \in L^2(0,T; H_0^1(\Omega, w)), \quad \forall T > 0,$$

and

$$u \in C(\mathbb{R}_+; L^2(\Omega)).$$

Furthermore there exists a maximal attractor $\mathcal{A}$ which is bounded in $H_0^1(\Omega, w)$, compact and connected in $L^2(\Omega)$. Let m be the integer such that

$$m - 1 < c' \left(\frac{\kappa_1}{d}\right)^{n/2} \int_\Omega w^{-n/2} \, dx \leq m,$$

Then the Hausdorff dimension of $\mathcal{A}$ is less than or equal to m.

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