A generalization of the Sobolev-Lieb-Thirring inequality

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1 Introduction

In this article we explain about a generalization of the Sobolev-Lieb-Thirring inequality and its application. In the proof of our theorem we use the \( \varphi \)-transform of Frazier and Jawerth.

In 1976 Lieb and Thirring proved the following inequality([8]).

**Theorem 1.1 (The Lieb-Thirring inequality)** Let \( V \) be a non-negative measurable function on \( \mathbb{R}^n \) and

\[
    \gamma \geq \frac{1}{2} \quad \text{for} \quad n = 1, \quad \gamma > 0 \quad \text{for} \quad n = 2, \quad \gamma \geq 0 \quad \text{for} \quad n \geq 3.
\]

Then we have

\[
    \sum_i |\lambda_i|^{\gamma} \leq c_{n,\gamma} \int_{\mathbb{R}^n} V^{n/2+\gamma} dx,
\]

where \( \lambda_1 \leq \lambda_2 \leq \cdots \) are the negative eigenvalues of the Schrödinger operator \( -\Delta - V \) on \( L^2(\mathbb{R}^n) \).

The case \( \gamma > 1/2, n = 1 \) or \( \gamma > 0, n \geq 2 \) was proved by Lieb and Thirring([8]). The case \( \gamma = 1/2, n = 1 \) was proved by Weidl([15]). The case \( \gamma = 0, n \geq 3 \) was established by Cwikel([1]), Lieb([7]) and Rozenbljum([9],[10]).

Furthermore Lieb and Thirring proved the following inequality as an application of Theorem 1.1.
Theorem 1.2 (The Sobolev-Lieb-Thirring inequality) Suppose that $n \in \mathbb{N}$, $\psi_i, |\nabla \psi_i| \in L^2(\mathbb{R}^n)$ ($i = 1, \ldots, N$), and that $\{\psi_i\}_{i=1}^N$ is orthonormal in $L^2(\mathbb{R}^n)$. Then we have

$$\int_{\mathbb{R}^n} \rho^{1+2/n} dx \leq c_n \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \psi_i|^2 dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\psi_i(x)|^2.$$

The Sobolev-Lieb-Thirring inequality has important applications such as the stability of matter or the estimates of the dimension of attractors of nonlinear equations (c.f.[5],[8],[14]).

2 Proof of Theorem 1.2

In this section we explain about the outline of an alternative proof of Theorem 1.2.

First we recall the definition of $A_p$-weights. By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. A locally integrable function $w > 0$ a.e. on $\mathbb{R}^n$ is an $A_p$-weight for some $p \in (1, \infty)$ if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_{Q} w(x) dx \left( \frac{1}{|Q|} \int_{Q} w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. We say that $w$ is an $A_1$-weight if there exists a positive constant $C$ such that

$$\frac{1}{|Q|} \int_{Q} w(y) dy \leq C w(x) \quad \text{a.e. } x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. We write $A_p$ for the class of $A_p$-weights. It is easy to show $A_1 \subset A_p$ for $1 < p < \infty$. An example of $A_p$-weight for $1 < p < \infty$ is given by $w(x) = |x|^\alpha \in A_p$ where $x \in \mathbb{R}^n$ and $-n < \alpha < n(p-1)$. Let $\Omega$ be a bounded $C^1$-domain in $\mathbb{R}^n, n \geq 2$. Then $w(x) = \text{dist}(x, \partial \Omega)^\alpha, (-1 < \alpha < p - 1)$, is another example of $A_p$-weight.

For $f \in L^1_{loc}(\mathbb{R}^n)$, we define the Hardy-Littlewood maximal operator as

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ such that $x \in Q$. For a nonnegative, locally integrable function $w$ on $\mathbb{R}^n$ and $p \in [1, \infty)$ we set

$$L^p(w) = \{ f : \text{measurable}, \|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty \}.$$

The proof of the following proposition is in [6].
Proposition 2.1  
(i) Let $1 < p < \infty$ and $w$ be a nonnegative, locally integrable function on $\mathbb{R}^n$. Then $M$ is bounded on $L^p(w)$ if and only if $w \in A_p$.

(ii) Let $0 < \tau < 1$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and $\dot{M}(f)(x) < \infty$ a.e.. Then $\dot{M}(f)(x)^\tau \in A_1$.

(iii) Let $1 < p < \infty$ and $w_1, w_2 \in A_1$. Then $w_1 w_2^{1-p} \in A_p$.

We consider a function $\varphi$ which satisfies the following properties.

(A1) $\varphi \in S(\mathbb{R}^n)$.

(A2) $\text{supp} \hat{\varphi} \subset \{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \}$.

(A3) $|\hat{\varphi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$.

(A4) $\sum_{\nu \in \mathbb{Z}} |\hat{\varphi}(2^{-\nu} \xi)|^2 = 1$ for all $\xi \neq 0$.

For $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $Q = \{(x_1, \ldots, x_n) : k_i \leq 2^\nu x_i < k_i + 1, i = 1, \ldots, n\}$, and $x \in \mathbb{R}^n$, we set $\varphi_Q(x) = 2^{\nu n/2} \varphi(2^\nu x - k)$. The cube $Q$ described above is called a dyadic cube.

Let $Q$ be the set of all dyadic cubes in $\mathbb{R}^n$.

Now we explain about the outline of a proof of Theorem 1.2. We may assume $\psi_i \in C_{0}^{\infty}(\mathbb{R}^n)$ for $i = 1, \ldots, N$. Let $V(x) = \delta \rho(x)^{2/n}$ where $\delta$ is a positive constant. Then we get $\int_{\mathbb{R}^n} V^{1+n/2} \, dx < \infty$. Set $v(x) = M(V^\kappa)(x)^{1/\kappa}$. Then (i) of Proposition 2.1 leads to

$$
\int_{\mathbb{R}^n} v^{1+n/2} \, dx = \int_{\mathbb{R}^n} M(V^\kappa)^{(1+n/2)/\kappa} \, dx \leq c_1 \int_{\mathbb{R}^n} V^{1+n/2} \, dx < \infty.
$$

Furthermore we have $v \in A_1$ and $V \leq v$ a.e..

The following two lemmas are essentially proved by Frazier and Jawerth, where $(f, g)$ denotes the inner product in $L^2(\mathbb{R}^n)$ (c.f. [11]).

Lemma 2.1 There exists an $\alpha > 0$ such that

$$
\alpha \sum_{Q \in Q} |Q|^{-2/n} |(f, \varphi_Q)|^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2 \, dx
$$

for all $f \in C_{0}^{\infty}(\mathbb{R}^n)$. 
Lemma 2.2 Let $v \in A_2$. Then there exists a $\beta > 0$ such that
\[
\int_{\mathbb{R}^n} |f|^2 v \, dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx
\]
for all $f \in C_0^\infty(\mathbb{R}^n)$.

By Lemmas 2.1 and 2.2 we have for $f \in C_0^\infty(\mathbb{R}^n)$
\[
\int_{\mathbb{R}^n} |\nabla f|^2 \, dx - \int_{\mathbb{R}^n} V |f|^2 \, dx \\
\geq \alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2/n} |(f, \varphi_Q)|^2 - \beta \sum_{Q \in \mathcal{Q}} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_Q v \, dx.
\]

Let
\[
I = \{Q \in \mathcal{Q} : \beta \int_Q v \, dx > \alpha |Q|^{-2/n} \}
\]
and $\{\mu_k\}_{1 \leq k}$ be the non-decreasing rearrangement of
\[
\left\{ \alpha |Q|^{-2/n} - \beta |Q|^{-1} \int_Q v \, dx \right\}_{Q \in I}.
\]

When
\[
\mu_k = \alpha |Q|^{-2/n} - \beta |Q|^{-1} \int_Q v \, dx,
\]
we define $\varphi_k = \varphi_Q$. Then we get
\[
\sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla \psi_i|^2 \, dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V |\psi_i|^2 \, dx \\
\geq \sum_{i=1}^N \sum_{Q \in \mathcal{Q}} |(\psi_i, \varphi_Q)|^2 \left\{ \alpha |Q|^{-2/n} - \beta |Q|^{-1} \int_Q v \, dx \right\} \\
\geq \sum_{k} \sum_{i=1}^N \mu_k |(\psi_i, \varphi_k)|^2 = \sum_k \mu_k \sum_{i=1}^N |(\psi_i, \varphi_k)|^2 \\
= - c \sum_k |\mu_k|.
\]

Now we use the following lemma in [13].

Lemma 2.3 There exists a positive constant $c$ such that
\[
\sum_k |\mu_k| \leq c \int_{\mathbb{R}^n} v^{1+n/2} \, dx,
\]
where $c$ depends only on $n$. 
Hence by Lemma 2.3 we have
\[\sum_{i=1}^{N} \int_{\mathbb{R}^{n}} |\nabla \psi_i|^2 \, dx - \sum_{i=1}^{N} \int_{\mathbb{R}^{n}} V|\psi_i|^2 \, dx \geq -c \int_{\mathbb{R}^{n}} V^{1+n/2} \, dx = -c \delta^{1+n/2} \int_{\mathbb{R}^{n}} \rho^{1+2/n} \, dx.\]

Therefore
\[\sum_{i=1}^{N} \int_{\mathbb{R}^{n}} |\nabla \psi_i|^2 \, dx \geq \delta \int_{\mathbb{R}^{n}} \rho^{1+2/n} \, dx - c \delta^{1+n/2} \int_{\mathbb{R}^{n}} \rho^{1+2/n} \, dx = (\delta - c \delta^{1+n/2}) \int_{\mathbb{R}^{n}} \rho^{1+2/n} \, dx.\]

If we take \(\delta\) small enough, then we get the inequality in Theorem 1.2.

3 Some generalizations

We have the following generalization of the Sobolev-Lieb-Thirring inequality for \(n \geq 3\) (c.f.[11, Lemma 3.2], [13]).

**Theorem 3.1** Let \(n \in \mathbb{N}, \ n \geq 3, \ w \in A_2\) and \(w^{-n/2} \in A_{n/2}\). Suppose that \(\psi_i \in L^2(\mathbb{R}^n), \ |\nabla \psi_i| \in L^2(w) \ (i = 1, \ldots, N),\) and \(\{\psi_i\}_{i=1}^{N}\) is orthonormal in \(L^2(\mathbb{R}^n)\). Then we have
\[\int_{\mathbb{R}^{n}} \rho(x)^{1+2/n} w(x) \, dx \leq c \sum_{i=1}^{N} \int_{\mathbb{R}^{n}} |\nabla \psi_i(x)|^2 w(x) \, dx,\]
where
\[\rho(x) = \sum_{i=1}^{N} |\psi_i(x)|^2\]
and \(c\) is a positive constant depending only on \(n\) and \(w\).

An example of \(w\) which satisfies the conditions in Theorem 3.1 is given by \(w(x) = |x|^\alpha\) for \(-n+2 < \alpha < 2\).

In the proof of Theorem 3.1 we use the following lemma.

**Lemma 3.1** Let \(w \in A_2\). Then there exists an \(\alpha > 0\) such that
\[\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2/n} |(f, \varphi_Q)|^2 \frac{1}{|Q|} \int_{Q} w \, dx \leq \int_{\mathbb{R}^{n}} |\nabla f|^2 w \, dx\]
for all \(f \in C_0^\infty(\mathbb{R}^n)\).
We omit the detail of the proof of Theorem 3.1.

By Theorem 3.1 we can prove the following $L^p$ version of the Sobolev-Lieb-Thirring inequality.

**Theorem 3.2** Let $n \in \mathbb{N}$, $n \geq 3$ and $2n/(n+2) < p < n$. Then there exists a positive constant $c$ such that for every family $\{\psi_i\}_{i=1}^{N}$ in $L^2(\mathbb{R}^n)$ which is orthonormal and $|\nabla \psi_i(x)| \in L^p(\mathbb{R}^n)$, $(i = 1, \ldots, N)$, we have

$$
\int_{\mathbb{R}^n} \rho(x)^{(1+2/n)p/2} dx \leq c \int_{\mathbb{R}^n} \left( \sum_{i=1}^{N} |\nabla \psi_i(x)|^2 \right)^{p/2} dx,
$$

where

$$
\rho(x) = \sum_{i=1}^{N} |\psi_i(x)|^2
$$

and $c$ depends only on $n$ and $p$.

**Proof**

Our proof is very similar to that of the extrapolation theorem in harmonic analysis (c.f. [2, Theorem 7.8]).

Let $2 < p < n$ and $2/p + 1/q = 1$. Let $u \in L^q$, $u \geq 0$ and $\|u\|_{L^q} = 1$. We take a $\gamma$ such that $n/(n-2) < \gamma < q$. Then we have $u \leq M(u^\gamma)^{1/\gamma}$ a.e and $M(u^\gamma)^{1/\gamma} \in A_1$. Furthermore let $\alpha = \frac{n}{(n-2)\gamma}$. Then $0 < \alpha < 1$ and

$$
M(u^\gamma)^{-n/(2\gamma)} = \{M(u^\gamma)^{\alpha}\}^{1-n/2} \in A_{n/2},
$$

where we used $M(u^\gamma)^{\alpha} \in A_1$ and (iii) of Proposition 2.1. Therefore we have

$$
\int \rho^{1+2/n} u dx \leq \int \rho^{1+2/n} M(u^\gamma)^{1/\gamma} dx \leq c \int \left( \sum_{i=1}^{N} |\nabla \psi_i|^2 \right)^{p/2} M(u^\gamma)^{1/\gamma} dx
$$

$$
\leq c \left( \int \left( \sum_{i=1}^{N} |\nabla \psi_i|^2 \right)^{p/2} dx \right)^{2/p} \left( \int M(u^\gamma)^{q/\gamma} dx \right)^{1/q}
$$

$$
\leq c \left( \int \left( \sum_{i=1}^{N} |\nabla \psi_i|^2 \right)^{p/2} dx \right)^{2/p} \left( \int u^{q} dx \right)^{1/q},
$$

where we used Theorem 3.1 and the inequality

$$
\int M(u^\gamma)^{q/\gamma} dx \leq c \int u^q dx = c.
$$
If we take the supremum for all \( u \in L^q, u \geq 0 \) and \( ||u||_{L^q} = 1 \), then we get
\[
\left( \int \rho^{(1+2/n)p/2} dx \right)^{2/p} \leq c \left( \int \left( \sum_{i=1}^{N} |\nabla \psi_i|^2 \right)^{p/2} dx \right)^{2/p}.
\]

Next we consider the case \( 2n/(n+2) < p < 2 \). Let
\[
f = \left( \sum_{i=1}^{N} |\nabla \psi_i|^2 \right)^{1/2}.
\]
We can take \( \gamma \) such that \((2 - p)n/2 < \gamma < p\). Then we have
\[
M(f^{\gamma})^{-(2-p)/\gamma} \in A_2
\]
because
\[
M(f^{\gamma})^{(2-p)/\gamma} \in A_1
\]
by (ii) of Proposition 2.1. Furthermore we have
\[
\{M(f^{\gamma})^{-(2-p)/\gamma}\}^{-n/2} = M(f^{\gamma})^{(2-p)n/(2\gamma)} \in A_1 \subset A_{n/2}.
\]
Therefore
\[
\int \rho^{(1+2/n)p/2} dx = \int \rho^{(1+2/n)p/2} M(f^{\gamma})^{-(2-p)p/(2\gamma)} M(f^{\gamma})^{(2-p)p/(2\gamma)} dx
\]
\[
\leq \left( \int \rho^{1+2/n} M(f^{\gamma})^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int M(f^{\gamma})^{p/\gamma} dx \right)^{1-p/2}
\]
\[
\leq c \left( \int f^2 M(f^{\gamma})^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int f^p dx \right)^{1-p/2}
\]
\[
\leq c \left( \int f^{(2-\gamma)/\gamma} M(f^{\gamma})^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int f^p dx \right)^{1-p/2}
\]
\[
\leq c \left( \int M(f^{\gamma})^{2/\gamma} M(f^{\gamma})^{-(2-p)/\gamma} dx \right)^{p/2} \left( \int f^p dx \right)^{1-p/2} \leq c \int f^p dx,
\]
where we used Theorem 3.1 in the second inequality.

We shall give a generalization of Theorem 3.1. We say a family \( \{\psi_i\}_{i=1}^{N} \subset L^2(\mathbb{R}^n) \) is suborthonormal if
\[
\sum_{i,j=1}^{N} \xi_i \xi_j \overline{\psi_i} \psi_j \leq \sum_{i=1}^{N} |\xi_i|^2
\]
for all \( \xi_i \in \mathbb{C}, i = 1, \ldots, N \) (c.f.[5]).
For $w \in A_2$ and $s > 0$ let $\mathcal{H}^s(w)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$||f||_{\mathcal{H}^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) \, dx + ||f||^2 \right\}^{1/2}.$$

For any $Q \in Q$ there exists a unique $Q' \in Q$ such that $Q \subset Q'$ and the side-length of $Q'$ is double of that of $Q$. We call $Q'$ the parent of $Q$.

We have the following generalization of Theorem 3.1([13]).

**Theorem 3.3** Let $n \in \mathbb{N}$, $s > 0$, $\max(1, \frac{n}{2s}) < p \leq 1 + \frac{n}{2s}$, and $w \in A_2$. If $2s < n$, then we assume that $w^{-n/(2s)} \in A_{n/(2s)}$. If $2s \geq n$, then we assume that $w^{-n/(2s)} \in A_p$ and

$$\int_{Q'} w \, dx \leq 2^{2s} \int_{Q} w \, dx$$

for all dyadic cubes $Q$ and its parent $Q'$.

Then for $\{\psi_i\}_{i=1}^N \subset \mathcal{H}^s(w)$ which is suborthonormal in $L^2(\mathbb{R}^n)$ we have

$$\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \psi_i(x)|^2 w(x) \, dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\psi_i(x)|^2$$

and $c$ is a positive constant depending only on $n, p, s$, and $w$.

**Remarks**

1. The case $s \in \mathbb{N}$ and $w \equiv 1$ is studied by Ghidaglia, Marion and Temam([5]).

2. The case $w \equiv 1$ is studied by Edmunds and Ilyin([3]) for $\{\psi_i\}_{i=1}^N$ which is orthonormal in $L^2(\mathbb{R}^n)$.

3. When $2s < n$, an example of $w$ is given by $w(x) = |x|^\alpha$ for $-n + 2s < \alpha < 2s$.

4. When $2s > n$, an example of $w$ is given by $w(x) = |x|^\alpha$ for $0 \leq \alpha < \min\{2s-n, n\}$.

5. When $2s = n$, our condition means $w \approx 1$. 


4 Estimate of the Hausdorff dimension of the attractor of a nonlinear equation

In this section we apply Theorem 3.1 to a nonlinear equation. In [14] the following result is proved.

**Theorem 4.1** Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j,$$

where $b_j \in \mathbb{R}$, $b_{2p-1} > 0$,

and

$$\kappa_1 \geq 0, \quad g'(s) \geq -\kappa_1, \quad \forall s \in \mathbb{R}.$$

Let $d > 0$ and $u_0 \in L^2(\Omega)$. Then the equation

$$\begin{cases}
\frac{\partial u}{\partial t} - d \Delta u + g(u) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\
u(x, t) = 0 & \text{on } \partial \Omega \times \mathbb{R}_+ \\
u(x, 0) = u_0(x) & x \in \Omega
\end{cases}$$

has a unique solution $u = u(x, t)$ such that

$$u \in L^2(0, T; H_0^1(\Omega)) \cap L^{2p}(0, T; L^{2p}(\Omega)), \quad \forall T > 0$$

and

$$u \in C(\mathbb{R}_+; L^2(\Omega)).$$

Furthermore there exists a maximal attractor $A$ which is bounded in $H_0^1(\Omega)$, compact and connected in $L^2(\Omega)$. Let $m$ be the integer such that

$$m - 1 < c \left( \frac{\kappa_1}{d} \right)^{n/2} |\Omega| \leq m,$$

where $c$ is a constant depending only on $n$. Then the Hausdorff dimension of $A$ is less than or equal to $m$.

We have the following result as an application of Theorem 3.1.
Theorem 4.2 Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded $C^2$-domain. Let
\[ g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_j \in \mathbb{R}, \quad b_{2p-1} > 0 \]
and
\[ \kappa_1 \geq 0, \quad g'(s) \geq -\kappa_1, \quad \forall s \in \mathbb{R}. \]
Let
\[ d(x) = \text{dist}(x, \partial\Omega), \]
\[ -1 + \frac{2}{n} < a < \frac{2}{n}, \quad w(x) = d(x)^a, \]
and $H^1_0(\Omega, w)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm
\[ ||f||_{H^1_0(\Omega, w)} = \left\{ \int_{\Omega} (|\nabla f|^2 + |f|^2) w \, dx \right\}^{1/2}. \]
Let $d > 0$ and $u_0 \in L^2(\Omega)$. Then the equation
\[
\begin{cases}
\frac{\partial u}{\partial t} - d \sum_{i=1}^{n} \partial_{x_i}(w(x)\partial_{x_i}u) + g(u) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\
u(x, t) = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\
u(x, 0) = u_0(x) & x \in \Omega
\end{cases}
\]
has a unique solution $u = u(x, t)$ such that
\[ u \in L^2(0, T; H^1_0(\Omega, w)), \quad \forall T > 0, \]
and
\[ u \in C(\mathbb{R}_+; L^2(\Omega)). \]
Furthermore there exists a maximal attractor $A$ which is bounded in $H^1_0(\Omega, w)$, compact and connected in $L^2(\Omega)$. Let $m$ be the integer such that
\[ m - 1 < c' \left( \frac{\kappa_1}{d} \right)^{n/2} \int_{\Omega} w^{-n/2} dx \leq m, \]
Then the Hausdorff dimension of $A$ is less than or equal to $m$.

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参考文献


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