On the algebraic Bethe Ansatz approach to the correlation functions of the XXZ spin-1/2 Heisenberg chain

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Abstract

We present a review of the method we have elaborated to compute the correlation functions of the XXZ spin-1/2 Heisenberg chain. This method is based on the resolution of the quantum inverse scattering problem in the algebraic Bethe Ansatz framework, and leads to a multiple integral representation of the dynamical correlation functions. We describe in particular some recent advances concerning the two-point functions: in the finite chain, they can be expressed in terms of a single multiple integral. Such a formula provides a direct analytic connection between the previously obtained multiple integral representations and the form factor expansions for the correlation functions.

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1 Introduction

Computing exact and manageable expressions for correlation functions is a central question in the theory of quantum integrable models [1-3]. This problem is of great importance, both from a theoretical and mathematical view point and for applications to various interesting physical situations. Apart from few cases, like free fermions [4-9] or conformal field theories [10], this issue is still far from its complete solution. Although several important advances have been obtained in the recent years, we are still lacking a general method that could give in particular a systematic way to evaluate compact expressions for two-point functions and their long distance asymptotic behaviour.

The aim of the present paper is to give a review of an approach to this problem elaborated in [11-13] and in [14-17], together with an account of the more recent progress obtained in [18-20].

In our search for a general method to compute correlation functions of quantum integrable systems, our strategy is to consider a simple but representative model for which it is possible to develop new concepts and tools towards this goal. An archetype of such a model is provided by the XXZ spin-$\frac{1}{2}$ Heisenberg chain in a magnetic field with Hamiltonian,

$$H = H^{(0)} - hS_z, \tag{1.1}$$

where

$$H^{(0)} = \sum_{m=1}^{M} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \right\}, \tag{1.2}$$

$$S_z = \frac{1}{2} \sum_{m=1}^{M} \sigma_m^z, \qquad [H^{(0)}, S_z] = 0. \tag{1.3}$$

Here Δ is the anisotropy parameter, h denotes the external classical magnetic field, and $\sigma_m^{x,y,z}$ are the local spin operators (in the spin-$\frac{1}{2}$ representation) associated with each site of the chain. The quantum space of states is $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, where $\mathcal{H}_m \sim \mathbb{C}^2$ is called local quantum space. The operators $\sigma_m^{x,y,z}$ act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere. For simplicity, the length of the chain M is chosen to be even and we assume periodic boundary conditions. Since the simultaneous reversal of all spins is equivalent to a change of sign of the magnetic field, it is enough to consider the case $h \geq 0$. In the thermodynamic limit $M \to \infty$ and at zero magnetic field, the model exhibits different regimes depending on the value of Δ [1]. The ground state is ferromagnetic for $\Delta < -1$, while it has magnetisation zero for $\Delta > -1$. In the last case the spectrum is gapless for $-1 < \Delta < 1$ (massless regime), while for $\Delta > 1$ the ground state is twice degenerated with a gap in the spectrum (massive regime).

We are basically interested in the two-point correlation functions of local spins, although the results presented here allow us to consider other correlation functions as well. If we restrict ourselves to the zero temperature situation, such a problem comes down to the computation of the average value of a product of two local spin operators in the ground state $| \psi_g \rangle$ of the Hamiltonian (1.1):

$$g_{\alpha\beta}(m) = \langle \psi_g | \sigma_1^\alpha \sigma_{m+1}^\beta | \psi_g \rangle, \qquad (\alpha, \beta) = (+, -), (-, +), (z, z). \tag{1.4}$$

Despite its apparent simplicity, such an object is highly non-trivial to handle. The first problem to solve is obviously to determine the ground state $| \psi_g \rangle$. A method to diagonalise the Hamiltonian was proposed by Bethe in 1931 [21] and developed later in [22-25]. The algebraic version

of the Bethe Ansatz was created in the framework of the Quantum Inverse Scattering Method by L.D. Faddeev and his school [26–28]. Different ways to study the correlation functions of this model were proposed in the series of works (see e.g. [11, 12, 14–17, 29–34]).

Multiple integral representation for the correlation functions were obtained for the first time from the q-vertex operator approach (also using corner transfer matrix technique) in the massive regime $\Delta \geq 1$ in 1992 [29] and conjectured in 1996 [30] in the massless regime $-1 \leq \Delta \leq 1$ (see also [31]). A proof of these results, together with their extension to non-zero magnetic field, was obtained in 1999 [11, 12] for both regimes using algebraic Bethe Ansatz and the actual resolution of the so-called quantum inverse scattering problem [11, 13]. In fact, these multiple integral representations have been constructed for the elementary building blocks (see Section 2), since any arbitrary correlation function can be expressed in terms of a linear combination of such blocks. One should note however that, although these formulas are quite explicit, the actual analytic computation of the corresponding multiple integrals is missing up to now. Moreover, the evaluation of the two-point correlation functions (1.4) at lattice distance m is a priori quite involved, since the number of terms in the corresponding linear combination of the elementary blocks growths exponentially with m (like 2^m). This makes the problem of asymptotic behaviour at large distance extremely difficult to solve in this settings from the present knowledge of the elementary blocks.

In the articles [14, 20] we have derived new multiple integral representations more adapted to the two-point correlation functions. One of them [20] is based on the direct re-summation of the above linear combinations of the elementary blocks. The second one [14] uses an explicit representation for the multiple action of the twisted transfer matrices (see (2.7)) on an arbitrary Bethe state. In both cases the number of multiple integrals describing the two-point functions (1.4) reduces from 2^m to m.

The development of these methods allowed us to perform further re-summation and to obtain representations for the two-point functions on the lattice in terms of a single multiple integral [18]. We call this type of representation *master equation*. The remarkable property of the master equation is that it gives a direct analytic link between two general approaches to the computation of correlation functions: in the context of the XXZ Heisenberg chain, the first method consists in acting with the local operators σ_1^α and σ_{m+1}^β on the ground state $\langle \psi_g |$ to produce a new state $\langle \psi(\alpha, \beta, m) |$, and then in computing the resulting scalar product $\langle \psi(\alpha, \beta, m) | \psi_g \rangle$; the second method consists in inserting between the two operators of local spin a sum over a complete set of states (for example eigenstates of the Hamiltonian), which gives a decomposition of the two-point function in the form

$$g_{\alpha\beta}(m) = \sum_i \langle \psi_g | \sigma_1^\alpha | i \rangle \langle i | \sigma_{m+1}^\beta | \psi_g \rangle. \tag{1.5}$$

Using the technique developed in [18, 19], we are able to re-sum completely the form factor expansion (1.5) and to show that it leads indeed to the master equation obtained by the first method. In fact, these two different approaches have a very simple interpretation in the context of the master equation. Namely, they correspond to two different ways to evaluate the contour integral, by computing the residues in the poles that are either inside or outside the integration contour. The first way leads to a representation of the correlation function $\langle \sigma_1^\alpha \sigma_{m+1}^\beta \rangle$ in terms of the previously obtained [14] multiple integrals. The second one gives us the form factor type expansion of the correlation function (i.e. an expansion in terms of matrix elements of local spin operators between the ground state and all excited states).

This method was generalised in [19] to the time-dependent (dynamical) correlation functions

$$g_{\alpha\beta}(m, t) = \langle \psi_g | \sigma_1^\alpha(0) \, \sigma_{m+1}^\beta(t) | \psi_g \rangle = \langle \psi_g | \sigma_1^\alpha \, e^{iHt} \, \sigma_{m+1}^\beta \, e^{-iHt} | \psi_g \rangle. \tag{1.6}$$

It is worth mentioning that, up to now, the only known exact results on the dynamical correlations concern the case of free fermions $\Delta = 0$ [5, 6, 9, 35–39]. It turns out, however, that the methods developed in [14, 18] can be directly applied to the computation of the time-dependent correlation functions. In particular, one can obtain a time-dependent analogue of the master equation and also a multiple integral representation for $g_{\alpha\beta}(m, t)$ in the thermodynamic limit, both in massive and massless regime.

This paper is organised as follows. In Section 2, we briefly recall how to obtain multiple integral representations for the elementary building blocks of the correlation functions using the algebraic Bethe Ansatz method [11–13], and introduce useful techniques and notations that will be used all along the article. In Section 3 we explain how to re-sum these elementary building blocks to obtain compact representations for the two-point functions and their generating functions [14, 15]. The problem of asymptotic behaviour for large distances is tackled in Section 4. There we consider the toy example of the so-called emptiness formation probability to show how the multiple integral representations of Section 3 can be analysed in the asymptotic limit of large distances, both in the massless [16, 17] and massive regimes. We also discuss how the methods we have developed could be extended to the case of the two-point functions. Section 5 and 6 are devoted to the derivation of the master equation for the correlation functions by the two equivalent approaches that have been mentioned above. In the last section we present our conclusions and perspectives.

2 Algebraic Bethe Ansatz and elementary blocks

Any n-point correlation function of the Heisenberg chain can be reconstructed as a sum of elementary building blocks defined in the following way:

$$F_m(\{\epsilon_j, \epsilon'_j\}) = \langle \psi_g \mid \prod_{j=1}^{m} E_j^{\epsilon'_j, \epsilon_j} \mid \psi_g \rangle. \tag{2.1}$$

Here $\mid \psi_g \rangle$ is the normalised ground state of the chain and $E_j^{\epsilon'_j, \epsilon_j}$ denotes the elementary operator acting on the quantum space $\mathcal{H}_j$ at site j as the 2×2 matrix of elements $E_{lk}^{\epsilon', \epsilon} = \delta_{l, \epsilon'} \delta_{k, \epsilon}$.

A multiple integral representation for these building blocks was obtained for the first time in [29, 30]. In this section, we briefly recall how it can be derived in the framework of algebraic Bethe Ansatz [11, 12]. In general, we have to solve the following successive problems:

(i) determination of the ground state $\langle \psi_g \mid$,

(ii) evaluation of the action the product of local operators on this ground state,

(iii) computation of the scalar product of the resulting state with $\mid \psi_g \rangle$,

(iv) thermodynamic limit.

The starting point of our method is to use in step (i) the description of the eigenstates obtained via algebraic Bethe Ansatz [26, 28]. They are constructed in this framework in terms of generalised creation and annihilation operators which are themselves highly non-local. Acting with local operators on such states in step (ii) is therefore a priori a non-trivial problem. One of the key-ingredient of our method, which enables us to compute this action explicitly, is the solution of the so-called quantum inverse scattering problem [11, 13]: local operators are reconstructed in terms of the generators of the so-called Yang-Baxter algebra, which contains in particular these creation/annihilation operators for the eigenstates. Step (ii) can then be completed using only the quadratic commutation relations satisfied by these generators [12]. The computation of the resulting scalar products in step (iii) may also present some technical

difficulties. In the case of the XXZ Heisenberg chain, it has been solved using the algebraic structure of the Yang-Baxter algebra [11,40]. Finally, step (iv) is based on the results of [24,25].

Note that this procedure remains essentially the same in the case of the two-point correlation functions (see Section 3). The main difference is that, in step (ii), the reconstruction of the corresponding local operators from the solution of the inverse problem gives rise to a more complicated combination of the generators of the Yang-Baxter algebra, so that the use of their commutation relations to determine their action on the eigenstates involves a more complicated combinatoric.

2.1 General framework

To compute the elementary blocks (2.1), or more generally any correlation function, the first step is to determine the eigenstates of the Hamiltonian (1.1) and in particular its ground state. In the framework of algebraic Bethe Ansatz [26], these eigenstates can be described in terms of generalised creation and annihilation operators which are elements of the so-called quantum monodromy matrix. In the case of the XXZ chain (1.1) the monodromy matrix is a 2×2 matrix,

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{2.2}$$

with operator-valued entries A, B, C and D which depend on a complex parameter λ (spectral parameter) and act in the quantum space of states $\mathcal{H}$ of the chain. It is defined as the ordered product

$$T(\lambda) = L_M(\lambda) \dots L_2(\lambda) L_1(\lambda), \tag{2.3}$$

where $L_n(\lambda)$ denotes the quantum L-operator at the site n of the chain:

$$L_n(\lambda) = \begin{pmatrix} \sinh(\lambda + \frac{\eta}{2} \sigma_n^z) & \sinh \eta \, \sigma_n^- \\ \sinh \eta \, \sigma_n^+ & \sinh(\lambda - \frac{\eta}{2} \sigma_n^z) \end{pmatrix}. \tag{2.4}$$

Here and in the following, the parameter η is related to the anisotropy parameter as $\Delta = \cosh \eta$.

The quantum operators A, B, C and D satisfy a set of quadratic commutation relations given by the R-matrix of the model, and generate the so-called Yang-Baxter algebra. These commutation relations imply in particular that the transfer matrices, defined as

$$\mathcal{T}(\lambda) = \operatorname{tr} T(\lambda) = A(\lambda) + D(\lambda), \tag{2.5}$$

commute for different values of the spectral parameter: $[\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0$. The Hamiltonian (1.2) at $h = 0$ is related to $\mathcal{T}(\lambda)$ by the 'trace identity'

$$H^{(0)} = 2 \sinh \eta \, \frac{d\mathcal{T}(\lambda)}{d\lambda} \mathcal{T}^{-1}(\lambda) \Big|_{\lambda = \frac{\eta}{2}} - 2M \cosh \eta. \tag{2.6}$$

Therefore, to diagonalise the Hamiltonian (1.1), it is enough to determine the common eigenstates and eigenvalues of these transfer matrices.

For technical reasons, it is actually convenient to introduce a slightly more general object, the twisted transfer matrix

$$\mathcal{T}_\kappa(\lambda) = A(\lambda) + \kappa D(\lambda), \tag{2.7}$$

where κ is a complex parameter. The particular case of $T_\kappa(\lambda)$ at $\kappa = 1$ corresponds to the usual (untwisted) transfer matrix $T(\lambda)$. It will be also convenient to consider an inhomogeneous version of the XXZ chain, for which

$$T(\lambda) = L_M(\lambda - \xi_M + \eta/2) \ldots L_2(\lambda - \xi_2 + \eta/2) L_1(\lambda - \xi_1 + \eta/2). \tag{2.8}$$

Here, $\xi_1, \ldots, \xi_M$ are complex parameters (inhomogeneity parameters) attached to each site of the lattice. The homogeneous model (1.1) corresponds to the case where $\xi_j = \eta/2$ for $j = 1, \ldots, M$.

In the framework of algebraic Bethe Ansatz, an arbitrary quantum state can be obtained from the states generated by multiple action of operators $B(\lambda)$ on the reference state $|0\rangle$ with all spins up (respectively by multiple action of operators $C(\lambda)$ on the dual reference state $\langle 0|$),

$$|\psi\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle, \qquad \langle\psi| = \langle 0|\prod_{j=1}^{N} C(\lambda_j), \qquad N = 0, 1, \ldots, M. \tag{2.9}$$

2.2 Description of the eigenstates

Let us consider here the subspace $\mathcal{H}^{(M/2-N)}$ of the space of states $\mathcal{H}$ with a fixed number N of spins down. In this subspace, the eigenstates $|\psi_\kappa(\{\lambda\})\rangle$ (respectively $\langle\psi_\kappa(\{\lambda\})|$) of the twisted transfer matrix $T_\kappa(\mu)$ can be constructed in the form (2.9), where the parameters $\lambda_1, \ldots, \lambda_N$ satisfy the system of twisted Bethe equations

$$\mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) = 0, \qquad j = 1, \ldots, N. \tag{2.10}$$

Here, the function $\mathcal{Y}_\kappa$ is defined as

$$\mathcal{Y}_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^{N} \sinh(\lambda_k - \mu + \eta) + \kappa\, d(\mu) \prod_{k=1}^{N} \sinh(\lambda_k - \mu - \eta), \tag{2.11}$$

and $a(\lambda)$, $d(\lambda)$ are the eigenvalues of the operators $A(\lambda)$ and $D(\lambda)$ on the reference state $|0\rangle$. In the normalisation (2.4) and for the inhomogeneous model (2.8), we have

$$a(\lambda) = \prod_{a=1}^{M} \sinh(\lambda - \xi_a + \eta), \qquad d(\lambda) = \prod_{a=1}^{M} \sinh(\lambda - \xi_a). \tag{2.12}$$

The corresponding eigenvalue of $T_\kappa(\mu)$ on $|\psi_\kappa(\{\lambda\})\rangle$ (or on a dual eigenstate) is

$$\tau_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^{N} \frac{\sinh(\lambda_k - \mu + \eta)}{\sinh(\lambda_k - \mu)} + \kappa\, d(\mu) \prod_{k=1}^{N} \frac{\sinh(\mu - \lambda_k + \eta)}{\sinh(\mu - \lambda_k)}. \tag{2.13}$$

The solutions of the system of twisted Bethe equations (2.10) have been analysed in [41]. In general, not all of these solutions correspond to eigenvectors of $T_\kappa(\mu)$.

DEFINITION 2.1. *A solution* $\{\lambda\}$ *of the system (2.10) is called admissible if*

$$d(\lambda_j) \prod_{\substack{k=1 \\ k\neq j}}^{N} \sinh(\lambda_j - \lambda_k + \eta) \neq 0, \qquad j = 1, \ldots, N, \tag{2.14}$$

and unadmissible otherwise. A solution is called off-diagonal if the corresponding parameters $\lambda_1, \ldots, \lambda_N$ *are pairwise distinct, and diagonal otherwise.*

One of the main result of [41] is that, for generic parameters κ and $\{\xi\}$, the set of the eigenstates corresponding to the admissible off-diagonal solutions of the system of twisted Bethe equations (2.10) form a basis in the subspace $\mathcal{H}^{(M/2-N)}$. It has been proven in [19] that this result is still valid in the homogeneous case $\xi_j = \eta/2$, $j = 1, \ldots, N$, at least if κ is in a punctured vicinity of the origin (i.e. $0 < |\kappa| < \kappa_0$ for κ_0 small enough). Note however that, for specific values of κ and $\{\xi\}$, the basis of the eigenstates in $\mathcal{H}^{(M/2-N)}$ may include some states corresponding to unadmissible solutions of (2.10) (in particular in the homogeneous limit at $\kappa = 1$).

At $\kappa = 1$, it follows from the trace identity (2.6) that the eigenstates of the transfer matrix coincide, in the homogeneous limit, with the ones of the Hamiltonian (1.1). The corresponding eigenvalues in the case of zero magnetic field can be obtained from (2.6), (2.13):

$$H^{(0)} |\psi(\{\lambda\})\rangle = \sum_{j=1}^{N} E(\lambda_j) \cdot |\psi(\{\lambda\})\rangle,\tag{2.15}$$

where the bare one-particle energy $E(\lambda)$ is equal to

$$E(\lambda) = \frac{2\sinh^2 \eta}{\sinh(\lambda + \frac{\eta}{2})\sinh(\lambda - \frac{\eta}{2})}.\tag{2.16}$$

One can similarly define the bare one-particle momentum. It is given by

$$p(\lambda) = i \log\left(\frac{\sinh(\lambda - \frac{\eta}{2})}{\sinh(\lambda + \frac{\eta}{2})}\right).\tag{2.17}$$

2.3 Action of local operators on eigenstates

A local operator $E_j^{\epsilon'_j, \epsilon_j}$, acting in a local quantum space $\mathcal{H}_j$ at site j, can also be expressed in terms of the entries of the monodromy matrix by solving the quantum inverse scattering problem [11, 13]:

$$E_j^{\epsilon'_j, \epsilon_j} = \prod_{\alpha=1}^{j-1} \mathcal{T}(\xi_\alpha) \cdot T_{\epsilon_j, \epsilon'_j}(\xi_j) \cdot \prod_{\alpha=1}^{j} \mathcal{T}^{-1}(\xi_\alpha).\tag{2.18}$$

This enables us to use the quadratic commutation relations for the generators A, B, C, D of the Yang-Baxter algebra to get the action of any product of local operators on arbitrary states of the form (2.9) [12]:

$$\langle 0| \prod_{k=1}^{N} C(\lambda_k) \cdot \prod_{j=1}^{m} T_{\epsilon_j, \epsilon'_j}(\lambda_{N+j}) = \sum_{\mathcal{P} \subset \{\lambda\}} \mathcal{F}_\mathcal{P}(\{\lambda\}, \{\epsilon_j, \epsilon'_j\}) \langle 0| \prod_{b \in \mathcal{P}} C(\lambda_b),\tag{2.19}$$

in which the sum is taken over subsets $\mathcal{P}$ of cardinal N of the set $\{\lambda_1, \ldots, \lambda_{N+m}\}$, and the coefficients $\mathcal{F}_\mathcal{P}(\{\lambda\}, \{\epsilon_j, \epsilon'_j\})$ can be computed generically. Thus, the elementary blocks (2.1), and more generally any correlation functions (see Section 3), can be expressed as some sums over scalar products of a Bethe state with an arbitrary state of the form (2.9).

2.4 Scalar products

We recall here the expressions for the scalar product of an eigenstate of the twisted transfer matrix with any arbitrary state of the form (2.9).

Let us first define, for arbitrary positive integers n, n' ($n \leq n'$) and arbitrary sets of variables $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_{n'}$ such that $\{\lambda\} \subset \{\nu\}$, the $n \times n$ matrix $\Omega_\kappa(\{\lambda\}, \{\mu\}|\{\nu\})$ as

$$(\Omega_\kappa)_{jk}(\{\lambda\}, \{\mu\}|\{\nu\}) = a(\mu_k) \, t(\lambda_j, \mu_k) \prod_{a=1}^{n'} \sinh(\nu_a - \mu_k + \eta)$$

$$- \kappa \, d(\mu_k) \, t(\mu_k, \lambda_j) \prod_{a=1}^{n'} \sinh(\nu_a - \mu_k - \eta), \quad (2.20)$$

with

$$t(\lambda, \mu) = \frac{\sinh \eta}{\sinh(\lambda - \mu) \sinh(\lambda - \mu + \eta)}. \quad (2.21)$$

PROPOSITION 2.1. [11, 18, 40] *Let $\{\lambda_1, \ldots, \lambda_N\}$ be a solution of the system of twisted Bethe equations (2.10), and $\mu_1, \ldots, \mu_N$ be generic complex numbers. Then*

$$\langle 0 | \prod_{j=1}^{N} C(\mu_j) | \psi_\kappa(\{\lambda\}) \rangle = \langle \psi_\kappa(\{\lambda\}) | \prod_{j=1}^{N} B(\mu_j) | 0 \rangle$$

$$= \frac{\prod\limits_{a=1}^{N} d(\lambda_a) \prod\limits_{a,b=1}^{N} \sinh(\mu_b - \lambda_a)}{\prod\limits_{a>b}^{N} \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)} \cdot \det_N \left(\frac{\partial}{\partial \lambda_j} \tau_\kappa(\mu_k|\{\lambda\}) \right) \quad (2.22)$$

$$= \frac{\prod\limits_{a=1}^{N} d(\lambda_a)}{\prod\limits_{a>b}^{N} \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)} \cdot \det_N \Omega_\kappa(\{\lambda\}, \{\mu\}|\{\lambda\}). \quad (2.23)$$

Remark 2.1. If the sets $\{\lambda\}$ and $\{\mu\}$ are different, the eigenstate $| \psi_\kappa(\{\lambda\}) \rangle$ is orthogonal to the dual eigenstate $\langle \psi_\kappa(\{\mu\}) |$. Otherwise

$$\langle \psi_\kappa(\{\lambda\}) | \psi_\kappa(\{\lambda\}) \rangle = \frac{\prod\limits_{a=1}^{N} d(\lambda_a)}{\prod\limits_{\substack{a,b=1 \\ a \neq b}}^{N} \sinh(\lambda_a - \lambda_b)} \cdot \det_N \Omega_\kappa(\{\lambda\}, \{\lambda\}|\{\lambda\}) \quad (2.24)$$

$$= (-1)^N \frac{\prod\limits_{a=1}^{N} d(\lambda_a)}{\prod\limits_{\substack{a,b=1 \\ a \neq b}}^{N} \sinh(\lambda_a - \lambda_b)} \cdot \det_N \left(\frac{\partial}{\partial \lambda_k} \mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) \right). \quad (2.25)$$

The equations (2.22)–(2.25) are valid for any arbitrary complex parameter κ, in particular at $\kappa = 1$. In this case we may omit the subscript κ and denote $(\psi, \tau, \mathcal{Y}, \Omega) = (\psi_\kappa, \tau_\kappa, \mathcal{Y}_\kappa, \Omega_\kappa)|_{\kappa=1}$.

Using these expressions for the scalar product and the norm of the Bethe state, one sees from equation (2.19) that the correlation functions can be expressed as (multiple) sums of determinants [12].

2.5 Elementary blocks in the thermodynamic limit

In the thermodynamic limit, the system of Bethe equations for the ground state turns into a single integral equation for the ground state spectral density $\rho_{\rm tot}(\lambda)$ [25]:

$$\rho_{\rm tot}(\lambda) + \int_{\mathcal{C}} K(\lambda - \mu)\, \rho_{\rm tot}(\mu)\, d\mu = \frac{i}{2\pi} t(\lambda, \eta/2), \tag{2.26}$$

where the contour $\mathcal{C}$, which depends on the value of the magnetic field h, is an interval of the real axis in the massless regime and of the imaginary axis in the massive regime ($\mathcal{C} = [-\Lambda_h, \Lambda_h]$), and the kernel K is given by

$$K(\lambda) = \frac{i \sinh 2\eta}{2\pi \sinh(\lambda + \eta) \sinh(\lambda - \eta)}. \tag{2.27}$$

For technical convenience, one can also define an inhomogeneous version $\rho(\lambda, \xi)$ of this ground state density as the solution of the equation

$$\rho(\lambda, \xi) + \int_{\mathcal{C}} K(\lambda - \mu)\, \rho(\mu, \xi)\, d\mu = \frac{i}{2\pi} t(\lambda, \xi). \tag{2.28}$$

Note that $\rho_{\rm tot}(\lambda) = \rho(\lambda, \eta/2)$. In the case of zero magnetic field, this integral equation can be solved explicitely and we have

$$|\Delta| < 1: \quad \Lambda_h = \Lambda = \infty, \tag{2.29}$$

$$\rho(\lambda, \xi) = \frac{i}{2\zeta \sinh \frac{\pi}{\zeta}(\lambda - \xi)}, \qquad (\zeta = i\eta > 0), \tag{2.30}$$

$$\Delta > 1: \quad \Lambda_h = \Lambda = -i\pi/2, \tag{2.31}$$

$$\rho(\lambda, \xi) = -\frac{1}{2\pi} \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 + q^{2n}}\right)^2 \frac{\vartheta_2(i(\lambda - \xi), q)}{\vartheta_1(i(\lambda - \xi), q)}, \quad (\zeta = -\eta > 0, \; q = e^{\eta}). \tag{2.32}$$

More generally, in the limit $M \to \infty$, sums over the solutions $\lambda_1, \ldots, \lambda_N$ of Bethe equations for the ground state become integrals over the density $\rho_{\rm tot}$:

$$\frac{1}{M} \sum_{j=1}^{N} f(\lambda_j) = \int_{\mathcal{C}} \rho_{\rm tot}(\lambda) f(\lambda)\, d\lambda + o(1/M), \tag{2.33}$$

for any smooth bounded function $f(\lambda)$. This leads to a multiple integral representation for the correlation functions. In particular, the m-point elementary blocks (2.1) can be written as a m-fold multiple integral of the form [12]

$$F_m(\{\epsilon_j, \epsilon'_j\}) = \left(\prod_{j=1}^{m} \int_{\mathcal{C}_j} d\lambda_j\right) \mathcal{F}(\{\lambda\}, \{\epsilon_j, \epsilon'_j\})\, \mathcal{S}(\{\lambda\}). \tag{2.34}$$

In this expression, the set of integration contours $\{\mathcal{C}_j, j = 1, \ldots, m\}$ depends on the regime, on the value of the magnetic field, and on the configuration $\{\epsilon_j, \epsilon'_j\}$ of the block we consider (see [12]). The integrand can be split into two parts: a purely algebraic quantity $\mathcal{F}(\{\lambda\}, \{\epsilon_j, \epsilon'_j\})$ which arises from the commutation relation of the monodromy matrix elements and does not depend on the ground state, and a quantity $\mathcal{S}(\{\lambda\})$ which is the same for all blocks and contains

all the informations about the ground state. The latter is actually a functional of the ground state density that comes from the thermodynamic limit of the normalised scalar product. In the general inhomogeneous case, it is given as

$$S(\{\lambda\}) = \prod_{1 \leq j < k \leq m} \frac{1}{\sinh(\xi_j - \xi_k)} \cdot \det_{1 \leq j,k \leq m} [\rho(\lambda_j, \xi_k)]. \tag{2.35}$$

We refer to [12] for an explicit expression of the algebraic part $\mathcal{F}(\{\lambda\}, \{\epsilon_j, \epsilon'_j\})$. Let us just mention here that one can essentially distinguish two types of integrals at this level: what we will call 'D-type' integral, that comes from the contribution of the action of an operator D on a state of the from (2.9), with its corresponding algebraic part and integration contour $\mathcal{C}_j = \mathcal{C}$, and the 'A-type' integral, associated to the action of operator A, with a different algebraic part and a contour $\mathcal{C}_j$ which is shifted compared to the contour $\mathcal{C}$ of the integral equation (2.26) for the ground state density. As the action of operator B is very similar to the successive action of operators A and D, it produces in the final result both types of integrals.

Let us finally note that the representation (2.34) for the elementary block (2.1) coincides exactly for zero magnetic field with the multiple integral representation obtained and conjectured in [29,30] (see also [31]) from the q-vertex operator approach, and generalises it to the case of a non-zero magnetic field for which the quantum affine symmetry used in [31] is broken.

3 Re-summation of the elementary blocks

The method presented in the last section is quite straightforward and gives formally the possibility to compute any correlation function. However, it has been developed for the computation of the expectation values of the monomials $T_{a_1 b_1}(\xi_1) \cdots T_{a_m b_m}(\xi_m)$, leading to the evaluation of elementary building blocks, whereas the study of the two-point functions involves big sums of such monomials.

Indeed, let us consider for example the correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$. Then, according to the solution of the inverse scattering problem (2.18), we need to calculate the expectation value

$$\langle \psi(\{\lambda\}) | C(\xi_1) \cdot \prod_{a=2}^{m} T(\xi_a) \cdot B(\xi_{m+1}) \cdot \prod_{b=1}^{m+1} T^{-1}(\xi_b) | \psi(\{\lambda\}) \rangle. \tag{3.1}$$

Since $| \psi(\{\lambda\}) \rangle$ is an eigenstate of the transfer matrix T, the action of $\prod_{b=1}^{m+1} T^{-1}(\xi_b)$ on this state merely produces a numerical factor. However, it is much more complicated to evaluate the action of $\prod_{a=2}^{m} T(\xi_a)$. Indeed, we have to act first with $C(\xi_1)$ on $\langle \psi(\{\lambda\}) |$ (or with $B(\xi_{m+1})$ on $| \psi(\{\lambda\}) \rangle$), which gives a sum of states which are no longer eigenstates of the transfer matrix, and on which the multiple action of T is not simple. In fact, in the framework of the approach of Section 2, the product $\prod_{a=2}^{m} (A + D)(\xi_a)$ would be computed as a sum of 2^{m-1} monomials, which eventually would lead to a huge sum of elementary blocks. This is not very convenient, in particular at large distance m. Therefore, to obtain manageable expressions for such correlation functions, it is of great importance to develop an alternative and compact way to express the multiple action of the transfer matrix on arbitrary states or, in other words, to make an effective re-summation of the corresponding sum of 2^{m-1} terms.

In this section, we explain two different ways to perform such re-summations.

3.1 Re-summation with auxiliary integrals

The results presented in this subsection were first obtained in [14]. We recall here the main steps of this re-summation.

Let us consider the multiple action of the twisted transfer matrices on an arbitrary dual state $\langle 0 \, | \, \prod_{j=1}^{N} C(\mu_j)$,

$$\langle 0 \, | \prod_{j=1}^{N} C(\mu_j) \prod_{a=1}^{m} \mathcal{T}_\kappa(x_a), \tag{3.2}$$

where $x_1, \ldots, x_m$ and $\mu_1, \ldots, \mu_N$ are generic complex numbers. Using the quadratic commutation relations between A, D and C, one can prove

PROPOSITION 3.1. [14] *Let* κ, $x_1, \ldots, x_m$ *and* $\mu_1, \ldots, \mu_N$ *be generic complex numbers. The action of* $\prod_{a=1}^{m} \mathcal{T}_\kappa(x_a)$ *on the state* $\langle 0 \, | \, \prod_{j=1}^{N} C(\mu_j)$ *can be written as*

$$\langle 0 \, | \prod_{j=1}^{N} C(\mu_j) \prod_{a=1}^{m} \mathcal{T}_\kappa(x_a)$$

$$= \sum_{n=0}^{\min(m,N)} \sum_{\substack{\{\mu\}=\{\mu_{\alpha_+}\}\cup\{\mu_{\alpha_-}\} \\ \{x\}=\{x_{\gamma_+}\}\cup\{x_{\gamma_-}\} \\ |\alpha_+|=|\gamma_+|=n}} R_n^\kappa(\{x_{\gamma_+}\},\{x_{\gamma_-}\},\{\mu_{\alpha_+}\},\{\mu_{\alpha_-}\}) \langle 0 \, | \prod_{a\in\gamma_+} C(x_a) \prod_{b\in\alpha_-} C(\mu_b). \tag{3.3}$$

In this expression the set of parameters $\{\mu\}$ *is divided into two subsets* $\{\mu\} = \{\mu_{\alpha_+}\} \cup \{\mu_{\alpha_-}\}$ *such that* $\{\mu_{\alpha_+}\} \cap \{\mu_{\alpha_-}\} = \emptyset$. *Similarly the set* $\{x\}$ *is also divided as* $\{x\} = \{x_{\gamma_+}\} \cup \{x_{\gamma_-}\}$, $\{x_{\gamma_+}\} \cap \{x_{\gamma_-}\} = \emptyset$. *These partitions are independent except that* $\#\{x_{\gamma_+}\} = \#\{\mu_{\alpha_+}\} = n$. *The sum in (3.3) is taken with respect to all such partitions, and the corresponding coefficient* $R_n^\kappa(\{x_{\gamma_+}\},\{x_{\gamma_-}\},\{\mu_{\alpha_+}\},\{\mu_{\alpha_-}\})$ *is given by*

$$R_n^\kappa = \left\{ \prod_{\substack{a>b \\ a,b\in\alpha_+}} \sinh(\mu_b - \mu_a) \prod_{\substack{a<b \\ a,b\in\gamma_+}} \sinh(x_b - x_a) \prod_{a\in\alpha_+} \prod_{b\in\alpha_-} \sinh(\mu_b - \mu_a) \right\}^{-1}$$

$$\times \prod_{a\in\gamma_-} \mathcal{T}_\kappa(x_a|\{x_{\gamma_+}\} \cup \{\mu_{\alpha_-}\}) \cdot \det_n \Omega_\kappa(\{x_{\gamma_+}\},\{\mu_{\alpha_+}\} \, | \, \{x_{\gamma_+}\} \cup \{\mu_{\alpha_-}\}). \tag{3.4}$$

The equations (3.3), (3.4) are the key-formulae of our re-summation. When applying these expressions to particular cases, one obtains directly new multiple integral representations for the two-point correlation functions which are essentially different from the ones that result from the elementary blocks approach.

One of the simplest applications of Proposition 3.1 concerns the generating function of the two-point correlation function of the third components of spin, which is defined as the expectation value

$$\langle Q_{l,m}^\kappa \rangle = \frac{\langle \psi(\{\lambda\}) \, | \, Q_{l,m}^\kappa \, | \, \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) \, | \, \psi(\{\lambda\}) \rangle} \tag{3.5}$$

of the operator

$$Q_{l,m}^\kappa = \prod_{n=l}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma_n^z \right) = \prod_{j=1}^{l-1} \mathcal{T}(\xi_j) \cdot \prod_{j=l}^{m} \mathcal{T}_\kappa(\xi_j) \cdot \prod_{j=1}^{m} \mathcal{T}^{-1}(\xi_j), \tag{3.6}$$

where $| \, \psi(\{\lambda\}) \rangle$ is an eigenstate of $\mathcal{T}(\mu)$ in the subspace $\mathcal{H}^{(M/2-N)}$. The two-point correlation function of the third components of local spins in the eigenstate $| \, \psi(\{\lambda\}) \rangle$ can be obtained

in terms of the second 'lattice derivative' and the second derivative with respect to κ of the generating function (3.5) at $\kappa = 1$:

$$\langle \sigma_l^z \, \sigma_{l+m}^z \rangle = \langle \sigma_l^z \rangle + \langle \sigma_{l+m}^z \rangle - 1$$

$$+ 2\frac{\partial^2}{\partial \kappa^2} \langle Q_{l,l+m}^\kappa - Q_{l,l+m-1}^\kappa - Q_{l+1,l+m}^\kappa + Q_{l+1,l+m-1}^\kappa \rangle \Big|_{\kappa=1}. \quad (3.7)$$

Due to the translational invariance of the correlation functions in the homogeneous model, we will simply consider the following expectation value:

$$\langle Q_{1,m}^\kappa \rangle = \prod_{j=1}^{m} \tau^{-1}(\xi_j | \{\lambda\}) \cdot \frac{\langle \psi(\{\lambda\}) | \prod_{j=1}^{m} \mathcal{T}_\kappa(\xi_j) | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) | \psi(\{\lambda\}) \rangle}. \quad (3.8)$$

In order to evaluate this generating function, one should first compute the multiple action of $\mathcal{T}_\kappa(\xi_j)$ in the r.h.s. of (3.8) by means of Proposition 3.1, and then project the result on the eigenstate $| \psi(\{\lambda\}) \rangle$ using (2.22) for the scalar product. Hereby the expression of the coefficient R_n^κ and of the matrices Ω, Ω_κ can be simplified using Bethe equations for the set $\{\lambda\}$ and the fact that $d(\xi_j) = 0$. Note also that we can restrict ourselves to the case $m < N$, since eventually we are going to compute the correlation function in the thermodynamic limit. The result can be written in the following form

$$\langle Q_{1,m}^\kappa \rangle = \sum_{n=0}^{m} \sum_{\substack{\{\lambda\}=\{\lambda_{\alpha_+}\}\cup\{\lambda_{\alpha_-}\} \\ \{\xi\}=\{\xi_{\gamma_+}\}\cup\{\xi_{\gamma_-}\} \\ |\alpha_+|=|\gamma_+|=n}} \prod_{a\in\gamma_-} \prod_{b\in\gamma_+} \frac{\sinh(\xi_b - \xi_a + \eta)}{\sinh(\xi_b - \xi_a)} \cdot \mathcal{F}_n^\kappa(\{x_{\gamma_+}\}, \{\lambda_{\alpha_+}\}, \{\lambda_{\alpha_-}\}). \quad (3.9)$$

Here we have combined all the factors in one function $\mathcal{F}_n$ and extracted explicitly the dependency on the subset $\{\xi_{\gamma_-}\}$. We refer to [14] for a more explicit expression.

Let us now suppose that $| \psi(\{\lambda\}) \rangle$ is the eigenstate of the inhomogeneous transfer matrix which tends, in the homogeneous limit, toward the ground state of the Hamiltonian (1.1). Then, in the thermodynamic limit, the sum over the partitions of the set $\{\lambda\}$ turns, for each given n, into an n-fold multiple integral over the support of the ground state density, just like in Section 2.5 for the elementary blocks. As for the sum over the partitions of the set $\{\xi\}$, it can be computed in terms of some auxiliary contour integrals. Indeed, it is easy to see that

$$\sum_{\substack{\{\xi\}=\{\xi_{\gamma_-}\}\cup\{\xi_{\gamma_+}\} \\ |\gamma_+|=n}} \prod_{a\in\gamma_-} \prod_{b\in\gamma_+} \frac{\sinh(\xi_b - \xi_a + \eta)}{\sinh(\xi_b - \xi_a)} \cdot \mathcal{F}_n^\kappa(\{x_{\gamma_+}\}, \{\lambda_{\alpha_+}\}, \{\lambda_{\alpha_-}\})$$

$$= \frac{1}{n!} \oint_{\Gamma\{\xi\}} \prod_{j=1}^{n} \frac{dz}{2\pi i} \prod_{a=1}^{n} \prod_{b=1}^{m} \frac{\sinh(z_a - \xi_b + \eta)}{\sinh(z_a - \xi_b)}$$

$$\times \frac{\prod_{\substack{a=1 \\ b\neq a}}^{n} \prod_{b=1}^{n} \sinh(z_a - z_b)}{\prod_{a=1}^{n} \prod_{b=1}^{n} \sinh(z_a - z_b + \eta)} \cdot \mathcal{F}_n^\kappa(\{z\}, \{\lambda_{\alpha_+}\}, \{\lambda_{\alpha_-}\}). \quad (3.10)$$

Here the contour $\Gamma\{\xi\}$ surrounds the points $\xi_1, \ldots, \xi_m$ and does not contain any other singularities of the integrand. Observe that this representation allows one to take the homogeneous limit directly by setting $\xi_j = \eta/2$ in the expression.

Thus, the sum over partitions in (3.9) can be written in terms of multiple integrals. The resulting representation for the generating function of the correlation function $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ has the following form [14]:

$$\langle Q_{1,m}^\kappa \rangle = \sum_{n=0}^{m} \frac{1}{(n!)^2} \oint_{\Gamma\{\xi\}} \prod_{j=1}^{n} \frac{dz}{2\pi i} \int_C d^n\lambda \prod_{a=1}^{n} \prod_{b=1}^{m} \frac{\sinh(z_a - \xi_b + \eta)\sinh(\lambda_a - \xi_b)}{\sinh(z_a - \xi_b)\sinh(\lambda_a - \xi_b + \eta)}$$

$$\times\, W_n(\{\lambda\}, \{z\}) \cdot \det_n M_\kappa(\{\lambda\}, \{z\}) \cdot \det_n \rho(\lambda_j, z_k), \quad (3.11)$$

with

$$W_n(\{\lambda\}, \{z\}) = \prod_{a=1}^{n}\prod_{b=1}^{n} \frac{\sinh(z_a - \lambda_b + \eta)\sinh(\lambda_b - z_a + \eta)}{\sinh(z_a - z_b + \eta)\sinh(\lambda_a - \lambda_b + \eta)}, \quad (3.12)$$

and

$$(M_\kappa)_{jk}(\{\lambda\}, \{z\}) = t(z_k, \lambda_j) + \kappa\, t(\lambda_j, z_k) \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j + \eta)\sinh(\lambda_j - z_a + \eta)}{\sinh(\lambda_j - \lambda_a + \eta)\sinh(z_a - \lambda_j + \eta)}. \quad (3.13)$$

The integration contour C and the density function $\rho(\lambda, z)$ are defined in (2.26)–(2.28).

If we had used the expressions of the elementary blocks derived in Section 2, we would have obtained the generating function $\langle Q_{1,m}^\kappa \rangle$ as a sum of 2^m terms, each of them being written as a m-multiple integral of the type (2.34). Instead, we have now a representation containing only m nontrivial terms. The n-th term is formulated as a $2n$-fold multiple integral, with n integrals over the support of the ground state density and n auxiliary contour integrals over some auxiliary variables z_j. We will see in Section 5 that these last integrals play the role of an effective re-summation of the form factor series.

Observe also that, in the homogeneous model, the dependency on the distance m enters each integral only as a power of a simple function. This fact might be used for the asymptotic analysis of these multiple integrals by the steepest descent method.

Other two-point functions can be considered in a similar manner. For example, the expectation value (3.1) gives us the correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$. It is clear that one can evaluate this correlation function by using first the equations (3.3), (3.4) of Proposition 3.1, by acting in a second step with the operator $B(\xi_{m+1})$ on the resulting states, and by finally computing the corresponding scalar products via (2.22). All the steps of this derivation are quite similar to the ones that we have just described in the case of the generating function $\langle Q_{1,m}^\kappa \rangle$. Let us merely give here the new multiple integral representation that we obtain by this method for the ground-state correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$ in the thermodynamic limit. For simplicity, we present the answer in the homogeneous limit and at zero magnetic field:

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \sum_{n=0}^{m-1} \frac{1}{n!(n+1)!} \oint_{\Gamma\{\frac{\eta}{2}\}} \prod_{j=1}^{n+1} \frac{dz_j}{2\pi i} \int_C d^{n+2}\lambda \left(\prod_{a=1}^{n+1} \frac{\sinh(z_a + \frac{\eta}{2})}{\sinh(z_a - \frac{\eta}{2})} \cdot \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \frac{\eta}{2})}{\sinh(\lambda_a + \frac{\eta}{2})} \right)^m$$

$$\times\, \frac{1}{\sinh(\lambda_{n+1} - \lambda_{n+2})} \cdot \left(\frac{\prod_{a=1}^{n+1} \sinh(\lambda_{n+1} - z_a + \eta)\sinh(\lambda_{n+2} - z_a)}{\prod_{a=1}^{n} \sinh(\lambda_{n+1} - \lambda_a + \eta)\sinh(\lambda_{n+2} - \lambda_a)} \right) \cdot \hat{W}_n(\{\lambda\}, \{z\})$$

$$\times \det_{n+1} \hat{M}_\kappa(\{\lambda\}, \{z\}) \cdot \det_{n+2} \left[\rho(\lambda_j, z_1), \ldots, \rho(\lambda_j, z_{n+1}), \rho(\lambda_j, \tfrac{\eta}{2}) \right], \quad (3.14)$$

where the contours $\mathcal{C}$ and $\Gamma\{\frac{\eta}{2}\}$ are the same as in (3.11). The analogue $\hat{W}_n(\{\lambda\}, \{z\})$ of the function $W_n(\{\lambda\}, \{z\})$ is

$$\hat{W}_n(\{\lambda\}, \{z\}) = \frac{\prod\limits_{a=1}^{n} \prod\limits_{b=1}^{n+1} \sinh(\lambda_a - z_b + \eta) \sinh(z_b - \lambda_a + \eta)}{\prod\limits_{a=1}^{n} \prod\limits_{b=1}^{n} \sinh(\lambda_a - \lambda_b + \eta) \prod\limits_{a=1}^{n+1} \prod\limits_{b=1}^{n+1} \sinh(z_a - z_b + \eta)}, \tag{3.15}$$

and the $(n+1) \times (n+1)$ matrix $\hat{M}_\kappa$ has the entries

$$(\hat{M}_\kappa)_{jk} = t(z_k, \lambda_j) - t(\lambda_j, z_k) \prod\limits_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j + \eta)}{\sinh(\lambda_j - \lambda_a + \eta)} \prod\limits_{b=1}^{n+1} \frac{\sinh(\lambda_j - z_b + \eta)}{\sinh(z_b - \lambda_j + \eta)}, \quad j \le n, \tag{3.16}$$

$$(\hat{M}_\kappa)_{n+1,k} = t(z_k, \tfrac{\eta}{2}). \tag{3.17}$$

3.2 Alternative method

There exists another way to reduce the number of terms in the multiple integral representations for the two-point functions. In fact, the re-summation which has just been described has been performed at the algebraic level: we have computed algebraically the multiple action of the twisted transfer matrices on an arbitrary state and, thus, we have avoided any mention of the elementary blocks. On the contrary, the method that will be presented below deals directly with the elementary blocks in the thermodynamic limit.

Let us consider again the generating function $\langle Q_{1,m}^\kappa \rangle$ for the correlation function of the third components of spin. It has been already mentioned that one can, in the multiple integral representations (2.34) for the elementary blocks, distinguish two types of integrals: the 'D-type' integrals (with the original contour $\mathcal{C}$), and the 'A-type' integrals (with a shifted contour). In fact, the generating function (3.8) can be decomposed as a sum over elementary blocks obtained as expectation values of products of operators A and D only. Such elementary blocks, containing only diagonal elementary matrices, can be in general written in the following form:

$$F_m(\{\epsilon_j, \epsilon_j\}) = \int\limits_{\mathcal{C}} d\lambda_1 \dots \int\limits_{\mathcal{C}} d\lambda_m \, \mathcal{S}(\{\lambda\})$$

$$\times \prod\limits_{j>k} \frac{\sinh(\lambda_j - \xi_k + (\epsilon_j - 1)\eta) \sinh(\lambda_k - \xi_j + (2 - \epsilon_k)\eta)}{\sinh(\lambda_j - \lambda_k + (3 - \epsilon_j - \epsilon_k)\eta)}, \tag{3.18}$$

where the indexes ϵ_j can take two values 1 or 2: $\epsilon_j = 1$ corresponds to an 'A-type' integral and $\epsilon_j = 2$ corresponds to a 'D-type' integral. For simplicity reason, we consider here only the zero magnetic field case, but a representation similar to (3.18), with more complicated integration contours, can also be written in the case of a non-zero external magnetic field. Hence, the generating function $\langle Q_{1,m}^\kappa \rangle$ can be expressed as a sum of 2^m such terms. It is easy to see that the terms which have the same number of 'A-type' integrals exhibit a quite similar structure. This observation permits to write the generating function $\langle Q_{1,m}^\kappa \rangle$ as power series on κ,

$$\langle Q_{1,m}^\kappa \rangle = \sum\limits_{s=0}^{m} \kappa^s G_s(m), \tag{3.19}$$

where the coefficient $G_s(m)$ collects all the terms containing s 'D-type' integrals and $m - s$ 'A-type' integrals. This coefficient can be expressed as the following sum,

$$G_s(m) = \sum_{\epsilon_1 + \cdots + \epsilon_m - m = s} F_m(\{\epsilon_j, \epsilon_j\}). \tag{3.20}$$

One can immediately remark from (3.18) that the ground state density functional $S(\{\lambda\})$ (2.35) is common for all the terms in this sum. After symmetrisation over the variables λ corresponding to the integrals of the same type and extraction of the common denominator

$$\Theta_m^s(\lambda_1, \ldots, \lambda_m) = \prod_{k=1}^{s} \prod_{j=s+1}^{m} \frac{1}{\sinh(\lambda_j - \lambda_k)}$$

$$\times \prod_{m \geq j > k > s} \frac{\sinh(\lambda_j - \lambda_k)}{\sinh(\lambda_j - \lambda_k + \eta) \sinh(\lambda_j - \lambda_k - \eta)}$$

$$\times \prod_{s \geq j > k \geq 1} \frac{\sinh(\lambda_j - \lambda_k)}{\sinh(\lambda_j - \lambda_k + \eta) \sinh(\lambda_j - \lambda_k - \eta)}, \tag{3.21}$$

we obtain the following representation:

$$G_s(m) = \frac{1}{s!(m-s)!} \int_C d\lambda_1 \ldots \int_C d\lambda_m \, \Theta_m^s(\lambda_1, \ldots, \lambda_m) \cdot \mathcal{G}_s(m, \{\lambda\}|\{\xi\}) \cdot S(\{\lambda_j\}). \tag{3.22}$$

The function $\mathcal{G}_s(m, \{\lambda\}|\{\xi\})$ in (3.22) is a rather complicated sum over permutations which corresponds to the sum (3.20) over all possible configurations of the algebraic part in the expression (3.18) of the elementary blocks. It is possible to express it in a simpler form if we notice that it satisfies the four following important properties:

1. The function $\mathcal{G}_s(m, \{\lambda\}|\{\xi\})$ is symmetric under the permutations of the variables ξ_1, $\xi_2, \ldots, \xi_m$.

2. The function $e^{(m-1)\lambda_j} \mathcal{G}_s(m, \{\lambda\}|\{\xi\})$ is a polynomial function of $e^{2\lambda_j}$ of degree $m - 1$.

3. For $m = 1$,

$$\mathcal{G}_0(1, \lambda_1|\xi_1) = \mathcal{G}_1(1, \lambda_1|\xi_1) = 1. \tag{3.23}$$

4. The function $\mathcal{G}_s(m, \{\lambda\}|\{\xi\})$ satisfies the following recursion relations,

$$\mathcal{G}_s(m, \{\lambda\}|\{\xi\})\Big|_{\lambda_j = \xi_k} = \prod_{\substack{a=1 \\ a \neq k}}^{m} \sinh(\lambda_j - \xi_a) \prod_{a \neq j} \sinh(\lambda_a - \xi_k)$$

$$\times \mathcal{G}_s(m - 1, \lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_m|\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots \xi_m), \quad j \leq s, \tag{3.24}$$

$$\mathcal{G}_s(m, \{\lambda\}|\{\xi\})\Big|_{\lambda_j = \xi_k} = \prod_{\substack{a=1 \\ a \neq k}}^{m} \sinh(\lambda_j - \xi_a) \prod_{a \neq j} \sinh(\lambda_a - \xi_k)$$

$$\times \mathcal{G}_{s-1}(m - 1, \lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_m|\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots \xi_m), \quad j > s. \tag{3.25}$$

These properties can be easily proved using the definition of $\mathcal{G}_s(m, \{\lambda\}|\{\xi\})$. They define this function in a unique way as for any m they define a polynomial of degree $m - 1$ in m points.

Recursion relations of the same kind as (3.24)-(3.25) were obtained for the first time by Korepin in [42] for the partition function of the six-vertex model with domain wall boundary conditions, and the corresponding unique solution was found by Izergin in [43]. The conditions 1-4 are very similar to the conditions that characterise the partition function except that they contain one more parameter s. However, the expression for the partition function obtained by Izergin satisfies these relations for any s. As the solution of the recursion relation is unique, we can conclude that the function $\mathcal{G}_s(m, \{\lambda\}|\{\xi\})$ is proportional to the partition function $Z_m(\{\lambda\}, \{\xi\})$ and does not depend on s. More precisely,

$$\mathcal{G}_s(m, \{\lambda\}|\{\xi\}) = \frac{1}{\sinh^m \eta} Z_m(\{\lambda\}, \{\xi\}), \tag{3.26}$$

where the partition function is given by the Izergin formula,

$$Z_m(\{\lambda\}, \{\xi\}) = \frac{\prod\limits_{j=1}^{m} \prod\limits_{k=1}^{m} \sinh(\lambda_j - \xi_k + \eta) \sinh(\lambda_j - \xi_k)}{\prod\limits_{j>k}^{m} \sinh(\lambda_j - \lambda_k) \sinh(\xi_k - \xi_j)} \cdot \det_m [t(\lambda_j, \xi_k)]. \tag{3.27}$$

We obtain finally the generating function $\langle Q_{1,m}^\kappa \rangle$ as a sum of $m+1$ terms, each of them being given as a m-fold multiple integral:

$$\langle Q_{1,m}^\kappa \rangle = \sum_{s=0}^{m} \kappa^s G_s(m), \tag{3.28}$$

$$G_s(m) = \frac{1}{s!(m-s)! \sinh^m \eta} \int\limits_C d^m \lambda \, \Theta_m^s(\lambda_1, \ldots, \lambda_m) \cdot Z_m(\{\lambda\}|\{\xi\}) \cdot S(\{\lambda\}). \tag{3.29}$$

It is interesting to mention that the first and the last terms in this sum give a representation for the emptiness formation probability which will be studied in details in the next section. One of the most interesting property of this representation is the presence under the integrals of the expression for the partition function of the corresponding six-vertex model with domain wall boundary conditions. This is a new and unexpected connection of this very important object with the correlation functions of the XXZ spin chain.

One can note that the two representations (3.11) and (3.28) of the generating function $\langle Q_{1,m}^\kappa \rangle$ that we have obtained in this section are quite different and present different advantages: the first terms of (3.11) are very simple, but further terms become more and more complicated, whereas all the terms of (3.28) have more or less the same structure. One can hope that this last remark may lead to a common strategy to compute their asymptotics.

Similar expressions can be obtained for the two-point functions. For example the correlation function $g_{+-}(m) = \langle \sigma_1^+ \sigma_{m+1}^- \rangle$ can be written as

$$g_{+-}(m) = \sum_{s=0}^{m-1} \tilde{g}_{+-}(m, s), \tag{3.30}$$

in which the coefficients $\tilde{g}_{+-}(m,s)$ are given as the following multiple integrals,

$$
\tilde{g}_{+-}(m,s) = \frac{1}{s!(m-1-s)!\sinh^{m-1}\eta} \int_C d\lambda_2 \ldots \int_C d\lambda_m \int_C d\lambda_+ \int_C d\lambda_-
$$

$$
\times \left(\prod_{k=2}^{s+1} \frac{\sinh(\lambda_- - \xi_k + \eta)\sinh(\lambda_k - \xi_1 + \eta)}{\sinh(\lambda_- - \lambda_k + \eta)} \right)
$$

$$
\times \left(\prod_{k=s+2}^{m} \frac{\sinh(\lambda_- - \xi_k + \eta)\sinh(\lambda_k - \xi_1)}{\sinh(\lambda_- - \lambda_k)} \right)
$$

$$
\times \left(\prod_{k=2}^{s+1} \frac{\sinh(\lambda_+ - \xi_k)\sinh(\lambda_k - \xi_{m+1})}{\sinh(\lambda_+ - \lambda_k)} \right)
$$

$$
\times \left(\prod_{k=s+2}^{m} \frac{\sinh(\lambda_+ - \xi_k)\sinh(\lambda_k - \xi_{m+1} + \eta)}{\sinh(\lambda_+ - \lambda_k - \eta)} \right)
$$

$$
\times \frac{\sinh(\lambda_+ - \xi_1)\sinh(\lambda_- - \xi_1 + \eta)}{\sinh(\lambda_+ - \lambda_-)} \cdot \Theta_{m-1}^s(\lambda_2, \ldots, \lambda_m)
$$

$$
\times Z_{m-1}(\{\lambda_2, \ldots, \lambda_m\}, \{\xi_2, \ldots, \xi_m\}) \cdot S(\{\lambda_2, \ldots, \lambda_m, \lambda_+, \lambda_-\}). \tag{3.31}
$$

A very similar representation can be also obtained directly for the two-point function $g_{zz}(m)$.

4 Towards asymptotic analysis

We have seen in the last section that it was possible to re-sum, at least partially, the multiple integral representation for the two-point function given by the sum over elementary blocks. This provides of course a more compact expression but, above all, an expression that seems, due to the particular form of the resulting multiple integrals, more suitable for the study of the asymptotic behaviour at large distances. In this section, we will see on a simple example how it is indeed possible to analyse this kind of integrals. We then discuss the problems that arise when one tries to extend this study to either representation (3.11) or (3.28) of the two-point function.

4.1 A simple example: the emptiness formation probability

There exists a particular correlation function for which it is possible to compute the main asymptotic behaviour: the so-called emptiness formation probability $\tau(m)$, which measures the probability of formation of some ferromagnetic sub-chain of length m in the (anti-ferromagnetic) ground state. It is defined as the expectation value

$$
\tau(m) = \langle \psi_g | \prod_{k=1}^{m} \frac{1 - \sigma_k^z}{2} | \psi_g \rangle \tag{4.1}
$$

on the normalised ground state $| \psi_g \rangle$ of the chain. Hence, this quantity corresponds to a single elementary block, which means that, in the framework of Section 2, it is given as a (single)

multiple integral of the type (2.34) with m integrations [12,31,44]:

$$\tau(m) = \lim_{\xi_1,\ldots\xi_m \to \eta/2} \prod_{a<b}^{m} \frac{1}{\sinh(\xi_a - \xi_b)}$$

$$\times \int_C d^m\lambda \; \frac{\prod_{j=1}^{m} \left\{ \prod_{k=1}^{j-1} \sinh(\lambda_j - \xi_k + \eta) \prod_{k=j+1}^{m} \sinh(\lambda_j - \xi_k) \right\}}{\prod_{a>b}^{m} \sinh(\lambda_a - \lambda_b + \eta)} \; \det_m[\rho(\lambda_j, \xi_k)]. \quad (4.2)$$

Due to its combinatorial simplicity, it has been widely studied recently (see for example [15,45–49]). However, the expression (4.2) is not convenient for the asymptotic analysis; in particular it is not symmetric. Its symmetrised version, obtained in [14], follows directly from the limit $\kappa \to \infty$ in representations (3.11) or (3.28) of the generating function $\langle Q_{1,m}^\kappa \rangle$:

$$\tau(m) = \lim_{\xi_1,\ldots\xi_m \to \eta/2} \frac{1}{m!} \int_C d^m\lambda \; \prod_{a,b=1}^{m} \frac{1}{\sinh(\lambda_a - \lambda_b + \eta)}$$

$$\times \prod_{a<b}^{m} \frac{\sinh(\lambda_a - \lambda_b)}{\sinh(\xi_a - \xi_b)} \cdot Z_m(\{\lambda\}, \{\xi\}) \cdot \det_m[\rho(\lambda_j, \xi_k)], \quad (4.3)$$

where $Z_m(\{\lambda\}, \{\xi\})$ denotes the partition function of the six-vertex model with domain wall boundary conditions given by (3.27). From this expression, it is possible to obtain the asymptotic behaviour of $\tau(m)$ using the saddle-point method. This was performed for the first time in [15] in the case of free fermions ($\Delta = 0$), but the method of [15] can be applied to the general case as well (see [17] for the study in the massless regime). We briefly recall here the main step of this computation and present the result in massless and massive regime.

To apply the saddle-point method to (4.3), it is convenient to express the integral in the following form:

$$\tau(m) = \int_{\mathcal{D}} d^m\lambda \; G_m(\{\lambda\}) \, e^{m^2 S_m(\{\lambda\})}, \quad (4.4)$$

with

$$S_m(\{\lambda\}) = -\frac{1}{m^2} \sum_{a>b}^{m} \log[\sinh(\lambda_a - \lambda_b + \eta)\sinh(\lambda_a - \lambda_b - \eta)]$$

$$+ \frac{1}{m} \sum_{a=1}^{m} \log[\sinh(\lambda_a + \eta/2)\sinh(\lambda_a - \eta/2)]$$

$$+ \frac{1}{m^2} \lim_{\xi_1\ldots\xi_m \to \eta/2} \log\left[\left(\frac{-2i\pi}{\sinh\eta}\right)^m \frac{\left(\det \rho(\lambda_j, \xi_k)\right)^2}{\prod_{a\neq b} \sinh(\xi_a - \xi_b)} \right] \quad (4.5)$$

and

$$G_m(\{\lambda\}) = \lim_{\xi_1\ldots\xi_m \to \eta/2} \frac{\det_m\left[\frac{i}{2\pi} t(\lambda_j, \xi_k) \right]}{\det_m \rho(\lambda_j, \xi_k)}. \quad (4.6)$$

In (4.4), the integration domain $\mathcal{D}$ is such that the variable of integration $\lambda_1, \ldots, \lambda_m$ are ordered in the interval $\mathcal{C} = [-\Lambda_h, \Lambda_h]$ (i.e. $-\Lambda_h < \lambda_1 < \cdots < \lambda_m < \Lambda_h$ in the massless case, and $-i\Lambda_h < i\lambda_1 < \cdots < i\lambda_m < i\Lambda_h$ in the massive case).

In the case of free fermions ($\Delta = 0$), $G_m(\{\lambda\}) \equiv 1$, and it is easy to see that S_m admits a unique maximum $S_m(\{\lambda'\})$ for a set of variables $\{\lambda'_1, \ldots, \lambda'_m\}$ satisfying the system of m saddle-point equations:

$$\partial_{\lambda_j} S_m(\{\lambda'\}) = 0, \qquad 1 \leq j \leq m. \tag{4.7}$$

In the limit $m \to \infty$, the distribution of these variables λ's at the saddle point can be described by a density function,

$$\rho_s(\lambda'_j) = \lim_{m \to \infty} \frac{1}{m(\lambda'_{j+1} - \lambda'_j)}, \tag{4.8}$$

and one can replace sums over the set $\{\lambda'\}$ by integrals:

$$\frac{1}{m} \sum_{j=1}^m f(\lambda'_j) \xrightarrow[m \to \infty]{} \int_C f(\lambda) \rho_s(\lambda) d\lambda, \tag{4.9}$$

$$\frac{1}{m} \sum_{\substack{j=1 \\ j \neq k}}^m \frac{f(\lambda'_j)}{\lambda'_j - \lambda'_k} \xrightarrow[m \to \infty]{} V.P. \int_C \frac{f(\lambda)}{\lambda - \lambda'_k} \rho_s(\lambda) d\lambda, \tag{4.10}$$

for any function f integrable on the contour C. Hence, the system (4.7) becomes a single integral equation for the density $\rho_s(\lambda')$, that can be solved explicitely by Fourier transform. Replacing, at the leading order in m, the expression of this saddle-point density in the integrals that approximate the sums in (4.5), one obtains that the main behaviour of the emptiness formation probability at the free fermion point in a magnetic field h ($|h| < 4$) is given by (see [15] for details)[1]

$$\frac{1}{m^2} \log \tau(m) \underset{m \to \infty}{\sim} S^{(0)} = \frac{1}{2} \log \left(\frac{4-h}{8} \right). \tag{4.11}$$

The general case is slightly more complicated, but follows the same procedure. The main problem is that, a priori, we do not know any asymptotic equivalent of the quantity $G_m(\lambda)$ when $m \to \infty$. Nevertheless, in the case of zero magnetic field, it is still possible to compute the asymptotic behaviour of (4.4) in the leading order, provided we make the following hypothesis: we assume that the integrand of (4.4) admits a maximum for a certain value $\lambda'_1, \ldots, \lambda'_m$ of the integration variables $\lambda_1, \ldots, \lambda_m$, that, for large m, the distribution of these parameters $\lambda'_1, \ldots, \lambda'_m$ can be described by a density function $\rho_s(\lambda')$ of the form (4.8) on the symmetric interval $[-\Lambda, \Lambda]$ (see (2.29), (2.31)), and that, at the leading order in m, we can replace the sums over the set of parameters $\{\lambda'\}$ by integrals over this density $\rho_s(\lambda')$ as in (4.9)-(4.10).

First, like in the free fermion case, it is easy to determine the maximum of the function $S_m(\{\lambda\})$. Indeed, let $\{\tilde{\lambda}\}$ be solution of the system

$$\partial_{\lambda_j} S_m(\{\tilde{\lambda}\}) = 0, \qquad 1 \leq j \leq m. \tag{4.12}$$

In the limit $m \to \infty$, if we suppose again that the parameters $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m$ become distributed according to a certain density $\tilde{\rho}_s(\lambda)$ and that sums over the $\tilde{\lambda}_j$ become integrals over this density,

---

[1] For $|h| \geq 4$ the ground state becomes ferromagnetic and the emptiness formation probability is equal to 0 (for $h \geq 4$) or to 1 (for $h \leq -4$).

the system (4.12) turns again into a single integral equation for $\tilde{\rho}_s$, that can be solved explicitly in the case of zero magnetic field:

$$\tilde{\rho}_s(\lambda) = \frac{i}{\pi} \sum_{n \in \mathbb{Z}} \frac{\cosh(n\zeta)}{\cosh(2n\zeta)} e^{-2n\lambda}, \qquad \text{(massive case } \Delta > 1, \ \zeta = -\eta > 0\text{)}, \qquad (4.13)$$

$$= \frac{\cosh \frac{\pi\lambda}{2\zeta}}{\zeta\sqrt{2}\cosh \frac{\pi\lambda}{\zeta}}, \qquad \text{(massless case } |\Delta| < 1, \ \zeta = i\eta > 0\text{)}. \qquad (4.14)$$

This gives for the maximum of $S_m(\{\lambda\})$ when $m \to \infty$[2]:

$$\lim_{m \to \infty} S_m(\{\tilde{\lambda}\}) = -\frac{\zeta}{2} - \sum_{n=1}^{\infty} \frac{e^{-n\zeta}}{n} \frac{\sinh(n\zeta)}{\cosh(2n\zeta)}, \qquad (\Delta = \cosh\zeta > 1), \qquad (4.15)$$

$$= \log \frac{\pi}{\zeta} + \frac{1}{2} \int_{\mathbb{R}-i0} \frac{d\omega}{\omega} \frac{\sinh \frac{\omega}{2}(\pi - \zeta) \cosh^2 \frac{\omega\zeta}{2}}{\sinh \frac{\pi\omega}{2} \sinh \frac{\omega\zeta}{2} \cosh \omega\zeta}, \qquad (|\Delta = \cos\zeta| < 1). \qquad (4.16)$$

The second step is to show that the factor $G_m(\{\lambda\})$ gives always a negligible contribution compared to $S_m(\{\lambda\})$ at this order in m, at least for any distribution of the variables λ_j satisfying the previous hypothesis of regularity. Indeed, we can use the integral equation (2.28) satisfied by the inhomogeneous spectral density for the ground state to express, for any set of variables $\{\lambda\}$, $G_m(\{\lambda\})$ in the form:

$$G_m(\{\lambda\}) = \lim_{\xi_1,\ldots\xi_m \to -\frac{i\zeta}{2}} \frac{\det_m \left[\rho(\lambda_j, \xi_k) + \int_C K(\lambda_j - \mu)\rho(\mu, \xi_k)d\mu \right]}{\det_m[\rho(\lambda_j, \xi_k)]}, \qquad (4.17)$$

where the kernel K is given by (2.27). If the distribution of $\{\lambda\}$ is regular enough in the interval $[-\Lambda, \Lambda]$, we can replace, in the limit $m \to \infty$, the integral

$$\int_C K(\lambda_j - \mu)\rho(\mu, \xi_k)d\mu \qquad (4.18)$$

in the determinant by the sum

$$\frac{1}{m} \sum_{l=1}^{m} K(\lambda_j - \lambda_l) \frac{\rho(\lambda_l, \xi_k)}{\hat{\rho}_s(\lambda_l)} \qquad (4.19)$$

where the density function $\hat{\rho}_s(\lambda)$ describes the distribution of the λ_j, $j = 1, \ldots, m$ in the limit $m \to \infty$. Therefore,

$$G_m(\lambda) \underset{m \to \infty}{\sim} \det_m \left(\delta_{jk} + \frac{K(\lambda_j - \lambda_k)}{m \, \hat{\rho}_s(\lambda_k)} \right). \qquad (4.20)$$

In the massive regime, this is merely the Fredholm determinant of the integral operator $\hat{I} + \hat{K}$, where $\hat{I}$ denotes the identity operator, and $\hat{K}$ the integral operator of kernel K (2.27). This determinant is given by the infinite product of its eigenvalues:

$$\lim_{m \to \infty} G_m(\{\lambda\}) = \det(\hat{I} + \hat{K}) = 2 \prod_{n=1}^{\infty} (1 + q^{2n})^2, \qquad q = e^{\eta} \quad \text{(massive regime)}. \qquad (4.21)$$

---

[2]At this main order in m, there exists a unique solution of the integral equation for $\tilde{\rho}_s$, and we know it corresponds to a maximum because $S_m(\{\lambda\}) \to -\infty$ on the boundary of $\mathcal{D}$.

In the massless regime, the determinant (4.20) and its inverse can be bounded via Hadamard inequality. Thus, in both regime, we can show that

$$\lim_{m \to \infty} \frac{1}{m^2} \log G_m(\{\lambda\}) = 0 \tag{4.22}$$

for any distribution of $\{\lambda\}$ with good properties of regularity, in particular for the saddle point. This means that, at the main order in m, the factor $G_m(\{\lambda\})$ does not contribute to the value of the maximum of the integrand, and that the latter is indeed given by the maximum (4.15)-(4.16) of $S_m(\{\lambda\})$, $\tilde{\rho}_s$ being identified with the saddle-point density ρ_s.

Finally we obtain the following result concerning the asymptotic behaviour of $\tau(m)$ for $m \to \infty$ (see [17] for the massless case):

$$S^{(0)}(\Delta) = \lim_{m \to \infty} \frac{\log \tau(m)}{m^2}, \tag{4.23}$$

$$= -\frac{\zeta}{2} - \sum_{n=1}^{\infty} \frac{e^{-n\zeta}}{n} \frac{\sinh(n\zeta)}{\cosh(2n\zeta)}, \qquad (\Delta = \cosh\zeta > 1), \tag{4.24}$$

$$= \log \frac{\pi}{\zeta} + \frac{1}{2} \int_{\mathbb{R}-i0} \frac{d\omega}{\omega} \frac{\sinh \frac{\omega}{2}(\pi - \zeta) \cosh^2 \frac{\omega\zeta}{2}}{\sinh \frac{\pi\omega}{2} \sinh \frac{\omega\zeta}{2} \cosh \omega\zeta}, \quad (-1 < \Delta = \cos\zeta < 1). \tag{4.25}$$

Note that this coincides with the exact known results obtained in [15, 47, 50] at the free fermion point and in [16, 46] at $\Delta = 1/2$, and is in agreement with the expected value in the Ising limit:

$$S^{(0)}(\Delta = 0) = -\frac{1}{2} \log 2 \qquad \text{(Free fermion case)}, \tag{4.26}$$

$$S^{(0)}(\Delta = \frac{1}{2}) = \frac{3}{2} \log 3 - 3 \log 2, \tag{4.27}$$

$$S^{(0)}(\Delta) \xrightarrow[\Delta \to \infty]{} -\infty \qquad \text{(Ising case)}. \tag{4.28}$$

Moreover, we can apply the same saddle-point procedure directly at the XXX point $\Delta = 1$ and check that

$$S^{(0)}(\Delta = 1) = S^{(0)}(\Delta \to 1^+) = S^{(0)}(\Delta \to 1^-)$$

$$= \log\left(\frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}\right) \approx \log(0.5991), \tag{4.29}$$

which is in good agreement with the numerical result $\log(0.598)$, obtained in [48].

In the massless regime, the leading asymptotic behaviour (4.25), was conjectured independently in [51]. In that article was also conjectured the first (power-law) sub-leading correction in the form:

$$\tau(m) \underset{m \to \infty}{\sim} A\, m^{-\gamma} e^{-m^2 S_0}, \qquad (-1 < \Delta = \cos\zeta < 1), \tag{4.30}$$

with

$$\gamma = \frac{1}{12} + \left(\frac{\zeta}{\pi}\right)^2 \frac{1}{3(1 - \zeta/\pi)}, \tag{4.31}$$

which is in agreement with the exact results at $\Delta = 0$ and $\Delta = 1/2$ (see [47], [16]). It would be interesting to check this latter conjecture by analysing corrections to the saddle-point method we have presented here.

4.2 The two-point functions: attempts and problems

The long-distance asymptotics of physical correlation functions, such as the two-point functions, have attracted long-standing interest. In the case of the XXZ model, some predictions were made already a long time ago.

In the massive regime ($\Delta > 1$), spin-spin correlation functions are expected to decay exponentially with the distance and the exact value of the correlation length was proposed in [52]. For the XXZ chain in the massless regime ($-1 < \Delta \leq 1$), zero temperature is a critical point and the correlation length becomes infinite in units of the lattice spacing. The leading long-distance effects can be predicted by conformal field theory and the correlation functions are expected to decay as a power of the distance. In particular, one expects that, at the leading order,

$$\langle \sigma_j^x \sigma_{j+n}^x \rangle = (-1)^n \frac{A}{n^{\pi-\zeta}} + \cdots, \tag{4.32}$$

$$\langle \sigma_j^z \sigma_{j+n}^z \rangle = -\frac{1}{\pi(\pi-\zeta)} \frac{1}{n^2} + (-1)^n \frac{A_z}{n^{\frac{\pi}{\pi-\zeta}}} + \cdots. \tag{4.33}$$

A conjecture for the non-universal correlation amplitudes A and A_z can be found in [53–55]. The exact value of the critical exponents in (4.32)-(4.33) was proposed for the first time in [56].

However, there does not exist at the moment any direct derivation of these predictions from the exact expressions of the correlation functions on the lattice. In the last subsection we have shown how to determine, at least in the main order, the asymptotic behaviour of the emptiness formation probability using the saddle-point method. We could expect to be able to apply the same technique to the new multiple integral representation of the two-point function obtained in Section 3.

In particular, one can notice immediately that each term of the representation (3.28) of the generating functional $\langle Q_{1,m}^\kappa \rangle$ has a structure very similar to (4.3). Indeed, it is possible to apply to the whole sum a slight modification of the saddle-point technique presented here. It shows that, as it should be, there is no contribution of order $\exp(\alpha m^2)$ when $m \to \infty$. However, to obtain the precise asymptotic behaviour of the two-point function, one should be able to analyse sub-leading corrections to this saddle-point method, which is technically quite difficult. It is not obvious in particular from these expressions that, in the massless regime, the leading asymptotic behaviour of the two-point function is only of power-law order.

If one considers instead (3.11), one can try to make a similar analysis for each term of the sum. It can be easily proved that, in the massless regime, the first terms of (3.11) decrease as powers of the distance. However, it is neither difficult to see that the next terms of the series are not negligible with respect to the first ones, which means that one should analyse the whole sum to obtain the correct power law asymptotic behaviour.

5 Complete re-summation for the finite chain

We have seen in the last section that, although the partial re-summations presented in Section 3 contain integrals that can be analysed in the main order via the saddle-point method, the asymptotic analysis of the sum itself is much more tricky. It is due to the fact that we have to take into account sub-leading corrections to the saddle-point to be able to obtain merely the main order asymptotic of either representation (3.11) or (3.28). From this point of view, it could be more convenient to deal only with one single (multiple) integral instead of a sum, as in the case of the emptiness formation probability.

It is actually possible to re-sum completely representation (3.9) to obtain, at the finite chain level, what we call the *master equation* for the two-point function. We will see in particular that

this master equation sheds a new light on the previous algebraic re-summation of Section 3.1 by connecting it to the expansion of the two-point function in terms of the form factors of the local spin operators.

5.1 Multiple action of transfer matrices: a complete re-summed formula

To perform the re-summation of the two-point function, we have used in Section 3.1 the explicit formula (3.3) for the multiple action of $\prod_{a=1}^{m} \mathcal{T}_\kappa(x_a)$ for an arbitrary set of complex numbers $\{x\}$ on an arbitrary state $\langle 0 | \prod_{j=1}^{N} C(\mu_j)$ [14]. Introducing auxiliary integrals as in (3.10), one can actually re-sum completely the sum over the partitions of the sets $\{\mu\}$ and $\{x\}$ in (3.3) and express this action in the form of a single multiple integral:

PROPOSITION 5.1. *Let κ, $x_1, \ldots, x_m$ and $\mu_1, \ldots, \mu_N$ be generic parameters. Then the action of $\prod_{a=1}^{m} \mathcal{T}_\kappa(x_a)$ on a state of the form $\langle 0 | \prod_{j=1}^{N} C(\mu_j)$ can be formally written as*

$$\langle 0 | \prod_{j=1}^{N} C(\mu_j) \prod_{a=1}^{m} \mathcal{T}_\kappa(x_a) = \frac{1}{N!} \oint_{\Gamma\{x\}\cup\Gamma\{\mu\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{a=1}^{m} \tau_\kappa(x_a|\{z\}) \cdot \prod_{a=1}^{N} \frac{1}{\mathcal{Y}_\kappa(z_a|\{z\})}$$

$$\times \prod_{\substack{j,k=1 \\ j<k}}^{N} \frac{\sinh(z_j - z_k)}{\sinh(\mu_j - \mu_k)} \cdot \det_{N} \Omega_\kappa(\{z\}, \{\mu\}|\{z\}) \cdot \langle 0 | \prod_{j=1}^{N} C(z_j), \quad (5.1)$$

where the integration contour $\Gamma\{x\} \cup \Gamma\{\mu\}$ surrounds the points[3] $x_1, \ldots, x_m$ and $\mu_1, \ldots, \mu_N$ and does not contain any other pole of the integrand.

Proof. The right-hand side in (5.1) can be computed by the sum over the residues at the points $x_1, \ldots, x_m$, $\mu_1, \ldots, \mu_N$. More precisely, it is given by

$$(2i\pi)^N \sum_{n=0}^{\min(m,N)} \sum_{\substack{\{\mu\}=\{\mu_{\alpha_+}\}\cup\{\mu_{\alpha_-}\} \\ \{x\}=\{x_{\gamma_+}\}\cup\{x_{\gamma_-}\} \\ |\alpha_+|=|\gamma_+|=n}} \text{Res}_{\{z\}=\{x_{\gamma_+}\}\cup\{\mu_{\alpha_-}\}}(\text{Integrand}), \quad (5.2)$$

and one can easily check that the residue of the integrand for $\{z\} = \{x_{\gamma_+}\} \cup \{\mu_{\alpha_-}\}$ is equal to $R_n^\kappa(\{x_{\gamma_+}\}, \{x_{\gamma_-}\}, \{\mu_{\alpha_+}\}, \{\mu_{\alpha_-}\})$. $\square$

To apply this formula to the effective computation of the two-point functions, one should give specific values to the parameters $x_1, \ldots, x_m$, $\mu_1, \ldots, \mu_N$: they will correspond in this context either to admissible solutions of the Bethe equation at $\kappa = 1$ (parametrising the eigenstate in which we compute expectation values) or to inhomogeneity parameters (set to $\eta/2$ in the homogeneous case) arising from the reconstruction of local spin operators. To ensure the existence of the corresponding contour $\Gamma\{x\} \cup \Gamma\{\mu\}$ for these particular values, one has to prove that there exists a surrounding of $[\{x_1, \ldots, x_m\} \cup \{\mu_1, \ldots, \mu_N\}]^N$ which does not contain any new singularity of the integrand. Since $C(z)$ is a polynomial in z in the normalisation (2.4), such singularities could occur only due to the product of the twisted Bethe equations $\mathcal{Y}_\kappa(z_a|\{z\})$ in the denominator. We will see in the following that these poles play a central role in the context of master equation.

---

[3]More precisely, for a set of complex variables $\{\nu_1, \ldots, \nu_l\}$, the notation $\Gamma\{\nu\}$ should be understood in the following way: $\Gamma\{\nu\}$ is the boundary of a set of poly-disks $\mathcal{D}_a(r)$ in $\mathbb{C}^N$, i.e. $\Gamma\{\nu\} = \cup_{a=1}^{l} \bar{\mathcal{D}}_a(r)$ with $\mathcal{D}_a(r) = \{z \in \mathbb{C}^N : |z_k - \nu_a| = r, \quad k = 1, \ldots, N\}$.

Solutions of the system of Bethe equations (2.10) were studied in [41] (see also Appendix A of [19] for the homogeneous case). Following the arguments of [18], one can easily prove that unadmissible and diagonal solutions do not correspond to singularities of the integrand, for either $\det \Omega_\kappa$ or $\langle 0 | \prod_{j=1}^N C(z_j)$ vanishes in this case. It was moreover proven in [18, 19] that, for $|\kappa|$ small enough, all admissible off-diagonal solutions are in the vicinities of the shifted inhomogeneity parameters $\xi_j - \eta$ (or of $-\eta/2$ in the homogeneous case) but are separated from these points as soon as $\kappa \neq 0$. For $|\kappa|$ small enough, they are also separated from any admissible off-diagonal solution $\lambda_1, \ldots, \lambda_N$ of the system of untwisted Bethe equations. Thus, just like in [18, 19], we can formulate the following lemma:

LEMMA 5.1. *Let* $\{\lambda\}$ *be an admissible off-diagonal solution of the system of untwisted Bethe equations, and* $\{\xi\}$ *be a set of inhomogeneity parameters. There exists* $\kappa_0 > 0$ *such that, for* $|\kappa| < \kappa_0$, *one can define a closed contour* $\Gamma\{\lambda\} \cup \Gamma\{\xi\}$ *which satisfies the following properties:*

1) it surrounds the points $\{\lambda\}$ *and* $\{\xi\}$, *while all admissible off-diagonal solutions of the system (2.10) are outside of this contour;*

2) the only poles which are inside and provide non-vanishing contributions to the integral (5.1) are $z_j = \lambda_k$ *and* $z_j = \xi_k$;

3) the only poles which are outside (within a set of periodic strips) and provide non-vanishing contributions to the integrand of (5.1) are the admissible off-diagonal solutions of the system of twisted Bethe equations (2.10).

Remark 5.1. Lemma 5.1 holds also in the homogeneous limit $\xi_j = \eta/2$.

5.2 Master equation for the two-point function

Proposition 5.1 associated to Lemma 5.1 enables us to obtain representations for the two-point functions of the finite chain as a single-multiple integral [18]. Let us consider, for a given eigenstate $| \psi(\{\lambda\}) \rangle$ of the untwisted transfer matrix, the expectation values

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \langle \sigma_1^z \rangle + \langle \sigma_{m+1}^z \rangle - 1 + 2\frac{\partial^2}{\partial \kappa^2} \langle Q_{1,m+1}^\kappa - Q_{1,m}^\kappa - Q_{2,m+1}^\kappa + Q_{2,m}^\kappa \rangle \Big|_{\kappa=1}, \quad (5.3)$$

$$\langle \sigma_1^\alpha \sigma_{m+1}^\beta \rangle = \lim_{\kappa \to 1} \langle \sigma_{1,\kappa}^\alpha \sigma_{\kappa,m+1}^\beta \rangle, \qquad (\alpha, \beta) = (-, +), (+, -). \quad (5.4)$$

For convenience, we have defined here the following κ-deformed spin operators:

$$\sigma_{j,\kappa}^\alpha = \sigma_j^\alpha \cdot \prod_{a=1}^{j} T(\xi_a) \cdot \prod_{a=1}^{j} T_\kappa^{-1}(\xi_a), \qquad \sigma_{\kappa,j}^\alpha = \prod_{a=1}^{j-1} T_\kappa(\xi_a) \cdot \prod_{a=1}^{j-1} T^{-1}(\xi_a) \cdot \sigma_j^\alpha, \quad (5.5)$$

and the operator $Q_{j,k}^\kappa$ is given by (3.6). Using the solution of the quantum inverse scattering problem (2.18) and acting with the resulting operators on $\langle \psi(\lambda) |$, notably by means of (5.1), we obtain the following result:

THEOREM 5.1. *Let* $\{\lambda\}$ *be an admissible off-diagonal solution of the system of untwisted Bethe equations, and let us consider the corresponding expectations values (5.3)-(5.4) in the inhomogeneous finite XXZ chain. Then there exists* $\kappa_0 > 0$ *such that, for* $|\kappa| < \kappa_0$, *the following representations hold:*

$$\langle Q_{1,m}^\kappa \rangle = \frac{1}{N!} \oint_{\Gamma\{\xi\} \cup \Gamma\{\lambda\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{a=1}^{m} \frac{\mathcal{T}_\kappa(\xi_a | \{z\})}{\mathcal{T}(\xi_a | \{\lambda\})} \cdot \prod_{a=1}^{N} \frac{1}{\mathcal{Y}_\kappa(z_a | \{z\})}$$

$$\times \det_N \Omega_\kappa(\{z\}, \{\lambda\} | \{z\}) \cdot \frac{\det_N \Omega(\{\lambda\}, \{z\} | \{\lambda\})}{\det_N \Omega(\{\lambda\}, \{\lambda\} | \{\lambda\})}, \quad (5.6)$$

$$\langle \sigma_{1,\kappa}^- \sigma_{\kappa,m+1}^+ \rangle = \frac{1}{(N-1)!} \oint_{\Gamma\{\xi\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i} \cdot \frac{\prod_{a=2}^{m} \tau_\kappa(\xi_a|\{z\})}{\prod_{a=1}^{m+1} \tau(\xi_a|\{\lambda\})} \cdot \prod_{a=1}^{N-1} \frac{1}{\mathcal{Y}_\kappa(z_a|\{z\})}$$

$$\times \prod_{j=1}^{N-1} \frac{1}{\sinh(z_j - \xi_{m+1})} \cdot \prod_{j=1}^{N} \frac{1}{\sinh(\lambda_j - \xi_1)} \cdot \frac{\det_N \Omega(\{\lambda\}, \{z\} \cup \{\xi_{m+1}\}|\{\lambda\})}{\det_N \Omega(\{\lambda\}, \{\lambda\}|\{\lambda\})}$$

$$\times \left\{ \sum_{\substack{\alpha,\beta=1 \\ \alpha<\beta}}^{N+1} (-1)^{\alpha+\beta} \left\{ \frac{a(\tilde{\lambda}_\alpha)\, d(\tilde{\lambda}_\beta)}{\sinh(\tilde{\lambda}_\alpha - \tilde{\lambda}_\beta - \eta)} \cdot \prod_{k=1}^{N} \left[\sinh(\lambda_k - \tilde{\lambda}_\alpha + \eta)\sinh(\lambda_k - \tilde{\lambda}_\beta - \eta) \right] \right. \right.$$

$$\left. - \frac{a(\tilde{\lambda}_\beta)\, d(\tilde{\lambda}_\alpha)}{\sinh(\tilde{\lambda}_\beta - \tilde{\lambda}_\alpha - \eta)} \cdot \prod_{k=1}^{N} \left[\sinh(\lambda_k - \tilde{\lambda}_\beta + \eta)\sinh(\lambda_k - \tilde{\lambda}_\alpha - \eta) \right] \right\}$$

$$\times \det_{N-1} \Omega_\kappa(\{z\}, \{\tilde{\lambda}\}\backslash\{\tilde{\lambda}_\alpha, \tilde{\lambda}_\beta\}|\{z\}) \Bigg\}. \quad (5.7)$$

$$\langle \sigma_{1,\kappa}^+ \sigma_{\kappa,m+1}^- \rangle = \frac{1}{(N+1)!} \oint_{\Gamma\{\xi\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N+1} \frac{dz_j}{2\pi i} \cdot \frac{\prod_{a=2}^{m} \tau_\kappa(\xi_a|\{z\})}{\prod_{a=1}^{m+1} \tau(\xi_a|\{\lambda\})} \cdot \prod_{a=1}^{N+1} \frac{1}{\mathcal{Y}_\kappa(z_a|\{z\})}$$

$$\times \prod_{j=1}^{N+1} \frac{1}{\sinh(z_j - \xi_{m+1})} \cdot \prod_{j=1}^{N} \frac{1}{\sinh(\lambda_j - \xi_1)} \cdot \frac{\det_{N+1} \Omega_\kappa(\{z\}, \{\lambda\} \cup \{\xi_1\}|\{z\})}{\det_N \Omega(\{\lambda\}, \{\lambda\}|\{\lambda\})}$$

$$\times \left\{ \sum_{\substack{\alpha,\beta=1 \\ \alpha<\beta}}^{N+2} (-1)^{\alpha+\beta} \left\{ \frac{a(\tilde{z}_\alpha)\, d(\tilde{z}_\beta)}{\sinh(\tilde{z}_\alpha - \tilde{z}_\beta - \eta)} \cdot \prod_{k=1}^{N+1} \left[\sinh(z_k - \tilde{z}_\alpha + \eta)\sinh(z_k - \tilde{z}_\beta - \eta) \right] \right. \right.$$

$$\left. - \frac{a(\tilde{z}_\beta)\, d(\tilde{z}_\alpha)}{\sinh(\tilde{z}_\beta - \tilde{z}_\alpha - \eta)} \cdot \prod_{k=1}^{N+1} \left[\sinh(z_k - \tilde{z}_\beta + \eta)\sinh(z_k - \tilde{z}_\alpha - \eta) \right] \right\}$$

$$\times \det_{N} \Omega(\{\lambda\}, \{\tilde{z}\}\backslash\{\tilde{z}_\alpha, \tilde{z}_\beta\}|\{\lambda\}) \Bigg\}, \quad (5.8)$$

where we have set $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N+1}) = (\lambda_1, \ldots, \lambda_N, \xi_1)$ *and* $(\tilde{z}_1, \ldots, \tilde{z}_{N+2}) = (z_1, \ldots, z_{N+1}, \xi_{m+1})$ *in (5.7) and in (5.8) respectively. The integration contours in (5.6)-(5.8) are such that the only singularities of the integrand which contribute to the integral are the points* $\xi_1, \ldots, \xi_m$ *and* $\lambda_1 \ldots, \lambda_N$.

In the following, this kind of representations will be called *master equation*.

Remark 5.2. Expressions (5.7) and (5.8) are obtained by acting with all the operators on the left, i.e. on the dual Bethe state $\langle \psi(\{\lambda\})|$. It is also possible to act on the right. In that case, we obtain for $\langle \sigma_{1,\kappa}^+ \sigma_{\kappa,m+1}^- \rangle$ a representation that is similar to (5.7), and for $\langle \sigma_{1,\kappa}^- \sigma_{\kappa,m+1}^+ \rangle$ a representation similar to (5.8).

Remark 5.3. Note that the κ-deformed two-point correlation functions $\langle \sigma_{1,\kappa}^\alpha \sigma_{\kappa,m+1}^\beta \rangle$, as well as the generating functional $\langle Q_{1,m}^\kappa \rangle$, are polynomials in κ. They are therefore completely

determined by the multiple integral expressions (5.6)-(5.8) which are valid at least in a vicinity of $\kappa = 0$. The limit $\kappa \to 1$ can be reached through analytic continuation.

Remark 5.4. All these results were formulated in the inhomogeneous case, but hold also in the homogeneous limit. We can of course choose $|\psi(\{\lambda\})\rangle$ to be the ground state of the Hamiltonian of the XXZ chain.

Remark 5.5. ¿From the master equation (5.6), it is easy to come back to the original series obtained in Section 3.1, by decomposing the multiple integral as

$$
\oint_{\Gamma\{\xi\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N} dz_j = \sum_{n=0}^{N} C_N^n \oint_{\Gamma\{\xi\}} \prod_{j=1}^{n} dz_j \oint_{\Gamma\{\lambda\}} \prod_{j=1}^{N-n} dz_j.
\tag{5.9}
$$

Note at this stage that, since the number of poles surrounded by $\Gamma\{\xi\}$ is m and since the integrand vanishes as soon as $z_j = z_k$, the sum in (5.9) is actually restricted to $n \le m$. The evaluation of the $N - n$ integrals over the contour $\Gamma\{\lambda\}$ as a sum over the residues leads to a sum over the partitions of the set $\{\lambda\}$ into two subset $\{\lambda_{\alpha_-}\}$ and $\{\lambda_{\alpha_+}\}$ of cardinal $N - n$ and n respectively. Since the remaining integrals are taken over the contour $\Gamma\{\xi\}$ that surrounds only the poles at $z_j = \xi_k$, one can set $d(z_j) = 0$ directly in the integrand. This gives a partial resummation, in which we can use the Bethe equation for $\{\lambda\}$ and take the thermodynamic limit as in Section 3.1.

A similar technique can be applied to (5.7) or (5.8).

5.3 Form factor expansion

It is also possible, instead of computing the integrals (5.6)-(5.8) by the sum over the residues at the poles inside the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$, to evaluate them using the poles *outside* this contour. Due to Lemma 5.1, we know that the only poles that can contribute are the admissible off-diagonal solutions of the system of twisted Bethe equations (2.10). This leads directly to the expansion of the correlation functions (5.3)-(5.4) over form factors.

Let us consider for example the two-point correlation function $\langle \sigma_1^- \sigma_{m+1}^+ \rangle$. Evaluating (5.7) by the residues at the poles given by the admissible off-diagonal solutions $\{\mu_1, \ldots, \mu_{N-1}\}$ of the κ-twisted Bethe equations outside the integration contour, one obtains

$$
\langle \sigma_{1,\kappa}^- \sigma_{\kappa,m+1}^+ \rangle = \sum_{\{\mu\}} \frac{\prod\limits_{a=2}^{m} \tau_\kappa(\xi_a|\{\mu\})}{\prod\limits_{a=1}^{m+1} \tau(\xi_a|\{\lambda\})}
$$

$$
\times \prod_{j=1}^{N-1} \frac{1}{\sinh(\mu_j - \xi_{m+1})} \cdot \prod_{j=1}^{N} \frac{1}{\sinh(\lambda_j - \xi_1)} \cdot \frac{\det_N \Omega(\{\lambda\}, \{\mu\} \cup \{\xi_{m+1}\}|\{\lambda\})}{\det_N \Omega(\{\lambda\}, \{\lambda\}|\{\lambda\})}
$$

$$
\times \left\{ \sum_{\substack{\alpha,\beta=1 \\ \alpha<\beta}}^{N+1} (-1)^{\alpha+\beta} \left\{ \frac{a(\tilde{\lambda}_\alpha)\, d(\tilde{\lambda}_\beta)}{\sinh(\tilde{\lambda}_\alpha - \tilde{\lambda}_\beta - \eta)} \cdot \prod_{k=1}^{N} \left[\sinh(\lambda_k - \tilde{\lambda}_\alpha + \eta)\sinh(\lambda_k - \tilde{\lambda}_\beta - \eta) \right] \right. \right.
$$

$$
\left. - \frac{a(\tilde{\lambda}_\beta)\, d(\tilde{\lambda}_\alpha)}{\sinh(\tilde{\lambda}_\beta - \tilde{\lambda}_\alpha - \eta)} \cdot \prod_{k=1}^{N} \left[\sinh(\lambda_k - \tilde{\lambda}_\beta + \eta)\sinh(\lambda_k - \tilde{\lambda}_\alpha - \eta) \right] \right\}
$$

$$
\times \det_{N-1} \Omega_\kappa(\{\mu\}, \{\bar{\lambda}\}\backslash\{\tilde{\lambda}_\alpha, \tilde{\lambda}_\beta\}|\{\mu\}) \right\}.
\tag{5.10}
$$

with $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N+1}) = (\lambda_1, \ldots, \lambda_N, \xi_1)$. In this expression, we can identify the following matrix elements of $\sigma_{1,\kappa}^-$ and $\sigma_{\kappa,m+1}^+$ between the Bethe state $|\psi(\{\lambda_j\}_{1\leq j\leq N})\rangle$ and the κ-twisted Bethe state $|\psi_\kappa(\{\mu_j\}_{1\leq j\leq N-1})\rangle$:

$$\langle\psi_\kappa(\{\mu\})|\sigma_{\kappa,m+1}^+|\psi(\{\lambda\})\rangle = \frac{\prod\limits_{a=1}^{m}\tau(\xi_a|\{\mu\})\quad\prod\limits_{j=1}^{N}d(\lambda_j)}{\prod\limits_{a=1}^{m+1}\tau(\xi_a|\{\lambda\})\prod\limits_{\substack{j,k=1\\j<k}}^{N}\sinh(\lambda_j-\lambda_k)\prod\limits_{\substack{j,k=1\\j<k}}^{N-1}\sinh(\mu_k-\mu_j)}$$

$$\times\prod_{j=1}^{N-1}\frac{1}{\sinh(\xi_{m+1}-\mu_j)}\det_N\Omega(\{\lambda\},\{\mu\}\cup\{\xi_{m+1}\}|\{\lambda\}),\quad(5.11)$$

$$\langle\psi(\{\lambda\})|\sigma_{1,\kappa}^-|\psi_\kappa(\{\mu\})\rangle = \frac{1}{\tau_\kappa(\xi_1|\{\mu\})}\frac{\prod\limits_{j=1}^{N-1}d(\mu_j)}{\prod\limits_{\substack{j,k=1\\j<k}}^{N-1}\sinh(\mu_k-\mu_j)\prod\limits_{\substack{j,k=1\\j<k}}^{N}\sinh(\lambda_j-\lambda_k)}$$

$$\times\prod_{j=1}^{N}\frac{1}{\sinh(\lambda_j-\xi_1)}\left\{\sum_{\substack{\alpha,\beta=1\\\alpha<\beta}}^{N+1}(-1)^{\alpha+\beta}\right.$$

$$\times\left\{\frac{a(\tilde{\lambda}_\alpha)\,d(\tilde{\lambda}_\beta)}{\sinh(\tilde{\lambda}_\alpha-\tilde{\lambda}_\beta-\eta)}\prod_{k=1}^{N}\left[\sinh(\lambda_k-\tilde{\lambda}_\alpha+\eta)\sinh(\lambda_k-\tilde{\lambda}_\beta-\eta)\right]\right.$$

$$\left.-\frac{a(\tilde{\lambda}_\beta)\,d(\tilde{\lambda}_\alpha)}{\sinh(\tilde{\lambda}_\beta-\tilde{\lambda}_\alpha-\eta)}\prod_{k=1}^{N}\left[\sinh(\lambda_k-\tilde{\lambda}_\beta+\eta)\sinh(\lambda_k-\tilde{\lambda}_\alpha-\eta)\right]\right\}$$

$$\left.\times\det_{N-1}\Omega(\{\mu\},\{\tilde{\lambda}\}\backslash\{\tilde{\lambda}_\alpha,\tilde{\lambda}_\beta\}|\{\mu\})\right\}.\quad(5.12)$$

In (5.12), we have defined again $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N+1})$ to be equal to $(\lambda_1, \ldots, \lambda_N, \xi_{m+1})$. Note that the expressions (5.11)-(5.12) of the matrix elements which appear in the summation (5.10) are the ones that are obtained when acting on the left with the corresponding local spin operators. This is due to the fact that the whole algebraic procedure we have used to derive the master equation is based from the very beginning on the principle of action on the left.

Hence, we obtain

$$\langle\sigma_{1,\kappa}^-\sigma_{\kappa,m+1}^+\rangle = \sum_{\{\mu\}}\frac{\langle\psi(\{\lambda\})|\sigma_{1,\kappa}^-|\psi_\kappa(\{\mu\})\rangle\cdot\langle\psi_\kappa(\{\mu\})|\sigma_{\kappa,m+1}^+|\psi(\{\lambda\})\rangle}{\langle\psi(\{\lambda\})|\psi(\{\lambda\})\rangle\cdot\langle\psi_\kappa(\{\mu\})|\psi_\kappa(\{\mu\})\rangle},\quad(5.13)$$

where the sum is taken over all the admissible solutions $\{\mu_1, \ldots, \mu_{N-1}\}$ of the system of $N-1$ κ-twisted Bethe equations. Observe that we did not need to use here the completeness of the corresponding κ-twisted Bethe states $|\psi_\kappa(\{\mu\})\rangle$ (see [19,41]) in $\mathcal{H}^{(M/2-N+1)}$, as the sum over the eigenstates of $\mathcal{T}_\kappa$ appears automatically as the result of the evaluation of the multiple integral (5.7) by the residues outside the integration contour.

Since the resulting expression (5.10)-(5.13) is again a polynomial in κ, we can claim from Remark 5.3 that the equality (5.13) is valid for all values of κ, in particular at $\kappa = 1$. For this value of κ and in the homogeneous limit, (5.13) represents precisely the form factor type

expansion of the correlation function $\langle \sigma_1^- \sigma_{m+1}^+ \rangle$, with respect to all the excited states of the Hamiltonian.

Of course, it is now clear that we can proceed in the opposite way, starting from the form factor expansion to obtain the master equation (5.7) and the re-summation of Section 3.1. We will discuss this point in more details in the next section.

6 Dynamical master equation

The approach described in the previous section can be generalised to the case of time-dependent (dynamical) correlation functions. Namely, one can derive a time-dependent analogue of the master equations (5.6)-(5.8) for the two-point functions on the finite chain,

$$\langle \sigma_1^\alpha(0)\, \sigma_{m+1}^\beta(t) \rangle = \frac{\langle \psi(\{\lambda\}) |\, \sigma_1^\alpha\, e^{iHt}\, \sigma_{m+1}^\beta\, e^{-iHt}\, | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) |\, \psi(\{\lambda\}) \rangle}, \tag{6.1}$$

where $| \psi(\{\lambda\}) \rangle$ denotes an eigenstate of the Hamiltonian (1.1) corresponding to an admissible off-diagonal solution of the Bethe equations. It is possible to obtain these time-dependent master equations either from the formula (5.1) for the multiple action of the twisted transfer matrices or via the form factor approach. Both approaches are described below.

6.1 Time-dependent generating function

Let us first consider the dynamical two-point correlation function of the third component of the spin in the eigenstate $| \psi(\{\lambda\}) \rangle$. Due to the property $[\sigma_m^z, S_z] = 0$, we have

$$\langle \sigma_1^z(0)\, \sigma_{m+1}^z(t) \rangle = \frac{\langle \psi(\{\lambda\}) |\, \sigma_1^z\, e^{iH^{(0)}t}\, \sigma_{m+1}^z\, e^{-iH^{(0)}t}\, | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) |\, \psi(\{\lambda\}) \rangle}. \tag{6.2}$$

We will explain here how to derive a time-dependent generalisation of the master equation (5.6) for a generating function of (6.2) similar to (3.5). This dynamical generating function is the expectation value of the following natural time-dependent generalisation of the operator[4] (3.6):

$$Q_{l,m}^\kappa(t) = T^{l-1}\left(\frac{\eta}{2}\right) \cdot T_\kappa^{m-l+1}\left(\frac{\eta}{2}\right) \cdot e^{itH_\kappa^{(0)}} \cdot T^{-m}\left(\frac{\eta}{2}\right) \cdot e^{-itH^{(0)}}, \tag{6.3}$$

where the κ-twisted Hamiltonian $H_\kappa^{(0)}$ is defined similarly as in (2.6):

$$H_\kappa^{(0)} = 2\sinh\eta\, \frac{dT_\kappa(\lambda)}{d\lambda}\, T_\kappa^{-1}(\lambda)\Big|_{\lambda=\frac{\eta}{2}} - 2M\cosh\eta. \tag{6.4}$$

Using the solution (2.18) of the quantum inverse problem and the fact that the transfer matrix T_κ commutes with the twisted Hamiltonian $H_\kappa^{(0)}$, it is easy to see that, just like in the time-independent case, the two-point function (6.2) is given as

$$\langle \sigma_1^z(0)\, \sigma_{m+1}^z(t) \rangle = 2\langle \sigma_1^z(0) \rangle - 1 + 2D^2 \frac{\partial^2}{\partial\kappa^2}\langle Q_{1,m}^\kappa(t) \rangle\Big|_{\kappa=1}, \tag{6.5}$$

where $\langle Q_{1,m}^\kappa(t) \rangle$ is the expectation value of (6.3) in the eigenstate $| \psi(\{\lambda\}) \rangle$,

$$\langle Q_{1,m}^\kappa(t) \rangle = \frac{\langle \psi(\{\lambda\}) |\, Q_{1,m}^\kappa(t)\, | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) |\, \psi(\{\lambda\}) \rangle}, \tag{6.6}$$

---

[4]In this section we consider the homogeneous chain only, since the local Hamiltonian $H^{(0)}$ is well defined only in this case.

and D^2 denotes the second lattice derivative defined as in (3.7).

In order to be able to use Proposition 5.1 to determine the action of the operator (6.3) on the eigenstate $|\psi(\{\lambda\})\rangle$, one should express the twisted evolution operator $e^{iH_\kappa^{(0)}t}$ in terms of a product of twisted transfer matrices T_κ. This can be done via the Trotter type formula [57,58],

$$e^{\pm it(H_\kappa^{(0)}+2M\cosh\eta)} = \lim_{L\to\infty}\left(T_\kappa\left(\frac{\eta}{2}+\varepsilon\right)\cdot T_\kappa^{-1}\left(\frac{\eta}{2}\right)\right)^{\pm L}, \qquad \varepsilon = \frac{1}{L}2it\sinh\eta. \tag{6.7}$$

Using now the identity

$$T_\kappa^{-1}(\eta/2) = \Big(\kappa\, a(\eta/2)d(-\eta/2)\Big)^{-1}T_\kappa(-\eta/2), \tag{6.8}$$

we arrive at the following representation for the operator $Q_{1,m}^\kappa(t)$:

$$Q_{1,m}^\kappa(t) = \lim_{L\to\infty}T_\kappa^L\left(\frac{\eta}{2}+\varepsilon\right)\cdot T_\kappa^{m-L}\left(\frac{\eta}{2}\right)\cdot T^{L-m}\left(\frac{\eta}{2}\right)\cdot T^{-L}\left(\frac{\eta}{2}+\varepsilon\right), \tag{6.9}$$

$$= \lim_{L\to\infty}\kappa^{m-L}T_\kappa^L\left(\frac{\eta}{2}+\varepsilon\right)\cdot T_\kappa^{L-m}\left(-\frac{\eta}{2}\right)\cdot T^{m-L}\left(-\frac{\eta}{2}\right)\cdot T^{-L}\left(\frac{\eta}{2}+\varepsilon\right). \tag{6.10}$$

Thus, the problem of evaluating the dynamical correlation function of the third components of the spin is reduced to the calculation of the multiple action of twisted transfer matrices on the state $\langle\psi(\{\lambda\})|$. Therefore one can use directly the results of the previous section, with the following generalisation of Lemma 5.1:

LEMMA 6.1. *Let $\{\lambda\}$ be an admissible off-diagonal solution of the system of untwisted Bethe equations. There exists $\kappa_0 > 0$ such that, for $0 < |\kappa| < \kappa_0$, one can define a closed contour $\Gamma\{\lambda\}\cup\Gamma\{\eta/2\}\cup\Gamma\{-\eta/2\}$ which satisfies the following properties:*

1) it surrounds the points $\{\lambda\}$, $\eta/2$ and $-\eta/2$, while all admissible off-diagonal solutions of the system (2.10) are outside of this contour;

2) for L large enough, the only poles which are inside and provide non-vanishing contributions to the integral (5.1) are $z_j = \lambda_k$, $z_j = \eta/2+\varepsilon$ and $z_j = -\eta/2$;

3) for L large enough, the only poles which are outside (within a set of periodic strips) and provide non-vanishing contributions to the integrand of (5.1) are the admissible off-diagonal solutions of the system of twisted Bethe equations (2.10).

Proceeding then to the limit $L\to\infty$, one obtains a time-dependent master equation for the generating function (6.6).

THEOREM 6.1. *Let $\{\lambda_1,\ldots,\lambda_N\}$ be an admissible off-diagonal solution of the system (2.10) at $\kappa = 1$. Then there exists $\kappa_0 > 0$ such that, for $0 < |\kappa| < \kappa_0$, the generating function $\langle Q_{1,m}^\kappa(t)\rangle$ (6.6) in the finite XXZ chain (1.1) is given by the multiple contour integral*

$$\langle Q_{1,m}^\kappa(t)\rangle = \frac{1}{N!}\oint_{\Gamma\{\pm\frac{\eta}{2}\}\cup\Gamma\{\lambda\}}\prod_{j=1}^N\frac{dz_j}{2\pi i}\cdot\prod_{b=1}^N e^{it[E(z_b)-E(\lambda_b)]+im[p(z_b)-p(\lambda_b)]}$$

$$\times\prod_{a=1}^N\frac{1}{\mathcal{Y}_\kappa(z_a|\{z\})}\cdot\det_N\Omega_\kappa(\{z\},\{\lambda\}|\{z\})\cdot\frac{\det_N\Omega(\{\lambda\},\{z\}|\{\lambda\})}{\det_N\Omega(\{\lambda\},\{\lambda\}|\{\lambda\})}. \tag{6.11}$$

In this expression, $E(\lambda)$ and $p(\lambda)$ denote respectively the bare one-particle energy and momentum (2.16) and (2.17); the integration contour is such that the only singularities of the integrand (5.6) within $\Gamma\{\pm\frac{\eta}{2}\}\cup\Gamma\{\lambda\}$ which contribute to the integral are the points $\{\pm\frac{\eta}{2}\}$ and $\{\lambda\}$.

Comparing the dynamical master equation (6.11) with the corresponding result (5.6) in the previous section, we see that the dependency on time appears in the integrand together with the bare energy. In the framework of the approach described above, this function arises as a Trotter type limit of the eigenvalues of the operators $\mathcal{T}$ and $\mathcal{T}_\kappa$. The function $E(z)$ has poles in the points $\pm\frac{\eta}{2}$, which explains why the integration contour $\Gamma\{\pm\frac{\eta}{2}\} \cup \Gamma\{\lambda\}$ in (6.11) needs to surround not only the point $\frac{\eta}{2}$, but also the point $-\frac{\eta}{2}$.

Observe also that, unlike in the time-independent case, the equation (6.11) holds only in a *punctured* vicinity of $\kappa = 0$, i.e. for $|\kappa|$ small enough, but non-zero. This is due to two reasons. First, the admissible solutions of the twisted Bethe equations are separated from the point $-\eta/2$ only if $\kappa \neq 0$. Second, the expectation value $\langle Q_{1,m}^\kappa(t) \rangle$ is no longer a polynomial in κ: due to the presence of the twisted evolution operator $e^{itH_\kappa^{(0)}}$ in (6.3), it has essential singularities at $\kappa = 0, \infty$. However, it is easy to see that $\langle Q_{1,m}^\kappa(t) \rangle$ remains a holomorphic function of κ everywhere except at these points, which means that the result (6.11) can be analytically continued from a punctured vicinity of $\kappa = 0$ to the whole complex plane $\mathbb{C}^*$, and in particular to the point $\kappa = 1$.

6.2 Correlation function $\langle \sigma^-(0)\,\sigma^+(t) \rangle$

We have seen in Section 5.3 that the explicit expansion of the correlation functions over the form factors can be obtained from the master equation. It was also mentioned there that one can follow an opposite strategy, i.e. sum up the form factors of the local spin operators to derive the contour integrals (5.6)-(5.8). This way to obtain the master equation is more direct and admits a very simple generalisation for the time-dependent case.

Consider, for example, the dynamical correlation function $\langle \sigma_1^-(0)\sigma_{m+1}^+(t) \rangle$. It would seem very natural to compute this correlation function by inserting between the operators $\sigma_1^-(0)$ and $\sigma_{m+1}^+(t)$ the complete set of the eigenstates of the Hamiltonian (1.1). The point is, however, that we have no convenient parametrisation for this complete set. Indeed, it is known that the set of the eigenstates corresponding to the admissible off-diagonal solutions of untwisted Bethe equations in the homogeneous case is not complete, and that one should take into account some unadmissible solutions as well. But, on the other hand, the vectors corresponding to unadmissible solutions are ill-defined.

Nevertheless, we know that the the eigenstates of the twisted transfer matrix $\mathcal{T}_\kappa$ corresponding to the admissible off-diagonal solutions of the twisted Bethe equations form a basis of the space of states, at least for $|\kappa|$ small enough, but non-zero. We can therefore use these states in order to sum up the form factor series. Indeed, let us consider, similarly as in (5.4), the κ-deformed time-dependent correlation function

$$\langle \sigma_{1,\kappa}^-(0)\sigma_{\kappa,m+1}^+(t) \rangle = \langle \sigma_{1,\kappa}^- e^{itH_\kappa^{(0)}-ithS_z} \sigma_{\kappa,m+1}^+ e^{-itH^{(0)}+ithS_z} \rangle, \tag{6.12}$$

with

$$\langle \sigma_1^-(0)\,\sigma_{m+1}^+(t) \rangle = \lim_{\kappa \to 1}\langle \sigma_{1,\kappa}^-(0)\sigma_{\kappa,m+1}^+(t) \rangle, \tag{6.13}$$

where $\sigma_{1,\kappa}^-$, $\sigma_{\kappa,m+1}^+$ are defined in (5.5) and $H_\kappa^{(0)}$ is given by (6.4). Inserting now the complete set of the twisted eigenstates between $e^{itH_\kappa^{(0)}-ithS_z}$ and $\sigma_{\kappa,m+1}^+$ in (6.12), we can immediately

write the time-dependent generalisation of the equation (5.13):

$$\langle \sigma_{1,\kappa}^-(0)\,\sigma_{\kappa,m+1}^+(t)\rangle = e^{-iht}\sum_{\{\mu\}}\prod_{j=1}^{N-1}e^{itE(\mu_j)}\prod_{j=1}^{N}e^{-itE(\lambda_j)}$$

$$\times\,\frac{\langle\psi(\{\lambda\})\,|\,\sigma_{1,\kappa}^-\,|\,\psi_\kappa(\{\mu\})\rangle\cdot\langle\psi_\kappa(\{\mu\})\,|\,\sigma_{\kappa,m+1}^+\,|\,\psi(\{\lambda\})\rangle}{\langle\psi(\{\lambda\})\,|\,\psi(\{\lambda\})\rangle\cdot\langle\psi_\kappa(\{\mu\})\,|\,\psi_\kappa(\{\mu\})\rangle}\,,\qquad(6.14)$$

where the sum is taken over all the admissible solutions $\{\mu_1,\dots,\mu_{N-1}\}$ of the system of $N-1$ κ-twisted Bethe equations. It remains now to repeat all the steps of the corresponding part of Section 5 in the opposite order (from (5.13) to (5.7)), and we obtain

$$\langle \sigma_{1,\kappa}^-(0)\,\sigma_{\kappa,m+1}^+(t)\rangle = \frac{e^{-iht}}{(N-1)!}\oint_{\Gamma\{\pm\frac{\eta}{2}\}\cup\Gamma\{\lambda\}}\prod_{j=1}^{N-1}\frac{dz_j}{2\pi i}\prod_{b=1}^{N-1}e^{itE(z_b)+imp(z_b)}\prod_{b=1}^{N}e^{-itE(\lambda_b)-imp(\lambda_b)}$$

$$\times\prod_{a=1}^{N-1}\frac{1}{\mathcal{Y}_\kappa(z_a|\{z\})}\cdot\prod_{j=1}^{N-1}\frac{1}{\sinh(z_j-\frac{\eta}{2})}\cdot\prod_{j=1}^{N}\frac{1}{\sinh(\lambda_j-\frac{\eta}{2})}\cdot\frac{\det_N\Omega(\{\lambda\},\{z\}\cup\{\frac{\eta}{2}\}\,|\,\{\lambda\})}{\det_N\Omega(\{\lambda\},\{\lambda\}\,|\,\{\lambda\})}$$

$$\times\left\{\sum_{\substack{\alpha,\beta=1\\\alpha<\beta}}^{N+1}(-1)^{\alpha+\beta}\left\{\frac{a(\tilde{\lambda}_\alpha)\,d(\tilde{\lambda}_\beta)}{\sinh(\tilde{\lambda}_\alpha-\tilde{\lambda}_\beta-\eta)}\cdot\prod_{k=1}^{N}\left[\sinh(\lambda_k-\tilde{\lambda}_\alpha+\eta)\sinh(\lambda_k-\tilde{\lambda}_\beta-\eta)\right]\right.\right.$$

$$\left.-\frac{a(\tilde{\lambda}_\beta)\,d(\tilde{\lambda}_\alpha)}{\sinh(\tilde{\lambda}_\beta-\tilde{\lambda}_\alpha-\eta)}\cdot\prod_{k=1}^{N}\left[\sinh(\lambda_k-\tilde{\lambda}_\beta+\eta)\sinh(\lambda_k-\tilde{\lambda}_\alpha-\eta)\right]\right\}$$

$$\left.\times\det_{N-1}\Omega_\kappa(\{z\},\{\tilde{\lambda}\}\backslash\{\tilde{\lambda}_\alpha,\tilde{\lambda}_\beta\}|\{z\})\right\}.\qquad(6.15)$$

In this expression, all the notations are just the same as in (5.7). Note that this result could have been obtained also by the method used in Section 6.1.

6.3 Dynamical correlation functions in the thermodynamic limit

It was explained in Remark 5.5 how one could, starting from the time-independent master equations, reproduce the integral representations (3.11)-(3.14) for the two-point functions in the thermodynamic limit. A similar method can be applied to the time-dependent case [19], although the existence of the essential singularities at $\pm\eta/2$ in the integrand makes this procedure more subtle.

For simplicity, we consider here only the case of the generating function $\langle Q_{1,m}^\kappa(t)\rangle$, $|\psi(\{\lambda\})\rangle$ being now the ground state of the Hamiltonian (1.1). Similarly as in Remark 5.5, we can decompose the multiple integral in (6.11) as

$$\oint_{\Gamma\{\pm\frac{\eta}{2}\}\cup\Gamma\{\lambda\}}\prod_{j=1}^{N}dz_j = \sum_{n=0}^{N}C_N^n\oint_{\Gamma\{\pm\frac{\eta}{2}\}}\prod_{j=1}^{n}dz_j\oint_{\Gamma\{\lambda\}}\prod_{j=n+1}^{N}dz_j.\qquad(6.16)$$

and evaluate the integrals over the contour $\Gamma\{\lambda\}$, which leads to a sum over the partitions of the set $\{\lambda\}$ into two disjoint subsets $\{\lambda\}=\{\lambda_{\alpha_+}\}\cup\{\lambda_{\alpha_-}\}$. In the time-independent case, we could then set $d(z)=0$ directly in the remaining part of the integrand, since all the terms proportional

to $d(z)$ were holomorphic in the point $\eta/2$. In the time-dependent case, the situation is a priori different: due to the essential singularities of the integrand in the points $\pm\eta/2$, the contribution of the function $d(z)$ in the vicinity of $\eta/2$ (respectively $a(z)$ in the vicinity of $-\eta/2$) does not vanish. Nevertheless, since $d(z)$ and $a(z)$ have zeros of order M at $z = \eta/2$ and $z = -\eta/2$ respectively, one can show that the contributions of the corresponding terms to the total result are bounded by $C^N/N!$, where C is some constant. Hence, these contributions vanish in the thermodynamic limit $M, N \to \infty$, $M/N = \text{const}$. Thus, in the thermodynamic limit, one can set $d(z) = 0$ in the vicinity of $z = \eta/2$ and $a(z) = 0$ in the vicinity of $z = -\eta/2$. This leads us to the following integral representation, in which the integrand is defined differently in the two half-planes of the complex plane [19]:

$$
\langle Q^\kappa_{1,m}(t) \rangle = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int_C d^n\lambda \oint_{\Gamma\{\pm\frac{\eta}{2}\}} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \cdot \prod_{a,b=1}^{n} \frac{\sinh(\lambda_a - z_b + \eta)\sinh(z_b - \lambda_a + \eta)}{\sinh(\lambda_a - \lambda_b + \eta)\sinh(z_a - z_b + \eta)}
$$

$$
\times \prod_{b=1}^{n} e^{it[E(z_b) - E(\lambda_b)] + im[p(z_b) - p(\lambda_b)]} \cdot \det_n M_\kappa(\{\lambda\}, \{z\}) \cdot \det_n[\mathcal{R}^\kappa_n(\lambda_j, z_k | \{\lambda\}, \{z\})]. \quad (6.17)
$$

In this expression, the contour $\Gamma\{\pm\frac{\eta}{2}\}$ surrounds the points $\pm\frac{\eta}{2}$ and does not contain any other singularities of the integrand.

Comparing this result with its time-independent analogue (3.11), we see that the determinant of densities $\det[\rho(\lambda_j, z_k)]$ has been replaced by a new thermodynamic quantity depending on the function $\mathcal{R}^\kappa_n(\lambda, z | \{\lambda_1, \dots, \lambda_n\}, \{z_1, \dots, z_n\})$. This function is defined differently in the vicinities of $\eta/2$ and $-\eta/2$:

$$
\mathcal{R}^\kappa_n(\lambda, z | \{\lambda\}, \{z\}) = \begin{cases} \rho(\lambda, z), & z \sim \eta/2; \\ -\kappa^{-1}\rho(\lambda, z + \eta) \prod_{b=1}^{n} \frac{\sinh(z - \lambda_b + \eta)\sinh(z_b - z + \eta)}{\sinh(\lambda_b - z + \eta)\sinh(z - z_b + \eta)}, & z \sim -\eta/2. \end{cases} \quad (6.18)
$$

Due to the factors $\exp(itE(z_b))$, the integrand in (6.17) has essential singularities in the points $\pm\eta/2$. However, in the case $t = 0$, these essential singularities disappear and the integrals around $-\eta/2$ vanish. The remaining part of the integrand has poles of order m at $z_j = \eta/2$. Hence, at $t = 0$, the sum over n in (6.17) is actually restricted to $n \le m$, and we reproduce the result of (3.11).

Conclusion

In this review, we have summarised recent results concerning the computation of correlation functions in the XXZ chain by the methods of the inverse scattering problem and the algebraic Bethe Ansatz. In conclusion, we would like to discuss some perspectives and problems to be solved.

One of the most interesting open problems is to prove the conformal field theory predictions concerning the asymptotic behaviour of the correlation functions. We have seen in Section 4 that the new integral representations presented in Section 3 seemed, from the view point of the asymptotic analysis, more promising than the original representations in terms of the elementary blocks of Section 2. Nevertheless, it is still not clear how to extract the asymptotics directly from these multiple integrals.

A possible way to solve this problem would be to find the thermodynamic limit of the master equations. It is natural to expect that, in this limit, one should obtain a representation for the

two-point functions in terms of a single functional integral, which could probably be estimated for the large time and distance.

Irrespective of this problem of the asymptotics, the master equation shows a direct analytic relation between the multiple integral representations and the form factor expansions for the correlation functions. It seems likely that similar representations exist for other model solvable by algebraic Bethe Ansatz. It would be in particular very interesting to obtain an analogue of this master equation in the case of the field theory models, which could provide an analytic link between short distance and long distance expansions of the correlation functions.

Another interesting further development would be to generalise all these results to the case of correlation functions at finite temperature. In this direction, a multiple integral representation similar to (3.11) was derived recently in [33] for the temperature time-independent correlation function, and it would be interested to obtain it in the time-dependent case as well. One can also wonder whether there exists a master equation similar to (5.6) for the temperature-dependent case. It would raise the interesting question of the form factor expansion at non-zero temperature.

It is also well known that, for the case of free fermions $\Delta = 0$, the dynamical correlation functions of the XXZ chain satisfy difference-differential classical exactly solvable equations [50, 59, 60]. It is natural to wonder whether this property holds also for general Δ, or at least for some specific cases. We hope that the multiple integral representations for the dynamical correlation functions open a way to study this problem.

Acknowledgements

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