Some results on Hessian measures for non-commuting vector fields.¹

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In this talk we present some extensions of the theory of Hessian measures developed in [4,5,6] to more general vector fields. Details of proofs are given in [8].

Let $X = (X_1, \cdots, X_m)$ be a system of vector fields in Euclidean space $\mathbb{R}^n, m \leq n$, given by

$$X_i = \sigma^{ij} D_j, \quad i = 1, \cdots, m,$$

(1)

where $\sigma^{ij} \in C^\infty(\mathbb{R}^n), i = 1, \cdots, m, j = 1, \cdots, m$. For $\Omega$ a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$, the Hessian and symmetrized Hessian of $u$, with respect to $X$, are defined respectively by

$$X^2 u = [X_i X_j u],$$

(2)

$$X_s^2 u = \left[\frac{1}{2}(X_i X_j + X_j X_i) u\right]_{i,j=1,\cdots,m}.$$ 

For a matrix $r \in \mathbb{R}^m \times \mathbb{R}^m, k = 1, \cdots, m$, we let $S_k(r)$ denote the sums of its $k \times k$ principal minors and define the corresponding operators $F_k$ by

$$F_k[u] = S_k(X_s^2 u)$$

(3)

A function $u \in C^2(\Omega)$ is called $k$--convex, with respect to $X$, if $F_j[u] \geq 0$ in $\Omega$, for all $j = 1, \cdots, k$. A function $u \in L^1_{\text{loc}}(\Omega)$ is called $k$--convex, with respect to $X$, if for each domain $\Omega' \subseteq \Omega$, there exists a sequence of $k$--convex functions $\{u_\ell\} \subseteq C^2(\Omega')$ such that $u_\ell \to u$ as $\ell \to \infty$, in $L^1(\Omega')$.

We denote the class of $k$--convex functions in $\Omega$ by $\phi^k(\Omega)$ or simply $\phi^k(\Omega)$, when $X$ is understood. The following properties of $k$--convex functions in the Euclidean case, $X_i = D_i, i = 1, \cdots, k$, are proved in [4,5].

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Theorem 1. For any $u \in \phi^k(\Omega)$ we have $Xu \in L^p_{\text{loc}}(\Omega)$ for any $p < nk(n-k)$ and there exists a Borel measure $\mu_k[u]$, extending $F_k[u]dx$ for $u \in C^2(\Omega)$, such that if $u_\ell \rightarrow u$ a.e. $(\Omega)$, then $\mu_k[u_\ell] \rightarrow \mu_k[u]$ weakly.

In [8], these results are extended, in part, to anti-self adjoint systems $X$, $(X_i^* = -X_i, i = 1, \cdots, m)$, satisfying the Hörmander condition that at each point of $\Omega$, the Lie algebra generated by $X$ spans $\mathbb{R}^n$. In particular we prove,

Theorem 2. Let $u \in \phi_X^k(\Omega)$ where $X$ satisfies the above conditions. Then $Xu \in L^p_{\text{loc}}(\Omega)$, for any $p < Qk(m-1)/Q(m-k)$, where $Q$ denotes the homogeneous dimension of $X$. If $k = 2$ and $X$ is of step 2, then the commutators $[X_i, X_j]u \in L^p_{\text{loc}}(\Omega), i, j = 1, \cdots, m$ and there exists a Borel measure $\mu_2[u]$, extending $F_2[u]dx$ for $u \in C^2(\Omega)$, such that if $u_\ell \rightarrow u$ a.e.(\Omega), then $\mu_2[u_\ell] \rightarrow \mu_2[u]$ weakly in $\Omega$.

A more general theory for quasilinear operators extending the case $k = 1$ is developed in [7]. The restriction to Step 2 may be weakened [8], but so far we are unaware of any extensions of the commutator regularity and weak continuity to the cases $k > 2$. The proof of Theorem 2 draws upon our techniques in [4,5,7] and stems from an interesting identity, discovered in the special case of the Heisenberg group $\mathcal{H}^1$ in [1]. Namely, if we define the function $G$ on $\mathbb{R}^m \times \mathbb{R}^m$ by

$$G(r) = S_g(r) + \frac{1}{2} \sum_{i<j}(r_{ij} - r_{ji})^2,$$

then, for any $u \in C^2(\Omega)$,

$$Y_j u := X_i \left[ \frac{\partial G}{\partial r_{ij}}(X^2 u) \right] = [X_i, X_j]u,$$

that is $Y_j, j = 1, \cdots, m$, are vector fields, (vanishing when $X$ is Step 2).

More generally, we can define subharmonic functions along the lines of [5,6]. In particular we define an upper semi-continuous function $u : \Omega \rightarrow [-\infty, \infty)$ to be subharmonic with respect to the operator $F_k$ if $F_k[u] \geq 0$ in the viscosity sense, that is for any quadratic polynomial $q$ for which the difference $u - q$ has a finite local maximum at a point $y \in \Omega$, we have $F_k[q](y) \geq 0$. A $k-$convex function is then equivalent to a subharmonic function. Furthermore it follows from [3,9], that if $X$ generates the Lie algebra of a Carnot group, then a proper subharmonic function, ($\Xi - \infty$ on a set of positive measure), is $k-$convex. The equivalence of various notions
of convexity in the case $k = m$, is treated in [2,9], where other references are also given. In this case, Theorem 2 can be improved to $Xu \in L^\infty_{\text{loc}}(\Omega)$ if $u \in \phi^k(\Omega)$.

The extension of Theorem 2 to arbitrary step can be expressed in terms of the vector fields $Y = (Y_1, \ldots, Y_m)$ defined by (??). For example, if $m = 2$, then the commutator $[X_1, X_2]u \in L^2_{\text{loc}}(\Omega)$ if $Yu \in L^1_{\text{loc}}(\Omega)$, while $\mu_2[u_\ell] \rightarrow \mu_2[u]$ weakly if also $\{Yu_\ell\}$ is uniformly bounded in $L^1_{\text{loc}}(\Omega)$.

Finally we remark that from (??), we have a nononotonicity property for arbitrary anti-self adjoint systems of vector fields $X$, which extends the Euclidean case in [4] and the case of the Heisenberg group in [1]. Namely defining, for $u \in C^2(\Omega)$,

$$G[u] = G(X^2u) + \frac{1}{2} Xu.Yu,$$

we obtain

$$\int_\Omega G[u] \geq \int_\Omega G[v],$$

for any functions $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$, satisfying $u \leq v$ in $\Omega$, $u = v$ on $\partial\Omega$ with the operator $F_2$ degenerate elliptic with respect to their sum $u + v$.

In certain cases, including the Heisenberg group in [1], Theorem 2 may be derived from (??), using the approach in [4], rather than that through integral estimates in [5].

References


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