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ASYMPTOTIC SOLUTIONS OF HAMILTON-JACOBI EQUATIONS IN THE WHOLE EUCLIDEAN SPACE

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Abstract. In this note we describe some of results on the large-time behavior of solutions of a class of Hamilton-Jacobi equations in the whole space $\mathbf{R}^n$, which have been obtained in a joint work with Y. Fujita and P. Loreti [FIL2].

1. Introduction and main results

Recently there has been a great interest on the asymptotic behavior of viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations or viscous Hamilton-Jacobi equations. Among others Fathi [F2] has first established a fairly general convergence result for the Hamilton-Jacobi equation

$$(1) \qquad u_t(x,t) + H(x, Du(x,t)) = 0$$

on a compact manifold $\mathcal{M}$ with smooth strictly convex Hamiltonian H. Associated with this problem is the additive eigenvalue problem for the Hamiltonian H (or the Hamilton-Jacobi equation $H(x, Du) = 0$)

$$(2) \qquad c + H(x, Dv) = 0 \quad \text{in } \mathcal{M},$$

where the unknown is the pair of a constant $c \in \mathbf{R}$ and a solution v of (2). Here and in what follows we adapt the notion of viscosity solution to that of weak solution for first order PDE. It is known (see [LPV]) that a constant c for which (2) has a viscosity solution v is uniquely determined. The result obtained in [F2] is loosely stated as follows: for any viscosity solution u of (1) there is a viscosity solution v of (2) such that $u(x,t) - ct \to v(x)$ uniformly on $\mathcal{M}$ as $t \to \infty$. His approach to this asymptotic problem is based on the weak KAM theorem [F1] and especially on Aubry-Mather sets. A PDE approach to the same asymptotic problem has been developed by Barles and Souganidis [BS]. Fathi's approach has been developed by Roquejoffre [R] and Davini-Siconolfi [DS].

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In [FIL1], jointly with Y. Fujita and P. Loreit, the auhtor has recently investigated the asymptotic problem for viscous Hamilton-Jacobi equations in $\mathbf{R}^n$ with Ornstein-Uhlenbeck operator and have established a convergence result similar to the one stated above. The equations treated in [FIL1] have the form

$$u_t - \Delta u + \alpha x \cdot Du + H(Du) = f(x). \tag{3}$$

In [FIL2], we have studied the Cauchy problem

$$u_t + \alpha x \cdot Du + H(Du) = f(x) \quad \text{in } \mathbf{R}^n \times (0, \infty), \tag{4}$$

and

$$u|_{t=0} = \phi. \tag{5}$$

In this note we describe the main results obtained in [FIL2]. To be precise, here u represents the real-valued unknown function on $\mathbf{R}^n \times [0, \infty)$, α is a given positive constant, H, f, ϕ are given real-valued functions on $\mathbf{R}^n$, u_t and Du denote the t-derivative and x-gradient of u, respectively, and $x \cdot y$ denotes the Euclidean inner product of $x, y \in \mathbf{R}^n$.

We assume the following conditions on H, f, ϕ throughout this note:

(A1) $H, f, \phi \in C(\mathbf{R}^n)$.

(A2) H is convex on $\mathbf{R}^n$.

(A3) $\displaystyle\lim_{|p| \to \infty} \frac{H(p)}{|p|} = \infty$.

PDE (4) can be seen as the dynamic programming equation of the control system in which the state equation is given by

$$\dot{X}(t) + \alpha X(t) = \xi(t) \quad \text{for } t \in (0, T), \quad X(0) = x,$$

where $0 < T < \infty$, $x \in \mathbf{R}^n$, and $\xi \in L^1(0, T)$ is a control, and in which the value function u is given by

$$u(x, T) = \inf_{\xi \in L^1(0,T)} \left\{ \int_0^T [f(X(t)) + L(-\xi(t))] \, dt + \phi(X(T)) \right\}, \tag{6}$$

where L denotes the convex conjugate H^* of H, i.e.,

$$L(\xi) := H^*(\xi) \equiv \sup\{\xi \cdot p - H(p) \mid p \in \mathbf{R}^n\} \quad \text{for } \xi \in \mathbf{R}^n.$$

As is well-known, the function L is continuous on $\mathbf{R}^n$ and satisfies

$$\lim_{|\xi| \to \infty} \frac{L(\xi)}{|\xi|} = \infty.$$

We assume furthermore that there is a convex function $l : \mathbf{R}^n \to \mathbf{R}$ having the properties:

(A4) $\lim_{|x|\to\infty} (L(x) - l(x)) = \infty$.

(A5) $\inf\{f(x) + l(-\alpha x) \mid x \in \mathbf{R}^n\} > -\infty$.

(A6) $\inf\{\phi(x) + \frac{1}{\alpha} l(-\alpha x) \mid x \in \mathbf{R}^n\} > -\infty$.

The role of the function l to describe the class of solutions, which we treat in this note, as (A6) gives a lower bound of the initial data ϕ through the function l.

In view of (A4) and (A5), we see that the function $x \mapsto f(x) + L(-\alpha x)$ attains a minimum over $\mathbf{R}^n$, and we set

$$(7) \qquad c = \min\{f(x) + L(-\alpha x) \mid x \in \mathbf{R}^n\} \quad \text{and} \quad f_c(x) = f(x) - c \quad \text{for } x \in \mathbf{R}^n.$$

We observe as well that

$$(8) \qquad Z := \{x \in \mathbf{R}^n \mid f(x) + L(-\alpha x) = c\}$$

is a compact subset of $\mathbf{R}^n$.

This set Z corresponds to the projected Aubry set although we will not introduce the projected Aubry set for (4) in this note. Our approach in this note is based on the fact that the projected Aubry Z for (4) comprises only equilibrium points.

A typical case where (A1)–(A6) are satisfied is: let H, f, and ϕ satisfy (A1)–(A3). Assume furthermore that there is a constant $C_0 > 0$ such that

$$f(x) \geq -C_0(|x| + 1), \quad \phi(x) \geq -C_0(|x| + 1) \quad \text{for } x \in \mathbf{R}^n.$$

In this situation, if we take l to be the function given by $l(x) = (\alpha + 1)C_0(|x| + 1)$, then conditions (A4)–(A6) hold.

For $(x, y, T) \in \mathbf{R}^n \times \mathbf{R}^n \times (0, \infty)$ let $\mathcal{C}(x, T)$ and $\mathcal{C}(x, y, T)$ denote the spaces of absolutely continuous functions $X : [0, T] \to \mathbf{R}^n$ satisfying, respectively, $X(0) = x$ and $(X(0), X(T)) = (x, y)$. Define the functions $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ and $\psi : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ by

$$(9) \quad d(x, y) = \inf\Big\{\int_0^T [f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))]\,\mathrm{d}t \;\Big|\; T > 0, X \in \mathcal{C}(x, y, T)\Big\},$$

and

$$(10) \qquad \psi(x) = \inf\Big\{\int_0^T [f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))]\,\mathrm{d}t + \phi(X(T)) \;\Big|\; T > 0,\ X \in \mathcal{C}(x, T)\Big\},$$

respectively.

Define the function $v : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ by

$$v(x) = \inf_{y \in Z} (d(x,y) + \psi(y)). \tag{11}$$

We remark that this function v can be written also as

$$v(x) = \inf\{d(x,y) + d(y,z) + \phi(z) \mid y \in Z,\ z \in \mathbf{R}^n\}.$$

Proposition 1. *The functions d, ψ, and v are real-valued continuous functions on $\mathbf{R}^n \times \mathbf{R}^n$, $\mathbf{R}^n$, and $\mathbf{R}^n$, respectively.*

Henceforth $B(x,R)$ denotes the closed ball of $\mathbf{R}^n$ with center at x and radius $R \geq 0$.

Theorem 2. *There is a unique viscosity solution $u \in C(\mathbf{R}^n \times [0,\infty))$ of* (4) *and* (5) *which satisfies for any $0 < T < \infty$,*

$$\lim_{r \to \infty} \inf\{u(x,t) + \frac{1}{\alpha} L(-\alpha x) \mid (x,t) \in (\mathbf{R}^n \setminus B(0,r)) \times [0,T)\} = \infty. \tag{12}$$

The main result in this note is the following.

Theorem 3. *Let $u \in C(\mathbf{R}^n \times [0,\infty))$ be the unique viscosity solution of* (4) *and* (5) *satisfying* (12). *Then*

$$\lim_{t \to \infty} \max_{x \in B(0,R)} |u(x,t) - (ct + v(x))| = 0 \quad \textit{for } R > 0. \tag{13}$$

We remark that formula (11) for the asymptotic solution v has been shown in [DS] for a fairly general Hamilton-Jacobi equation in the periodic setting. The function v is a viscosity solution of

$$c + \alpha x \cdot Dv + H(Dv) = f(x) \quad \text{in } \mathbf{R}^n. \tag{14}$$

For instance, this follows from Theorem 3 and the stability of viscosity solutions of (4) under locally uniform convergence.

2. Outline of proof of the convergence result

This section will be devoted to proving Theorem 3. The approach explained here is different from that of [FIL2]. We will not prove the formula (11) for the asymptotic solution in this note. We may assume by replacing f by $f_c \equiv f - c$ if necessary that $c = 0$.

The following two lemmas give basic estimates on the solution u of (4) and (5) given by the formula (6). We omit giving a proof of these lemmas and refer to [FIL2] for a proof.

Lemma 4. *We have*

$$u(x,t) \geq -\frac{1}{\alpha} l(-\alpha x) - C \quad \textit{for all } (x,t) \in \mathbf{R}^n \times [0,\infty),$$

where C is a constant depending only on ϕ and l.

Lemma 5. *For each $R > 0$ the function u is bounded, uniformly continuous on $B(0,R) \times [0,\infty)$.*

We set

$$G(x,p) = \alpha x \cdot p + H(p) - f(x) \quad \text{for } (x,p) \in \mathbf{R}^n \times \mathbf{R}^n.$$

Observe that for $x \in Z$,

$$\begin{aligned} G(x,p) &= \max_{\xi \in \mathbf{R}^n}((\alpha x + \xi) \cdot p - L(\xi) - f(x)) \\ &= \max_{\xi \in \mathbf{R}^n}(\xi \cdot p - L(\xi - \alpha x) - f(x)) \geq -L(-\alpha x) - f(x) = 0. \end{aligned}$$

From this, it is easily seen that for each $x \in Z$, the function $t \mapsto u(x,t)$ is a viscosity subsolution of $u_t = 0$ in $(0,\infty)$, which implies that the function $t \mapsto u(x,t)$ is nonincreasing on $[0,\infty)$ for any $x \in Z$. Hence, in view of Lemma 4 or 5, we see that the limit $\lim_{t\to\infty} u(x,t)$ exists for all $x \in Z$. Using Dini's Lemma, we infer that the convergence of $u(x,t)$, as $t \to \infty$, is uniform for $x \in Z$.

We introduce the half relaxed limits of u as $t \to \infty$ as follows:

$$\begin{aligned} v^+(x) &= \limsup_{t\to\infty} u(x,t) \equiv \lim_{r\to 0+} \sup\{u(y,s) \mid |y-x| < r,\ s > 1/r\}, \\ v^-(x) &= \liminf_{t\to\infty} u(x,t) \equiv \lim_{r\to 0+} \inf\{u(y,s) \mid |y-x| < r,\ s > 1/r\}. \end{aligned}$$

As is well-known, we have in the viscosity sense

$$\begin{aligned} G(x, Dv^+(x)) &\leq 0 \quad \text{in } \mathbf{R}^n, \\ G(x, Dv^-(x)) &\geq 0 \quad \text{in } \mathbf{R}^n. \end{aligned}$$

The uniform convergence of u on the set Z, which has been shown above, can be stated just as

$$\text{(15)} \qquad \lim_{t\to\infty} u(x,t) = v^+(x) = v^-(x) \quad \text{for all } x \in Z.$$

By the definition, we have

$$v^-(x) \leq v^+(x) \quad \text{for all } x \in \mathbf{R}^n.$$

Indeed, in order to conclude that the function $u(x,t)$ converges to a function v uniformly on bounded sets as $t \to \infty$, it is enough to show that $v^+(x) \leq v^-(x)$ for all $x \in \mathbf{R}^n$.

Therefore, it remains to prove that

$$v^+(x) \le v^-(x) \quad \text{for all } x \in \mathbf{R}^n \setminus Z. \tag{16}$$

To this end, we fix any $\varepsilon > 0$ and, in view of (15) and Lemma 5, choose a compact neighborhood K of Z so that

$$v^+(x) \le v^-(x) + \varepsilon \quad \text{for all } x \in K.$$

We set $\psi(x) = -\frac{1}{\alpha}L(-\alpha x)$ for $x \in \mathbf{R}^n$. For any point x of differentiability of the function ψ, we have

$$D\psi(x) = DL(-\alpha x),$$

and hence the function $\xi \mapsto \xi \cdot D\psi(x) - L(\xi - \alpha x)$ attains a maximum at $\xi = 0$, i.e.,

$$G(x, D\psi(x)) = -L(-\alpha x) - f(x).$$

Noting that $p \mapsto G(x,p)$ and L are convex for any $x \in \mathbf{R}^n$, we see that ψ is a viscosity solution of

$$G(x, D\psi(x)) = -L(-\alpha x) - f(x) \quad \text{in } \mathbf{R}^n.$$

By (A4) and (A5) and by the definition of Z, there is a constant $\delta > 0$ such that in the viscosity sense

$$G(x, D\psi(x)) \le -\delta \quad \text{in } \mathbf{R}^n \setminus K.$$

We fix any $\lambda \in (0,1)$ and $A > 0$ and set

$$w_{\lambda,A}(x) = \min\{(1-\lambda)v^+(x) + \lambda\psi(x), \psi(x) + A\} \quad \text{for } x \in \mathbf{R}^n.$$

Observe by the convexity of the Hamiltonian G that $w_{\lambda,A}$ is a viscosity solution of

$$G(x, Dw_{\lambda,A}(x)) \le -\lambda\delta \quad \text{in } \mathbf{R}^n \setminus K.$$

By virtue of Lemma 4, we have

$$v^+(x) \ge v^-(x) \ge -\frac{1}{\alpha}l(-\alpha x) - C \quad \text{for all } x \in \mathbf{R}^n.$$

From this and (A4), we find that for some $R > 0$ and all $x \in \mathbf{R}^n \setminus B(0,R)$,

$$w_{\lambda,A}(x) = \psi(x) + A \le v^-(x).$$

We apply a standard comparison theorem to v^- and $w_{\lambda,A}$ in the domain $\operatorname{int} B(0,R) \setminus K$, to obtain

$$w_{\lambda,A} \le v^-(x) + \varepsilon \quad \text{for all } x \in B(0,R) \setminus K,$$

which guarantees that

$$w_{\lambda,A}(x) \le v^-(x) + \varepsilon \quad \text{for all } x \in \mathbf{R}^n \setminus K.$$

Sending $\lambda \to 0$ and $A \to \infty$ yields

$$v^+(x) \le v^-(x) + \varepsilon \text{ for all } x \in \mathbf{R}^n \setminus K.$$

This together with the choice of K, we have

$$v^+(x) \le v^-(x) + \varepsilon \quad \text{for all } x \in \mathbf{R}^n.$$

This is enough to conclude that $v^+(x) \le v^-(x)$ for all $x \in \mathbf{R}^n$. This completes the proof.

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