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Homogenization of fully nonlinear PDEs and backward SDEs

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概要

In this note, we discuss a probabilistic approach to homogenization of fully nonlinear second-order PDEs of parabolic type. We also study the rate of convergence of solutions, which can be regarded as a byproduct of our stochastic representation of solutions based on backward stochastic differential equations.

1 Problem.

Let us consider the Cauchy problem with small parameter $\varepsilon > 0$ of the form

$$(1.1) \qquad \begin{cases} -u_t + H(\varepsilon^{-1}x, u, u_x, u_{xx}) = 0\,, & \text{in} \quad [0,T) \times \mathbb{R}^d\,, \\ u(T,x) = h(x)\,, & \text{on} \quad \mathbb{R}^d\,, \end{cases}$$

where u_t stands for the partial derivative of u with respect to t, and u_x and u_{xx} denote its first and second derivatives with respect to x, respectively. The continuous function $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}$, called *Hamiltonian*, is assumed to be $\mathbb{Z}^d$-periodic with respect to its first variable. We also assume that $h(\cdot)$ is a bounded and uniformly continuous function. It is well known that (1.1) has a unique solution in the viscosity sense if H is proper (possibly degenerate elliptic) and satisfies some other structure conditions (see [6]).

Our aim is to prove the following convergence theorem (homogenization) under certain conditions on H.

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Theorem 1.1. *Let $\{u^\varepsilon(t,x);\varepsilon>0\}$ be the family of viscosity solutions to (1.1). Then, as ε goes to zero, it converges to a unique viscosity solution $u^0(t,x)$ of the following PDE*

$$
(1.2)\qquad \begin{cases} -u_t+\overline{H}(u,u_x,u_{xx})=0\,, & \text{in } [0,T)\times\mathbb{R}^d\,,\\ u(T,x)=h(x)\,, & \text{on } \mathbb{R}^d\,. \end{cases}
$$

Here, the effective Hamiltonian $\overline{H}=\overline{H}(y,p,X)$ is defined by the cell problem

$$
(1.3)\qquad \overline{H}=H(\eta,y,p,X+v_{\eta\eta}(\eta))\,,\qquad (v(\cdot),\overline{H}):\textit{unknown.}
$$

Such kind of homogenization problems have been largely studied by the so-called perturbed test function method based on the theory of viscosity solution (see [1], [2], [8], [9] for details). On the other hand, it seems to be worth studying (1.1)-(1.3) from probabilistic view point, for the class of fully nonlinear equations of this form contains important and interesting examples that are closely related to stochastic problems. Hamilton-Jacobi-Bellman equations (HJB equations, for short) are the most typical ones. There are also a number of literatures concerning homogenization of second-order PDEs treated by probabilistic methods. In particular, for the investigation of nonlinear PDEs, the notion of backward stochastic differential equation (BSDE) is useful (see [3], [4], [7], [10], [12] for the homogenization of semi-linear and quasi-linear equations by BSDE approaches, as well as [5] for that of fully nonlinear HJB equations). We remark that the literature [11], which this note is based on, also uses BSDE approach to prove the homogenization of fully nonlinear second-order PDEs.

The novelty of this note (and therefore that of [11]) is that under the assumption that H is uniformly elliptic and convex in the last variable, we obtain an estimate of convergence rate of solutions at the same time (Theorem 1.2 below). As far as fully nonlinear second-order equations concerned, to the best of our knowledge, such kind of rate of convergence have not been studied neither by the viscosity solution method nor by the probabilistic one.

Theorem 1.2. *Let $\delta\in(0,1)$ be the Hölder exponent of the second derivatives of solution u^0 to (1.2), i.e. $u^0\in C^{1+\delta/2,2+\delta}([0,T]\times\mathbb{R}^d)$. Then, for every compact subset Q of $[0,T]\times\mathbb{R}^d$, there exists a constant $C>0$ independent of $\varepsilon>0$ such that the following holds :*

$$
\sup_{(t,x)\in Q}|u^\varepsilon(t,x)-u^0(t,x)|\le C\,\varepsilon^{\frac{2\delta}{2+\delta}}\,.
$$

Remark 1.3. Under Assumption 2.1 below, it is known that (1.2) has a unique classical solution in the Hölder space $C^{1+\delta/2,2+\delta}([0,T]\times\mathbb{R}^d)$.

This note is organized as follows. In the next section, we give the precise assumption on H we suppose throughout this note. In Section 3, we discuss a stochastic representation of solutions by BSDEs. This interpretation makes us possible to treat homogenization of fully nonlinear equations in a probabilistic way. Section 4 is devoted to the proof of Theorem 1.2.

2 Assumption.

Throughout this note, the terminal function $h(\cdot)$ is assumed to be of C_b^3-class. Concerning the Hamiltonian H in (1.1), we make the following assumption.

Assumption 2.1. There exist K and $\nu > 0$ such that H satisfies the following conditions.

(A1) H is of C^2-class and all second derivatives are bounded.

(A2) H is convex in X.

(A3) For every (η, y, p, X) and $\xi \in \mathbb{R}^d$,

$$\nu|\xi|^2 \le H(\eta,y,p,X) - H(\eta,y,p,X+\xi\otimes\xi) \le \nu^{-1}|\xi|^2,$$

where $\xi\otimes\xi$ stands for the $(d\times d)$-matrix defined by $(\xi\otimes\xi)_{ij} := \xi^i\xi^j$.

(A4) For every (y,p,X), (y',p',X') and η,

$$|H(\eta,y,p,X) - H(\eta,y',p',X')| \le K\{|y-y'| + |p-p'| + |X-X'|\}.$$

(A5) For every η, η' and (y,p,X),

$$|H(\eta,y,p,X) - H(\eta',y,p,X)| \le K(1+|p|+|X|)|\eta-\eta'|.$$

3 Stochastic representation.

In this section, we introduce an appropriate family of controlled BSDEs in order to obtain a stochastic representation of solutions to (1.1). For this purpose, we prepare the following lemma.

Lemma 3.1. *Let us set $E := \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d\times d}$. Then, there exist a bounded continuous function a on $\mathbb{R}^d\times E$ taking its values in the set of symmetric matrices $\mathbb{S}^d\subset\mathbb{R}^{d\times d}$*

and a continuous function $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times E \longrightarrow \mathbb{R}$ *such that* H *can be written as follows :*

$$H(x,y,p,X) = \max_{\zeta \in E}\{-\sum_{i,j=1}^{d} a^{ij}(x,\zeta)X_{ij} - f(x,y,p,\zeta)\}, \tag{3.1}$$

where the maximum of the right-hand side is attained when $\zeta = (-y,-p,-X)$. *Moreover, we can take* $a = (a^{ij})$ *and* f *such that* a^{ij} *is Lipschitz continuous uniformly in* x, *and* f *is Lipschitz continuous uniformly in* (y,p) *and satisfies under the notation* $\zeta = (\alpha,\beta,\gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ *the following inequalities :*

$$-K(1+\min\{|y|,|\alpha|\}+\min\{|p|,|\beta|\}) \le f(x,y,p,\zeta) \le \tilde{K}(1+|y|+|p|+|\zeta|), \tag{3.2}$$

where $\tilde{K}$ *is a constant depending only on* K.

Sketch of the proof. We define a^{ij} and f by

$$\begin{aligned} a^{ij}(x,\zeta) &:= \tilde{H}_{X_{ij}}(x,\zeta), \\ f(x,y,p,\zeta) &:= \tilde{H}_{X_{ij}}(x,\zeta)\gamma_{ij} - \tilde{H}(x,\zeta) + K|\alpha+y| + K|\beta+p|, \end{aligned}$$

where $\tilde{H}(\eta,y,p,X) := H(\eta,-y,-p,-X)$. Then, by convexity and uniform Lipschitz continuity of H, we can easily check (3.1) as well as all properties of a and f stated in this lemma. □

Now, let us take any complete probability space $(\Omega,\mathcal{F},P)$ with d-dimensional Brownian motion $W = (W_t)_{0\le t\le T}$ and set $W_{t,s} := W_s - W_t$, $\mathcal{F}_{t,s} := \sigma(W_{t,r}; t \le r \le s) \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of all P-null sets. We fix an arbitrary point $(t,x) \in [0,T]\times\mathbb{R}^d$ and consider the following system of forward-backward stochastic differential equations (FBSDEs) :

$$\begin{cases} dX_s^{\varepsilon,\zeta} = \sigma(\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s)\,dW_{t,s}, \\ -dY_s^{\varepsilon,\zeta} = f(\varepsilon^{-1}X_s^{\varepsilon,\zeta},Y_s^{\varepsilon,\zeta},Z_s^{\varepsilon,\zeta},\zeta_s)\,ds - \sigma^*(\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s)Z_s^{\varepsilon,\zeta}\,dW_{t,s}, \\ X_t^{\varepsilon,\zeta} = x, \qquad Y_T^{\varepsilon,\zeta} = h(X_T^{\varepsilon,\zeta}), \end{cases} \tag{3.3}$$

where $\zeta : \Omega \times [t,T] \longrightarrow E$ is a given $\mathcal{F}_{t,s}$-adapted control process satisfying the integrability condition $E\int_0^T |\zeta_s|^2 ds < \infty$. Notice that $\sigma = (\sigma^{ij}) : \mathbb{R}^d \times E \longrightarrow \mathbb{R}^{d\times d}$ is a bounded and Lipschitz continuous function such that $\sum_{k=1}^d (\sigma^{ik}\sigma^{jk})(x,\zeta) = 2a^{ij}(x,\zeta)$. Then, we can show the following theorem (see [11], Theorem 1.3 for its proof).

Theorem 3.2. *Let $u^\varepsilon(t,x)$ be a solution of (1.1), and let $(X^{\varepsilon,\zeta}, Y^{\varepsilon,\zeta}, Z^{\varepsilon,\zeta})$ be a unique pair of solutions to (3.3). Then, we have the following representation formula*

$$u^\varepsilon(t,x) = \inf_\zeta Y_t^{\varepsilon,\zeta}\,, \tag{3.4}$$

where the infimum is taken over all admissible control processes.

4 Probabilistic approach to homogenization.

The aim of this section is to give the sketch of proof of Theorem 1.2. To avoid heavy notation, we set

$$v(\eta, s, x) := v(\eta, u^0(s,x), u_x^0(s,x), u_{xx}^0(s,x))\,, \qquad (s,x) \in [0,T] \times \mathbb{R}^d\,,$$

where $v(\eta, y, p, X)$ is a solution to the cell problem (1.3) with (y,p,X) frozen. Then, by applying Ito's formula to $Y_s^{\varepsilon,\zeta} - u^0(s, X_s^{\varepsilon,\zeta}) - \varepsilon^2 v(\varepsilon^{-1}X_s^{\varepsilon,\zeta}, s, X_s^{\varepsilon,\zeta})$, we can expect the convergence of the form

$$\lim_{\varepsilon\downarrow 0} \inf_\zeta E[\,Y_s^{\varepsilon,\zeta} - u^0(s, X_s^{\varepsilon,\zeta}) - \varepsilon^2 v(\varepsilon^{-1}X_s^{\varepsilon,\zeta}, s, X_s^{\varepsilon,\zeta})\,] = 0\,.$$

Unfortunately, the above observation cannot be justified since v is not differentiable with respect to (s,x). Nevertheless, for each fixed (s,x), v is twice differentiable in η. So, we can prove the convergence by using local arguments (i.e. by freezing the *slow* variable $(s, X_s^{\varepsilon,\zeta})$).

For this purpose, we first set $\overline{Y}_s^{\varepsilon,\zeta} := Y_s^{\varepsilon,\zeta} - u^0(s, X_s^{\varepsilon,\zeta})\,,\ \overline{Z}_s^{\varepsilon,\zeta} := Z_s^{\varepsilon,\zeta} - u_x^0(s, X_s^{\varepsilon,\zeta})\,.$ Then, $(\overline{Y}_s^{\varepsilon,\zeta}, \overline{Z}_s^{\varepsilon,\zeta})$ satisfies the following linear BSDE :

$$\begin{cases} -d\overline{Y}_s^{\varepsilon,\zeta} = \{\overline{\theta}(s, X_s^{\varepsilon,\zeta}, \varepsilon^{-1}X_s^{\varepsilon,\zeta}, \zeta_s) + \phi_s^{\varepsilon,\zeta}\overline{Y}_s^{\varepsilon,\zeta} + \psi_s^{\varepsilon,\zeta}\overline{Z}_s^{\varepsilon,\zeta}\}\,ds \\ \qquad\qquad\qquad -\sigma^*(\varepsilon^{-1}X_s^{\varepsilon,\zeta}, \zeta_s)\overline{Z}_s^{\varepsilon,\zeta}\,dW_{t,s}\,, \\ \overline{Y}_T^{\varepsilon,\zeta} = 0\,, \end{cases}$$

where the function $\overline{\theta} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times E \longrightarrow \mathbb{R}$ and bounded processes $(\phi_s^{\varepsilon,\zeta})$, $(\psi_s^{\varepsilon,\zeta})$ are defined as follows :

$$\begin{aligned} \overline{\theta}(s,x,\eta,\zeta) &:= \overline{H}(u^0(s,x), u_x^0(s,x), u_{xx}^0(s,x)) \\ &\qquad + a^{ij}(\eta,\zeta)u_{x^ix^j}^0(s,x) + f(\eta, u^0(s,x), u_x^0(s,x), \zeta)\,, \\ \phi_s^{\varepsilon,\zeta} &:= \int_0^1 f_y(\varepsilon^{-1}X_s^{\varepsilon,\zeta}, \lambda Y_s^{\varepsilon,\zeta} + (1-\lambda)u^0(s, X_s^{\varepsilon,\zeta}), u_x^0(s, X_s^{\varepsilon,\zeta}), \zeta_s)\,d\lambda\,, \\ \psi_s^{\varepsilon,\zeta} &:= \int_0^1 f_p(\varepsilon^{-1}X_s^{\varepsilon,\zeta}, Y_s^{\varepsilon,\zeta}, \lambda Z_s^{\varepsilon,\zeta} + (1-\lambda)u_x^0(s, X_s^{\varepsilon,\zeta}), \zeta_s)\,d\lambda\,. \end{aligned}$$

From the general theory of linear BSDEs, $\overline{Y}_t^{\varepsilon,\zeta}$ can be written as

$$\overline{Y}_t^{\varepsilon,\zeta} = E\int_t^T \Gamma_s^{\varepsilon,\zeta}\,\overline{\theta}(s, X_s^{\varepsilon,\zeta}, \varepsilon^{-1}X^{\varepsilon,\zeta}, \zeta_s)\,ds\,, \tag{4.1}$$

where $\Gamma_s^{\varepsilon,\zeta} > 0$ is an $\mathcal{F}_{t,s}$-adapted process such that

$$\sup_{\varepsilon>0} E \sup_{t\le s\le T} |\Gamma_s^{\varepsilon,\zeta}|^q < \infty\,, \qquad \forall q \ge 1\,.$$

Note that it is possible to write down this process explicitly (see [11]).

Next, for any given $N \in \mathbb{N}$ and $n > 0$, we consider the N-partition of the time duration

$$(t,T] = \bigcup_{j=0}^{N-1} \Delta_j := \bigcup_{j=0}^{N-1} (s_j, s_{j+1}]\,, \qquad s_j = t + \frac{j\,(T-t)}{N}\,, \quad j = 0,1,\ldots,N\,,$$

and the disjoint decomposition of the ball $B(n) := \{\, x \in \mathbb{R}^d;\ |x| \le n \,\} = \bigcup_{k=1}^{N'} B_k$, where $B_k \in \mathcal{B}(\mathbb{R}^d)$ $(k = 1,2,\ldots,N')$ are constructed by a finite open covering of $B(n)$ with radius less than $1/(2n)$. Then, we have the following lower estimate of $\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta}$.

Proposition 4.1. *For every $q > 1$ and $x_k \in B_k$ $(k = 1,\ldots,N')$, we have*

$$\begin{aligned}\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta} &+ C\,(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta}\,) \\ &> -\sup_\zeta \left| \sum_{j=0}^{N-1}\sum_{k=1}^{N'} E\int_{s_j}^{s_{j+1}} \Gamma_s^{\varepsilon,\zeta} 1_{\{X_{s_j}^{\varepsilon,\zeta}\in B_k\}} V(s_j, x_k, \varepsilon^{-1}X_s^{\varepsilon,\zeta}, \zeta_s)\,ds \right|\,,\end{aligned} \tag{4.2}$$

where $\delta > 0$ is the exponent appearing in Theorem 1.2 and we have set $V(s,x,\eta,\zeta) := \sum_{i,j=1}^d a^{ij}(\eta,\zeta) v_{\eta^i\eta^j}(\eta,s,x)$.

Sketch of the proof. We set

$$A_n = \{\, \sup_{t\le s\le T} |X_s^{\varepsilon,\zeta}| \le n \,\}\,, \quad B_{n,N} = \{\, \max_{0\le j\le N-1} \sup_{s\in\Delta_j} |X_s^{\varepsilon,\zeta} - X_{s_j}^{\varepsilon,\zeta}| \le 1/n \,\}\,.$$

Then, for each fixed $q > 1$, Chebyshev's inequality yields

$$P(A_n^c) \le \frac{C(1+|x|)^{2q}}{n^{2q}}\,, \qquad P(B_{n,N}^c) \le \sum_{j=0}^{N-1} Cn^{2q}\,|s_{j+1} - s_j|^q = \frac{Cn^{2q}(T-t)^q}{N^{q-1}}\,, \tag{4.3}$$

where $C > 0$ is a universal constant independent of n, N, ε, etc. Since $u^0 \in C^{1+\delta/2,2+\delta}([0,T]\times\mathbb{R}^d)$, we can also show that

$$|\overline{\theta}(s,x,\eta,\zeta) - \overline{\theta}(s',x',\eta,\zeta)| \le C\{|s-s'|^{\delta/2} + |x-x'|^{\delta}\}\,. \tag{4.4}$$

Now, for each $k = 1, \ldots, N'$, we set $C_{j,k} := \{ X^{\varepsilon,\zeta}_{s_j} \in B_k \}$ and fix $x_k \in B_k$ arbitrarily. Then, taking into account that $A_n \subset \bigcup_{k=1}^{N'} C_{j,k}$ and $C_{j,k} \cap C_{j,k'} = \emptyset$ (if $k \neq k'$), for every $s \in \Delta_j$, we have

$$\begin{aligned}
&\overline{\theta}(s, X^{\varepsilon,\zeta}_s, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) \\
&\quad = \sum_{k=1}^{N'} 1_{A_n \cap B_{n,N}} 1_{C_{j,k}} \{ \overline{\theta}(s, X^{\varepsilon,\zeta}_s, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) - \overline{\theta}(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) \} \\
&\qquad + \sum_{k=1}^{N'} 1_{A_n \cap B_{n,N}} 1_{C_{j,k}} \overline{\theta}(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) + 1_{(A_n \cap B_{n,N})^c} \overline{\theta}(s, X^{\varepsilon,\zeta}_s, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) .
\end{aligned}$$

Furthermore, since $\overline{\theta}(s, x, \eta, \zeta) \geq -V(s, x, \eta, \zeta)$,

$$\begin{aligned}
&\overline{\theta}(s, X^{\varepsilon,\zeta}_s, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) \\
&\quad \geq \sum_{k=1}^{N'} 1_{A_n \cap B_{n,N}} 1_{C_{j,k}} \{ \overline{\theta}(s, X^{\varepsilon,\zeta}_s, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) - \overline{\theta}(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) \} \\
&\qquad - \sum_{k=1}^{N'} 1_{C_{j,k}} V(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) + \sum_{k=1}^{N'} 1_{(A_n \cap B_{n,N})^c} 1_{C_{j,k}} V(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) \\
&\qquad - 1_{(A_n \cap B_{n,N})^c} V(s, X^{\varepsilon,\zeta}_s, \varepsilon^{-1} X^{\varepsilon,\zeta}_s, \zeta_s) \\
&\quad =: \Psi^j_1(s) - \Psi^j_2(s) + \Psi^j_3(s) - \Psi^j_4(s) .
\end{aligned}$$

By plugging the right-hand side into (4.1),

$$\overline{Y}^{\varepsilon,\zeta}_t \geq \sum_{j=0}^{N-1} E \int_{s_j}^{s_{j+1}} \Gamma^{\varepsilon,\zeta}_s \{ \Psi^j_1(s) - \Psi^j_2(s) + \Psi^j_3(s) - \Psi^j_4(s) \} \, ds .$$

We estimate the right-hand side one by one. Remark fist that on the event $A_n \cap B_{n,N} \cap C_{j,k}$,

$$|X^{\varepsilon,\zeta}_s - x_k| \leq |X^{\varepsilon,\zeta}_s - X^{\varepsilon,\zeta}_{s_j}| + |X^{\varepsilon,\zeta}_{s_j} - x_k| \leq 2/n \qquad \text{for all } \ s \in \Delta_j .$$

Then, by (4.4), we have

$$\begin{aligned}
\Big| E \int_{\Delta_j} \Gamma^{\varepsilon,\zeta}_s \Psi^j_1(s) \, ds \Big| &\leq K' E \Big[\int_{\Delta_j} \Gamma^{\varepsilon,\zeta}_s 1_{A_n \cap B_{n,N}} \sum_{k=1}^{N'} 1_{C_{j,k}} \{ |s - s_j|^{\delta/2} + |X^{\varepsilon,\zeta}_s - x_k|^{\delta} \} \, ds \Big] \\
&\leq C \, (s_{j+1} - s_j) \, (|s_{j+1} - s_j|^{\delta/2} + n^{-\delta}) .
\end{aligned}$$

By using (4.3), the inequalities

$$\begin{aligned}
\Big| E \int_{\Delta_j} \Gamma^{\varepsilon,\zeta}_s \Psi^j_4(s) \, ds \Big| &\leq |V|_{L^\infty} (s_{j+i} - s_j) \sqrt{P((A_n \cap B_{n,N})^c)} \sqrt{E \sup_{t \leq s \leq T} |\Gamma^{\varepsilon,\zeta}_s|^2} \\
&\leq C \, |V|_{L^\infty} (s_{j+i} - s_j) \, \{ n^{-q} (1 + |x|)^q + n^q N^{(1-q)/2} \}
\end{aligned}$$

hold, from which we obtain

$$\left| E \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta} \Psi_3^j(s)\, ds \right| \le C\, |V|_{L^\infty} (s_{j+i} - s_j) \{ n^{-q}(1+|x|)^q + n^q N^{(1-q)/2} \}$$

since $\sum_{k=1}^{N'} 1_{C_{j,k}} |V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)| \le |V|_{L^\infty} < \infty$. Thus, we have

$$\text{(4.5)} \quad \overline{Y}_t^{\varepsilon,\zeta} \ge -\sum_{j=0}^{N-1} E \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta} \Psi_2^j(s)\, ds - C\,(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta}),$$

where $C > 0$ depends only on $|x|$, δ, K', T and $|V|_{L^\infty}$. The above inequality doesn't depend on the choice of (ζ_s). Hence, we have completed the proof. □

We can also prove the inequality of the opposite direction in the same manner (the proof will by a little more complicated since we have to choose a "nice" control according to the parameter $\varepsilon > 0$. See [11], Proposition 2.5).

Proposition 4.2. *Let N, $N' \in \mathbb{N}$, $n > 0$, $q > 1$, etc. be the same parameters as in Proposition 4.1. Then,*

$$\text{(4.6)} \quad \inf_{\zeta} \overline{Y}_t^{\varepsilon,\zeta} - C\,(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta})$$
$$< \sup_{\zeta} \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E \int_{s_j}^{s_{j+1}} \Gamma_s^{\varepsilon,\zeta} 1_{\{X_{s_j}^{\varepsilon,\zeta} \in B_k\}} V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \right| .$$

Lemma 4.3. *For every N, $N' \in \mathbb{N}$, we have*

$$\sup_{\zeta} \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E \int_{\Delta_j} 1_{C_{j,k}} \Gamma_s^{\varepsilon,\zeta} V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \right| \le (\varepsilon + \varepsilon^2)\, C + \varepsilon^2 C N .$$

Sketch of the proof. We set $\overline{v}^{j,k}(\eta) = v(\eta, s_j, x_k) - v(0, s_j, x_k)$. Clearly, $\overline{v}_\eta^{j,k}(\eta) = v_\eta(\eta, s_j, x_k)$, $\overline{v}_{\eta\eta}^{j,k}(\eta) = v_{\eta\eta}(\eta, s_j, x_k)$. Thus, by Ito's formula,

$$\begin{aligned}
&\Gamma_{s_{j+1}}^{\varepsilon,\zeta} \overline{v}^{j,k}(\varepsilon^{-1} X_{s_{j+1}}^{\varepsilon,\zeta}) - \Gamma_{s_j}^{\varepsilon,\zeta} \overline{v}^{j,k}(\varepsilon^{-1} X_{s_j}^{\varepsilon,\zeta}) \\
&= \frac{1}{\varepsilon^2} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, ds + \frac{1}{\varepsilon} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, (\sigma^* \overline{v}_\eta^{j,k})(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, dW_{t,s} \\
&\quad + \frac{1}{\varepsilon} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, \sigma(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, \psi_s^{\varepsilon,\zeta} \cdot \overline{v}_\eta^{j,k}(\varepsilon^{-1} X_s^{\varepsilon,\zeta})\, ds \\
&\quad + \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1} X_s^{\varepsilon,\zeta})\, \psi_s^{\varepsilon,\zeta}\, dW_{t,s} + \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1} X_s^{\varepsilon,\zeta})\, \phi_s^{\varepsilon,\zeta}\, ds .
\end{aligned}$$

Remark here that each of stochastic integral terms appearing in the right-hand side is a $\mathcal{F}_{t,s}$-martingale and $C_{j,k} \in \mathcal{F}_{s_j}$. Taking expectation of both sides, we have

$$
\begin{aligned}
E \int_{\Delta_j} & 1_{C_{j,k}} \Gamma_s^{\varepsilon,\zeta} V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \\
&= -\varepsilon\, E\Big[1_{C_{j,k}} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta} \sigma(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, \psi_s^{\varepsilon,\zeta} \cdot \overline{v}_\eta^{j,k}(\varepsilon^{-1} X_s^{\varepsilon,\zeta})\, ds \Big] \\
&\quad - \varepsilon^2 E\Big[1_{C_{j,k}} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta} \overline{v}^{j,k}(\varepsilon^{-1} X_s^{\varepsilon,\zeta})\, \phi_s^{\varepsilon,\zeta}\, ds \Big] \\
&\quad + \varepsilon^2 E\, 1_{C_{j,k}} \{ \Gamma_{s_{j+1}}^{\varepsilon,\zeta} \overline{v}^{j,k}(\varepsilon^{-1} X_{s_{j+1}}^{\varepsilon,\zeta}) - \Gamma_{s_j}^{\varepsilon,\zeta} \overline{v}^{j,k}(\varepsilon^{-1} X_{s_j}^{\varepsilon,\zeta}) \}\,.
\end{aligned}
$$

Thus, we can deduce the desired inequality by summing up over all j, k, and taking supremum over all controls. □

The proof of Theorem 1.2. From Propositions 4.1, 4.2 and Lemma 4.3, we obtain the following estimate :

$$
\big| \inf_{\zeta} \overline{Y}_t^{\varepsilon,\zeta} \big| \le C(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta} + \varepsilon + \varepsilon^2 + \varepsilon^2 N)\,,
$$

where $C > 0$ may depend on $T > 0$ and $|x|$ but is independent of N, n, $q > 1$ and $\varepsilon > 0$.

Fix arbitrarily $\gamma_1,\ \gamma_2 > 0$ and define $n \in \mathbb{R}_+$ and $N \in \mathbb{N}$ by

$$
n := \varepsilon^{-\gamma_1}\,, \quad N := [\varepsilon^{-\gamma_2}] + 1\,.
$$

Then,

$$
\big| \inf_{\zeta} \overline{Y}_t^{\varepsilon,\zeta} \big| \le C(\varepsilon^{\gamma_1 q} + \varepsilon^{\gamma_2 (q-1)/2 - \gamma_1 q} + \varepsilon^{\delta\gamma_2/2} + \varepsilon^{\delta\gamma_1} + \varepsilon + \varepsilon^2 + \varepsilon^{2-\gamma_2})\,, \tag{4.7}
$$

from which we get the following inequality :

$$
\big| \inf_{\zeta} \overline{Y}_t^{\varepsilon,\zeta} \big| \le C\, \varepsilon^{F(\gamma_1,\gamma_2,q)}\,,
$$

where $F(\gamma_1,\gamma_2,q) := \min\{ \gamma_2(q-1)/2 - \gamma_1 q\,,\ \delta\gamma_1\,,\ 2 - \gamma_2 \}$. By straightforward computation, for each fixed $q > 1$,

$$
\begin{aligned}
F_{\max}(q) :=& \max\{ F(\gamma_1,\gamma_2,q)\,;\ 0 < \gamma_1 < (q-1)\gamma_2/2q\,, \quad 0 < \gamma_2 < 2 \} \\
=& \frac{2\delta(q-1)}{2q + \delta + \delta q}\,.
\end{aligned}
$$

Since the last term is increasing with respect to q and converges to $2\delta/(\delta+2)$ as $q \to +\infty$, we finally obtain

$$\big|\inf_{\zeta} \overline{Y}_t^{\varepsilon,\zeta}\big| \le \lim_{q\to+\infty} C\,\varepsilon^{F_{\max}(q)} \le C\,\varepsilon^{\frac{2\delta}{2+\delta}}\,.$$

We have completed the proof of Theorem 1.2. □

Remark 4.4. If v and u^0 are sufficiently smooth (e.g. $v(\eta, y, p, X) \in C^2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d})$ and $u^0(t,x) \in C_b^{2,4}([0,T]\times\mathbb{R}^d)$), then the local argument we used above is not necessary and the rate of convergence can be improved. In fact, let us consider the case where the Hamiltonian H is linear with respect to (y,p,X) :

$$H(\eta, y, p, X) := -\sum_{i,j=1}^{d} a^{ij}(\eta)X_{ij} - \sum_{i=1}^{d} b^i(\eta)p_i - c(\eta)y\,.$$

Then, the corresponding FBSDE can be written as

$$\begin{cases} dX_s^\varepsilon = b(\varepsilon^{-1}X_s^\varepsilon)\,ds + \sigma(\varepsilon^{-1}X_s^\varepsilon)\,dW_{t,s}\,, & X_t^\varepsilon = x\,, \\ -dY_s^\varepsilon = c(\varepsilon^{-1}X_s^\varepsilon)\,Y_s^\varepsilon\,ds - \sigma^*(\varepsilon^{-1}X_s^\varepsilon)\,Z_s^\varepsilon\,dW_{t,s}\,, & Y_T^\varepsilon = h(X_T^\varepsilon)\,, \end{cases}$$

where we have set $\sigma\sigma^* = 2a$. Then, it is well known that the effective Hamiltonian $\overline{H}$ in (1.2) is characterized by

$$\overline{H}(\eta, y, p, X) := -\sum_{i,j=1}^{d} \overline{a}^{ij}X_{ij} - \sum_{i=1}^{d} \overline{b}^i p_i - \overline{c}\,y\,,$$

$$\overline{g} = \int_{[0,1)^d} g(\eta)m(\eta)\,d\eta\,, \qquad g = a^{ij},\ b^i,\ c\,,$$

where $m(\eta)\,d\eta$ is the invariant measure on $[0,1)^d$ associated with the differential operator $L := a^{ij}(\eta)\partial_{x^i}\partial_{x^j}$.

Now let $v = v(\eta, y, p, X)$ be a unique solution of the cell problem (1.3) such that $v(0, y, p, X) = 0$. Then, v satisfies

$$v(\eta, \lambda_1\Theta_1 + \lambda_2\Theta_2) = \lambda_1 v(\eta, \Theta_1) + \lambda_2 v(\eta, \Theta_2)\,, \quad \forall\lambda_i \in \mathbb{R}\,,\ \ \Theta_i = (y_i, p_i, X_i)\,,\ i = 1,2.$$

In particular, v is infinitely differentiable with respect to (y,p,X).

Now, let u^0 be a solution to (1.2) and we assume that $u^0 \in C_b^{2,4}([0,T]\times\mathbb{R}^d)$. Then, by Ito's formula, we can easily see

$$\big|\,Y_s^\varepsilon - u^0(s, X_s^\varepsilon) - \varepsilon^2 v(\varepsilon^{-1}X_s^\varepsilon, s, X_s^\varepsilon)\,\big| \le C(\varepsilon + \varepsilon^2)\,,$$

which is (formally) the case where $\delta = 2$ in Theorem 1.2

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