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HJB equations in Hilbert spaces related to optimal control of stochastic Navier-Stokes equations

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1 Introduction

In this note we discuss recent results on Hamilton-Jacobi-Bellman (HJB) equations associated with optimal control of stochastic Navier-Stokes equations. Such equations appear in feedback control theory of fluid mechanics and have potential utility in several important applications [20, 26]. Here we will also show how they can be applied to establish a Large Deviation result for stochastic Navier-Stokes equations.

Not much is known about equations of this type. In [17, 25] first order HJB equations associated to control of deterministic Navier-Stokes equations were considered and existence and uniqueness of viscosity solutions have been proved. Kolmogorov equations for stochastic Navier-Stokes equations have been studied by Komech and Vishik (see [32] and the references therein), more recently by Flandoli and Gozzi [14] for the two-dimensional stochastic Navier-Stokes equations, and by Da Prato and Debussche [7] for the three-dimensional case. Only existence of strict and mild solutions has been proved in [7]. A semilinear equation associated to a special optimal control problem has been investigated by Da Prato and Debussche in [6] from the point of view of mild solutions using an exponential change of variables that reduced the equation to a more treatable one. We approach the problem from the point of view of viscosity solutions. Even though we only discuss the theory of semilinear equations the results can be easily extended to fully nonlinear equations by rather standard existing techniques.

We will consider an optimal control problem for the 2-dimensional stochastic Navier-Stokes (SNS) equations with periodic boundary conditions. Let $U=[0,L]\times[0,L]$, and let $\nu>0$. We define the spaces

$$\mathbf{H}= \text{ the closure of } \left\{\mathbf{x}\in \mathbf{H}^1_p\left(U;\mathbb{R}^2\right), \operatorname{div}\mathbf{x}=0, \int_U \mathbf{x}=0\right\} \text{ in } \mathbf{L}^2\left(U;\mathbb{R}^2\right),$$

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$$\mathbf{V} = \left\{ \mathbf{x} \in \mathbf{H}^1_p\left(U;\mathbb{R}^2\right), \operatorname{div}\mathbf{x} = 0, \int_U \mathbf{x} = 0 \right\},$$

where for an integer $k \geq 1$, $\mathbf{H}^k_p(U;\mathbb{R}^2)$ is the space of $\mathbb{R}^2$ valued functions $\mathbf{x}$ that are in $\mathbf{H}^k_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ and such that $\mathbf{x}(y+Le_i) = \mathbf{x}(y)$ for every $y \in \mathbb{R}^2$ and $i = 1,2$. We will denote by $\langle\cdot,\cdot\rangle$, and $\|\cdot\|$ respectively the inner product and the norm in $\mathbf{L}^2(U;\mathbb{R}^2)$. The space $\mathbf{H}$ inherits the same inner product and norm. Let $\mathbf{P}_H$ be the orthogonal projection in $\mathbf{L}^2(U;\mathbb{R}^2)$ onto $\mathbf{H}$. Define $\mathbf{A}\mathbf{x} = -\mathbf{P}_H\Delta\mathbf{x}$, with the domain $D(\mathbf{A}) = \mathbf{H}^2_p(U;\mathbb{R}^2) \cap \mathbf{V}$, and we denote $\mathbf{B}(\mathbf{x},\mathbf{y}) = \mathbf{P}_H[(\mathbf{x}\cdot\nabla)\mathbf{y}]$. For $\gamma = 1,2$ we denote by $\mathbf{V}_\gamma$ the domain of $\mathbf{A}^{\frac{\gamma}{2}}, D(\mathbf{A}^{\frac{\gamma}{2}})$, equipped with the norm

$$\|\mathbf{x}\|_\gamma = \|\mathbf{A}^{\frac{\gamma}{2}}\mathbf{x}\|.$$

The space $\mathbf{V}_1$ coincides with $\mathbf{V}$. We recall that because of the periodic boundary conditions (see for instance [30])

$$\langle B(\mathbf{x},\mathbf{x}),\mathbf{A}\mathbf{x}\rangle = 0 \quad \text{for } \mathbf{x} \in \mathbf{V}_2.$$

Let $\mathbf{Q} : \mathbf{H} \to \mathbf{H}$ be an operator that is self-adjoint, $\mathbf{Q} \geq 0$, and

$$\operatorname{tr}(\mathbf{Q}) < +\infty.$$

Denote $\mathbf{Q}_1 = \mathbf{A}^{\frac{1}{2}}\mathbf{Q}\mathbf{A}^{\frac{1}{2}}$. We will require throughout the paper that

$$\operatorname{tr}(\mathbf{Q}_1) < +\infty. \tag{1.1}$$

We also assume throughout the paper that $\mathbf{\Theta}$ is a complete, separable metric space.

We will work with the canonical sample space for the controlled SNS equations. For $0 \leq t \leq T$ let $\Omega_t = \{\omega \in C([t,T];\mathbf{H}) : \omega(t) = 0\}$. The Wiener process $\mathbf{W}$ is defined on Ω_t by $\mathbf{W}(\tau)(\omega) = \omega(\tau)$. Let $\mathcal{F}_{t,s}$ be the σ-algebra generated by paths of $\mathbf{W}$ up to time s in Ω_t, and let $\mathbb{P}_t$ be the Wiener measure on Ω_t (see [8, 23]). Then $(\Omega_t, \mathcal{F}_{t,T}, \mathcal{F}_{t,s}, \mathbb{P}_t)$ is the canonical sample space for the Wiener process $\mathbf{W}$.

We say that $\mathbf{a}(\cdot) : [t,T] \times \Omega_t \to \mathbf{\Theta}$, is an admissible control on $[t,T]$ if $\mathbf{a}(\cdot)$ is an $\mathcal{F}_{t,s}$-progressively measurable process. The set of all admissible controls on $[t,T]$ will be denoted by $\mathcal{U}_t$.

Given an initial time $t \geq 0$ and the terminal time $T \geq t$ the abstract controlled stochastic Navier-Stokes equations describe the evolution of the velocity vector field $\mathbf{X} : [t,T] \times U \times \Omega \to \mathbb{R}^2$ that satisfies the Ito type equation

$$\begin{cases} d\mathbf{X}(s) = \left(-\nu\mathbf{A}\mathbf{X}(s) - \mathbf{B}\left(\mathbf{X}(s),\mathbf{X}(s)\right) + \mathbf{f}\left(s,\mathbf{a}(s)\right)\right)ds + \mathbf{Q}^{\frac{1}{2}}d\mathbf{W}(s) & \text{in } (t,T] \times \mathbf{H}, \\ \mathbf{X}(t) = \mathbf{x} \in \mathbf{H}, \end{cases} \tag{1.2}$$

where $\mathbf{f} : [0,T] \times \mathbf{\Theta} \to \mathbf{V}$. We refer to [24, 32, 18] for results on such SNS equations.

The optimal control problem consists in the minimization, over all controls $\mathbf{a}(\cdot) \in \mathcal{U}_t$, of a cost functional

$$J(t,\mathbf{x};\mathbf{a}(\cdot)) = \mathbb{E}\left\{ \int_t^T l(s,\mathbf{X}(s),\mathbf{a}(s))ds + g(\mathbf{X}(T)) \right\}.$$

The dynamic programming approach to the control problem involves the study of the value function

$$\mathcal{V}(t,\mathbf{x}) = \inf_{\mathbf{a}(\cdot)\in\mathcal{U}_t} J(t,\mathbf{x};\mathbf{a}(\cdot))$$

and its to characterization as a solution of the associated Hamilton-Jacobi-Bellman partial differential equation

$$\begin{cases} u_t + \frac{1}{2}\mathrm{tr}\left(\mathbf{Q}D^2u\right) - \langle \nu\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{x}), Du\rangle + \inf_{\mathbf{a}\in\Theta}\left\{\langle \mathbf{f}(t,\mathbf{a}), Du\rangle + l(t,\mathbf{x},\mathbf{a})\right\} = 0 \\ u(T,\mathbf{x}) = g(\mathbf{x}) \qquad \text{for } (t,\mathbf{x}) \in (0,T)\times\mathbf{H}. \end{cases} \tag{1.3}$$

The idea then is to use the HJB equation to construct optimal feedback controls, obtain verification theorems, do numerical computations. This program has not been carried out yet in infinite dimensions. Theoretical results on (1.3) are its first step.

Our theory applies to a more general class of infinite dimensional HJB equations

$$\begin{cases} u_t + \frac{1}{2}\mathrm{tr}\left(\mathbf{Q}D^2u\right) - \langle \nu\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{x}), Du\rangle + F(t,\mathbf{x},Du) = 0 \\ u(T,\mathbf{x}) = g(\mathbf{x}) \qquad \text{for } (t,\mathbf{x}) \in (0,T)\times\mathbf{H}. \end{cases} \tag{1.4}$$

The results presented here in Sections 2 and 3 have been obtained in [18].

2 Viscosity solutions of the HJB equation

The definition of viscosity solution is slightly different from the one given in [18] where only special radial functions were used as test functions. They both give the same theory but the current one is easier to work with when it comes to Perron's method and relaxed limits. They borrow some ideas from [21], [3] and [5]. Ishii in [21] used a convex function defined only on a proper subspace as part of test functions to deal with unboundedness in the equation. His definition has been successfully used in [25] to treat some equations that may come from control of deterministic Navier-Stokes equations. A similar idea based on the use of energy functions also appeared recently in [12]. Crandall and Lions in [5] and Cannarsa and Tessitore in [3] used special radial functions and the coercivity of the unbounded operators in the state equations to improve the regularity of points where maxima and minima occur in the definition of viscosity solution. This idea has been successfully adapted to second order equations in [16, 18, 19], and also in [17]. Our definition merges these two approaches.

Definition 2.1 *A function ψ is a test function for equation (1.4) if $\psi = \varphi + \delta(t)h(\|\mathbf{x}\|_1)$, where*

- *$h \in C^2([0,+\infty))$ and is such that $h'(0) = 0, h''(0) > 0, h'(r) > 0$ for $r \in (0,+\infty)$.*
- *$\varphi \in C^{1,2}((0,T)\times\mathbf{H})$, and is such that $\varphi, \varphi_t, D\varphi, D^2\varphi$ are uniformly continuous on $[\epsilon, T-\epsilon]\times\mathbf{H}$ for every $\epsilon > 0$,*

- *$\delta \in C^1((0,T))$ is such that $\delta > 0$ on $(0,T)$.*

The function $h(\mathbf{x}) = h(\|\mathbf{x}\|_1)$ is not Fréchet differentiable in $\mathbf{H}$. Therefore the terms $\langle \mathbf{Ax} + \mathbf{B}(\mathbf{x},\mathbf{x}), Dh(\mathbf{x})\rangle$ and $\mathrm{tr}(\mathbf{Q}D^2h(\mathbf{x}))$ have to be understood properly. From the point of view of the HJB equation it would be best to set it up in the space $(0,T)\times\mathbf{V}$. However because of the associated control problem we want to keep $\mathbf{H}$ as our reference space. We define

$$Dh(\mathbf{x}) = \frac{h'(\|\mathbf{x}\|_1)}{\|\mathbf{x}\|_1)}\mathbf{Ax},$$

$$D^2h(\mathbf{x}) = h'(\|\mathbf{x}\|_1)\left(\frac{\mathbf{A}}{\|\mathbf{x}\|_1} - \frac{\mathbf{Ax}\otimes\mathbf{Ax}}{\|\mathbf{x}\|_1^3}\right) + h''(\|\mathbf{x}\|_1)\frac{\mathbf{Ax}\otimes\mathbf{Ax}}{\|\mathbf{x}\|_1^2}$$

and in what follows we will write

$$D\psi = D\varphi + Dh, \quad D^2\psi = D^2\varphi + D^2h$$

even though this is a slight abuse of notation since as we mentioned before h is not Fréchet differentiable in $\mathbf{H}$.

We assume that $F : [0,T]\times\mathbf{V}\times\mathbf{H}\to\mathbb{R}$.

Definition 2.2 *A function $u : (0,T)\times\mathbf{V}\to\mathbb{R}$ that is weakly sequentially upper-semicontinuous (respectively, lower-semicontinuous) on $(0,T)\times\mathbf{V}$ is called a viscosity subsolution (respectively, supersolution) of (1.4) if for every test function ψ, whenever $u-\psi$ has a local maximum (respectively $u+\psi$ has a local minimum) in the topology of $|\cdot|\times\|\cdot\|_1$ at $(t,\mathbf{x})$ then $\mathbf{x}\in\mathbf{V_2}$ and*

$$\psi_t(t,\mathbf{x}) + \frac{1}{2}\mathrm{tr}(\mathbf{Q}D^2\psi(t,\mathbf{x})) - \langle \nu\mathbf{Ax} + B(\mathbf{x},\mathbf{x}), D\psi(t,\mathbf{x})\rangle + F(t,\mathbf{x},D\psi(t,\mathbf{x})) \geq 0$$

(respectively

$$-\left(\psi_t(t,\mathbf{x}) + \frac{1}{2}\mathrm{tr}(\mathbf{Q}D^2\psi(t,\mathbf{x})) - \langle \nu\mathbf{Ax} + B(\mathbf{x},\mathbf{x}), D\psi(t,\mathbf{x})\rangle\right) + F(t,\mathbf{x},-D\psi(t,\mathbf{x})) \leq 0.)$$

A function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

It can be shown (see [22] for such an argument) that the maxima and minima in the above definition can be assumed to be strict and global. Moreover, if we control the growth of u at infinity we can control the growth of h at infinity. We also remark that even though $\|\mathbf{x}\|_1$ is not differentiable at 0, the function $h(\|\mathbf{x}\|_1)\in C^2(V)$.

3 Existence and uniqueness of solutions

We begin with a comparison theorem for equation (1.4). This result has been proved in [18] (see Theorem 5.2 there).

Theorem 3.1 *Suppose that there exist a modulus of continuity ω, and moduli ω_r such that for every $r > 0$ we have*

$$|F(t,\mathbf{x},\mathbf{p}) - F(t,\mathbf{y},\mathbf{p})| \le \omega_r(\|\mathbf{x}-\mathbf{y}\|_1) + \omega(\|\mathbf{x}-\mathbf{y}\|_1\|\mathbf{p}\|), \quad \text{if } \|\mathbf{x}\|_1, \|\mathbf{y}\|_1 \le r, \tag{3.1}$$

$$|F(t,\mathbf{x},\mathbf{p}) - F(t,\mathbf{x},\mathbf{q})| \le \omega((1+\|\mathbf{x}\|_1)\|\mathbf{p}-\mathbf{q}\|), \tag{3.2}$$

$$|F(t,\mathbf{x},\mathbf{p}) - F(s,\mathbf{x},\mathbf{p})| \le \omega_r(|t-s|), \quad \text{if } \|\mathbf{x}\|_1, \|\mathbf{y}\|_1, \|\mathbf{p}\|_1 \le r, \tag{3.3}$$

$$|g(\mathbf{x}) - g(\mathbf{y})| \le \omega_r(\|\mathbf{x}-\mathbf{y}\|), \quad \text{if } \|\mathbf{x}\|_1, \|\mathbf{y}\|_1 \le r. \tag{3.4}$$

Let $u, v : (0,T) \times \mathbf{V} \to \mathbb{R}$ be respectively a viscosity subsolution, and a viscosity supersolution of (1.4). Let

$$u(t,\mathbf{x}),\ -v(t,\mathbf{x}),\ |g(\mathbf{x})| \le C(1+\|\mathbf{x}\|_1^k) \tag{3.5}$$

for some $k > 0$, and let

$$\begin{cases} (i) & \lim_{t\uparrow T}(u(t,\mathbf{x}) - g(\mathbf{x}))^+ = 0 \\ (ii) & \lim_{t\uparrow T}(v(t,\mathbf{x}) - g(\mathbf{x}))^- = 0 \end{cases} \tag{3.6}$$

uniformly on bounded subsets of $\mathbf{V}$. Then $u \le v$ on $(0,T] \times \mathbf{V}$.

The next theorem gives existence of solutions of the HJB equation (1.3). It is taken from [18] (see Proposition 6.2 and Theorem 6.3 there) where it was shown for a different definition of viscosity solution. To prove it with the new definition we just have to follow the proof of Theorem 6.3 in [18] and do some minor technical modifications due to the introduction of the new test functions h.

Theorem 3.2 *Suppose that*
(i) The functions $l : \mathbf{V} \times \Theta \to \mathbb{R}$, and $g : \mathbf{H} \to \mathbb{R}$ are continuous and there exist $k \ge 0$ and for every $r > 0$ a modulus σ_r such that for every $t \in [0,T], \mathbf{a}(\cdot) \in \mathcal{U}_t$

$$|l(\mathbf{x},\mathbf{a})|, |g(\mathbf{x})| \le C(1+\|\mathbf{x}\|_1^k) \tag{3.7}$$

$$|l(\mathbf{x},\mathbf{a}) - l(\mathbf{y},\mathbf{a})| \le \sigma_r(\|\mathbf{x}-\mathbf{y}\|_1) \quad \text{if } \|\mathbf{x}\|_1, \|\mathbf{y}\|_1 \le r, \tag{3.8}$$

$$|g(\mathbf{x}) - g(\mathbf{y})| \le \sigma_r(\|\mathbf{x}-\mathbf{y}\|) \quad \text{if } \|\mathbf{x}\|_1, \|\mathbf{y}\|_1 \le r. \tag{3.9}$$

(ii) The function $\mathbf{f} : [0,T] \times \Theta \to \mathbf{V}$ is bounded, continuous, and $\mathbf{f}(\cdot,\mathbf{a})$ is uniformly continuous, uniformly for $\mathbf{a} \in \Theta$.

Then for every $r > 0$ there exists a modulus ω_r such that $\mathcal{V}$ satisfies

$$|\mathcal{V}(t_1,\mathbf{x}) - \mathcal{V}(t_2,\mathbf{y})| \le \omega_r(|t_1-t_2| + \|\mathbf{x}-\mathbf{y}\|) \tag{3.10}$$

for $t_1, t_2 \in [0,T]$ and $\|\mathbf{x}\|_1, \|\mathbf{y}\|_1 \le r$, and

$$|\mathcal{V}(t,\mathbf{x})| \le C(1+\|\mathbf{x}\|_1^k). \tag{3.11}$$

Moreover the value function $\mathcal{V}$ is the unique viscosity solution of the HJB equation (1.3) satisfying (3.5) and (3.6).

4 Discontinuous viscosity solutions and Perron's method

For a function v we denote

$$v^*(t,\mathbf{x}) = \limsup\{u(s,\mathbf{y}) : s \to t, \|\mathbf{y}-\mathbf{x}\| \to 0\},$$

$$v_*(t,\mathbf{x}) = \liminf\{u(s,\mathbf{y}) : s \to t, \|\mathbf{y}-\mathbf{x}\| \to 0\}.$$

Definition 4.1 *A locally bounded function $u : (0,T)\times\mathbf{V} \to \mathbb{R}$ is a discontinuous viscosity subsolution of (1.4) if whenever $(u-\delta(\cdot)h(\|\cdot\|_1))^* - \varphi$ has a local maximum in the topology of $|\cdot|\times\|\cdot\|$ at a point $(t,\mathbf{x})$ for test functions $\varphi, \delta(s)h(\|\mathbf{y}\|_1)$ such that*

$$u(s,\mathbf{y}) - \delta(s)h(\|\mathbf{y}\|_1) \to -\infty \quad \text{as } \|\mathbf{y}\|_1 \to \infty \quad \text{locally uniformly in } s \tag{4.1}$$

then $\mathbf{x} \in \mathbf{V}_2$ and

$$\psi_t(t,\mathbf{x}) + \frac{1}{2}\mathrm{tr}(\mathbf{Q}D^2\psi(t,\mathbf{x})) - \langle \nu\mathbf{A}\mathbf{x} + B(\mathbf{x},\mathbf{x}), D\psi(t,\mathbf{x})\rangle + F(t,\mathbf{x},D\psi(t,\mathbf{x})) \geq 0, \tag{4.2}$$

where $\psi(s,\mathbf{y}) = \varphi(s,\mathbf{y}) + \delta(s)h(\|\mathbf{y}\|_1)$.

A locally bounded function $u : (0,T)\times\mathbf{V} \to \mathbb{R}$ is a discontinuous viscosity supersolution of (1.4) if whenever $(u+\delta(\cdot)h(\|\cdot\|_1))_* - \varphi$ has a local minimum in the topology of $|\cdot|\times\|\cdot\|$ at a point $(t,\mathbf{x})$ for test functions $\varphi, \delta(s)h(\|\mathbf{y}\|_1)$ such that

$$u(s,\mathbf{y}) + \delta(s)h(\|\mathbf{y}\|_1) \to +\infty \quad \text{as } \|\mathbf{y}\| \to \infty \quad \text{locally uniformly in } s \tag{4.3}$$

then $\mathbf{x} \in \mathbf{V}_2$ and

$$\psi_t(t,\mathbf{x}) + \frac{1}{2}\mathrm{tr}(\mathbf{Q}D^2\psi(t,\mathbf{x})) - \langle \nu\mathbf{A}\mathbf{x} + B(\mathbf{x},\mathbf{x}), D\psi(t,\mathbf{x})\rangle + F(t,\mathbf{x},D\psi(t,\mathbf{x})) \leq 0, \tag{4.4}$$

where $\psi(s,y) = \varphi(s,y) - \delta(s)h(\|\mathbf{y}\|_1)$.

A discontinuous viscosity solution of (1.4) is a function which is both a discontinuous viscosity subsolution and a discontinuous viscosity supersolution.

The maxima and minima in the above definition can be assumed to be global and strict in the $|\cdot|\times\|\cdot\|$ norm.

The following comparison theorem can be proved by an argument similar to the proof of Theorem 3.1.

Theorem 4.2 *Let the assumptions of Theorem 3.1 be satisfied. Let $u, v : (0,T)\times\mathbf{V} \to \mathbb{R}$ be respectively a discontinuous viscosity subsolution, and a discontinuous viscosity supersolution of (1.4) satisfying (3.5) for some $k > 0$, and (3.6). Then $u \leq v$ on $(0,T]\times\mathbf{V}$. Moreover, if $u = v$ then u is locally uniformly continuous in $|\cdot|\times\|\cdot\|$ norm on bounded subsets of $[\epsilon,T]\times\mathbf{V}$ for every $\epsilon > 0$.*

Discontinuous viscosity solutions allow us to implement a version of Perron's method for equations (1.4). The result below will be proved in a future publication.

Theorem 4.3 *Let (3.1), (3.2), (3.3), and (3.4) hold, let* $F : [0,T] \times \mathbf{V} \times \mathbf{H} \to \mathbb{R}$ *be continuous in the* $|\cdot| \times \|\cdot\|_1 \times \|\cdot\|_{-1}$ *norm, and let*

$$|g(\mathbf{x})| \leq C(1 + \|\mathbf{x}\|^k) \text{ for some } C > 0. \tag{4.5}$$

Let u_0 *be a discontinuous viscosity subsolution of (1.4), and* v_0 *be a discontinuous viscosity supersolution of (1.4) such that*

$$u_0, -v_0 \leq C(1 + \|\mathbf{x}\|^k) \text{ for some } C > 0 \tag{4.6}$$

and

$$\lim_{t \uparrow T} \{|u_0(t,\mathbf{x}) - g(\mathbf{x})| + |v_0(t,\mathbf{x}) - g(\mathbf{x})|\} = 0$$
$$\text{uniformly on bounded sets of } \mathbf{V}. \tag{4.7}$$

Then the function

$$u(t,\mathbf{x}) = \sup\{w(t,\mathbf{x}) : u_0 \leq w \leq v_0, w \text{ is a discontinuous viscosity subsolution of (1.4)}\}$$

is the unique viscosity solution of (1.4) in the sense of Definition 2.2 satisfying (4.6) and (4.7). Moreover u *is locally uniformly continuous in* $|\cdot| \times \|\cdot\|$ *norm on bounded subsets of* $[\epsilon, T] \times \mathbf{V}$ *for every* $\epsilon > 0$.

5 Half-relaxed limits

Half-relaxed limits were introduced in the context of viscosity solutions in finite dimensions by Barles and Perthame [2]. Unfortunately, due to lack of local compactness, this method is not easily extendable to infinite dimensions and in fact it may not work (see [1, 28]). An infinite dimensional version of the Barles-Perthame procedure has been proposed in [22]. The method we present here is an adaptation to the current situation of the method introduced in [22]. The reselts of this section will appear in [29]

Let $F_n : [0,T] \times \mathbf{V} \times \mathbf{H} \to \mathbb{R}$ be continuous, locally bounded uniformly in n, and degenerate elliptic. Denote

$$F^+(t,\mathbf{x},\mathbf{p}) = \lim_{m\to\infty} \sup\{F_n(s,\mathbf{y},\mathbf{q}) : n \geq m, |t-s| + \|\mathbf{x}-\mathbf{y}\|_1 + \|\mathbf{p}-\mathbf{q}\|_{-1} \leq \frac{1}{m}\}$$

and

$$F_-(t,\mathbf{x},\mathbf{p}) = \lim_{m\to\infty} \inf\{F_n(s,\mathbf{y},\mathbf{q}) : n \geq m, |t-s| + \|\mathbf{x}-\mathbf{y}\|_1 + \|\mathbf{p}-\mathbf{q}\|_{-1} \leq \frac{1}{m}\}.$$

Theorem 5.1 *Let $k \geq 0$ and let $\epsilon_n \to 0$. Let u_n be viscosity subsolutions, (respectively, supersolutions) in the sense of Definition 2.2 of*

$$(u_n)_t + \frac{\epsilon_n}{2}\mathrm{tr}\left(\mathbf{Q}D^2u_n\right) - \langle \nu\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{x}), Du_n\rangle + F_n(t,\mathbf{x},Du_n) = 0 \quad in\ (0,T)\times\mathbf{V}. \tag{5.1}$$

Then the function

$$u^+(t,\mathbf{x}) = \lim_{m\to\infty} \sup\{u_n(s,\mathbf{y}) : n \geq m, |t-s| + \|\mathbf{x}-\mathbf{y}\|_1 \leq \frac{1}{m}\}$$

(respectively,

$$u_-(t,\mathbf{x}) = \lim_{m\to\infty} \inf\{u_n(s,\mathbf{y}) : n \geq m, |t-s| + \|\mathbf{x}-\mathbf{y}\|_1 \leq \frac{1}{m}\})$$

is a discontinuous viscosity subsolution (respectively, supersolution) of

$$(u^+)_t - \langle \nu\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{x}), Du^+\rangle + F_-(t,\mathbf{x},Du^+) = 0 \quad in\ (0,T)\times\mathbf{V}$$

(respectively,

$$(u_-)_t - \langle \nu\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{x}), Du_-\rangle + F^+(t,\mathbf{x},Du_-) = 0 \quad in\ (0,T)\times\mathbf{V}.)$$

If in addition $F^+ = F_- =: F$, F and g satisfy the assumptions of Theorem 3.1,

$$|u_n(t,\mathbf{x})| \leq C(1 + \|\mathbf{x}\|_1^k)$$

for some $C, k \geq 0$, and

$$\lim_{t\uparrow T} |u_n(t,\mathbf{x}) - g(\mathbf{x})| = 0 \tag{5.2}$$

uniformly on bounded subsets of $\mathbf{V}$, uniformly in n, then $u^+ = u_- =: u$, u is the unique viscosity solution of

$$\begin{cases} u_t - \langle \nu\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{x}), Du\rangle + F(t,\mathbf{x},Du) = 0 \\ u(T,\mathbf{x}) = g(\mathbf{x}) \qquad\qquad for\ (t,\mathbf{x}) \in (0,T)\times\mathbf{V} \end{cases} \tag{5.3}$$

in the sense of Definition 2.2 satisfying (3.5) and (3.6), and u is locally uniformly continuous in $|\cdot| \times \|\cdot\|$ norm on bounded subsets of $[\epsilon,T]\times\mathbf{V}$ for every $\epsilon > 0$. Moreover the functions u_n converge to u pointwise on $(0,T]\times\mathbf{V}$ and the convergence is uniform on bounded subsets of $[\epsilon,T]\times\mathbf{V}_\gamma$ for every $\gamma > 1, \epsilon > 0$.

The method of half-relaxed limits works in more generality when operators $\mathbf{A}$ and $\mathbf{B}$ are allowed to vary with n. In particular, a version of Theorem 5.1 holds if the u_n are viscosity solutions of appropriately defined finite dimensional equations with $\mathbf{A}$ and $\mathbf{B}$ replaced by properly defined "Galerkin-type" approximations $\mathbf{A}_n$ and $\mathbf{B}_n$.

6 Large deviation principle

The theory of large deviations deals with certain asymptotic properties of random variables. Here are the basic definitions and facts that we will need later.

In this section H is a separable Hilbert space, and $\{X_n\}$ is a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in H.

Definition 6.1 *A function $I : H \to [0, +\infty]$ with compact level sets is called a rate function on H.*

Definition 6.2 *We say that the sequence $\{X_n\}$ satisfies the large deviation principle on H with rate function I if the following conditions hold.*

- *For every closed subset F of H*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{X_n \in F\} \le -I(F) := -\inf_{x\in F} I(x).$$

- *For every open subset G of H*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{X_n \in G\} \ge -I(G).$$

Definition 6.3 *The sequence $\{X_n\}$ is called exponentially tight if for every $M \in (0, +\infty)$ there exists a compact set $K \subset H$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\{X_n \in H \setminus K\} \le -M.$$

Theorem 6.4 (Bryc) *(see [9], Theorem 1.3.8) Let $\{X_n\}$ be exponentially tight and let for every $g \in C_b(H)$ (the space of continuous and bounded functions on H) the (Laplace) limit*

$$\Lambda(g) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{-ng(X_n)} \tag{6.1}$$

exist. Then the sequence $\{X_n\}$ satisfies the large deviation principle on H with rate function

$$I(x) = -\inf_{g\in C_b(H)} \{g(x) + \Lambda(g)\}.$$

7 Large deviations for SNS equations

We will show how to apply the theory of viscosity solutions to establish large deviationn principle at single times for solutions of SNS equations with small noise intensities.

The use of viscosity solutions in large deviation type problems is not new in finite dimensions (see for instance [15] and the references therein). Unfortunately in infinite dimensional spaces such techniques were not available until quite recently. A few years ago Feng and Kurtz [13] proposed a very general framework for large deviations based on viscosity solutions in abstract spaces. However they only use viscosity solutions of the limiting first-order equation and the rest of the method relies on convergence of nonlinear semigroups and stochastic analysis making it rather cumbersome to apply. Similar approach is used in [10, 11] for infinite dimensional diffusions. We propose a purely PDE based technique that relies on our method of half-relaxed limits and, in the spirit, is a generalization of the finite dimensional method.

We refer the reader to [8] for some results on large deviations for infinite dimensional processes, to [4, 27] for results on large deviations for SNS, and to [9, 31] for the general theory of large deviations.

Let $0 < t < T$. We want to establish the large deviation principle on $\mathbf{H}$ for the processes $\mathbf{X}_n(T)$, where the $\mathbf{X}_n(\cdot)$ satisfy SNE

$$\begin{cases} d\mathbf{X}_n(s) = (-\nu\mathbf{A}\mathbf{X}_n(s) - \mathbf{B}(\mathbf{X}_n(s),\mathbf{X}_n(s)) + \mathbf{f}(s))\,ds + \frac{1}{\sqrt{n}}\mathbf{Q}^{\frac{1}{2}}d\mathbf{W}(s) & \text{for } t < s \le T, \\ \mathbf{X}_n(t) = \mathbf{x} \in \mathbf{V}, \end{cases} \tag{7.1}$$

where $\mathbf{W}$ is the canonical Wiener process defined on the canonical sample space $(\Omega_t, \mathcal{F}_{t,T}, \mathcal{F}_{t,s}, \mathbb{P}_t)$ as described in Section 1. We assume that $\mathrm{tr}(\mathbf{Q}_1) < +\infty$ and that $\mathbf{f} : [0,T] \to \mathbf{V}$ is continuous. Under these assumptions (7.1) has a unique strong solution, see [24, 32, 18]

We want to use Theorem 6.4 to establish the large deviation result, i.e. we need toshow the Laplace limit for $\{\mathbf{X}_n(T)\}$ and its exponential tightness. We sketch below the main steps of this procedure referring the readers to [29] for the details.

Imitating the proofs of Theorems 3.1 and 3.2 it can be shown that if $g \in C_b(H)$ then the function

$$u_n(t,\mathbf{x}) = -\frac{1}{n}\log \mathbb{E}e^{-ng(\mathbf{X}_n(T))}$$

is the unique viscosity solution of

$$\begin{cases} (u_n)_t + \frac{1}{2n}\mathrm{tr}\,(\mathbf{Q}D^2u_n) - \frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}Du_n\|^2 - \langle \nu\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{x}), Du_n\rangle + \langle \mathbf{f}(t), Du_n\rangle = 0, \\ u_n(T,\mathbf{x}) = g(\mathbf{x}) \quad \text{in } (0,T)\times\mathbf{V}. \end{cases} \tag{7.2}$$

(Unfortunately this fact cannot be derived directly from Theorems 3.1 and 3.2.) In particular it can be proved that comparison theorem for discontinuous viscosity solutions holds for (7.2) and the limit equation (7.3).

The Laplace limit (6.1) is equivalent to showing that $u_n(t,\mathbf{x})$ converge. This can be accomplished with the help of half-relaxed limits. The limit equation is

$$\begin{cases} u_t - \frac{1}{2}\|\mathbf{Q}^{\frac{1}{2}}Du\|^2 - \langle \nu\mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{x}), Du\rangle + \langle \mathbf{f}(t), Du\rangle = 0 & \text{in } (0,T)\times\mathbf{V}, \\ u(T,\mathbf{x}) = g(\mathbf{x}) & \text{for } \mathbf{x}\in\mathbf{V}. \end{cases} \tag{7.3}$$

As we have mentioned above comparison holds for discontinuous viscosity solutions of this equation. Moreover using comparison and appropriate barrier functions one can show that the functions u_n are uniformly bounded and satisfy (5.2). Therefore all assumptions of Theorem 5.1 are satisfied and so $u_n(t,\mathbf{x}) \to u(t,\mathbf{x})$, where u is the unique viscosity solution of (7.3).

It now remains to show exponential tightness of the processes $\mathbf{X}_n(T)$. This is an easy consequence of estimates of exponential moments of the $\mathbf{X}_n(T)$. It follows from Theorem 3.1 of [32] (page 395) that there exist constants $C_1, C_2 \geq 0$, where C_2 also depends on $\|\mathbf{x}\|_1$, such that

$$\mathbb{E}e^{C_1 n\|\mathbf{X}_n(T)\|_1^2} \leq C_2.$$

Let now $M > 0$ and let $K = \{\mathbf{x} : \|\mathbf{x}\|_1 \leq R\}$. The set K is compact in $\mathbf{H}$. The above estimate implies that

$$e^{C_1 n R^2}\mathbb{P}\{\|\mathbf{X}_n(T)\|_1 > R\} \leq C_2$$

which yields

$$\frac{1}{n}\log \mathbb{P}\{\|\mathbf{X}_n(T)\|_1 > R\} \leq \frac{1}{n}\log C_1 - C_1 R^2 \leq -M$$

if R is big enough. This gives the exponential tightness.

Therefore Theorem 6.4 establishes the large deviation principle on $\mathbf{H}$ for the processes $\mathbf{X}_n(T)$. We also obtain an explicit representation formula for the rate function I in terms of the function u. This formula can be further expanded if we interpret u as the value function of a deterministic optimal control problem.

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