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STRONG COMPARISON PRINCIPLE OF SEMICONTINUOUS VISCOSITY SOLUTIONS TO SOME NONLINEAR ELLIPTIC EQUATIONS

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This note is based on a joint work with Professor Yoshikazu Giga of University of Tokyo [5].

1. Introduction

In this note we are concerned with a nonlinear elliptic equation of the form

$$F(Du(x), D^2u(x)) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is a domain in $\mathbf{R}^n$. The function $u : \Omega \to \mathbf{R}$ is unknown and F is a given function. Here Du and D^2u denote, respectively, the gradient of u and the Hessian of u in variables x. The function $F : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ is continuous, where $\mathbf{S}^n$ denotes the space of all real $n \times n$ symmetric matrices.

Our goal is to establish the strong comparison principle for viscosity solutions of (1.1). By the strong comparison principle we mean the principle that a subsolution u agrees with a supersolution v in Ω if $u \leq v$ in Ω and $u(x_0) = v(x_0)$ at some point $x_0 \in \Omega$. A typical example of $F = F(p, X)$ we consider here is of the form

$$F(p,X) = -\mathrm{trace}\left\{\left(I - \frac{p \otimes p}{1+|p|^2}\right) X\right\}$$

so that (1.1) becomes

$$-\sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \quad \text{in} \quad \Omega. \tag{1.2}$$

The equation (1.2) is called the (graph) minimal surface equation.

We shall establish the strong comparison principle for some elliptic equations including the graph minimal surface equation. A solution we consider here is a viscosity solution which may not be continuous.

It is well known that for linear elliptic equations the strong comparison principle is equivalent to the strong maximum principle since linear combinations of solutions are still solutions. The strong maximum principle of classical solutions for linear elliptic equations has been well studied (cf. [12], [7]). There

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are some results on the strong maximum principle for weak solution (distribution sense) of quasilinear possibly degenerate equations (see e.g. [14], [11], [7]). For viscosity solutions Kawohl and Kutev [10] prove the strong maximum principle under continuity condition for subsolutions or supersolutions. Later, Bardi and Da Lio [1] improve this result without continuity assumption for solutions and they establish the strong maximum principle for a large class including the graph minimal surface equation and even for degenerate elliptic equations, for example, for the p-Laplacian equation with $p > 1$. For a level set equation of the minimal surface equation a special form of a strong maximum principle for level sets of solutions was established by [6]. On the other hand, there are a few results on the strong comparison principle for nonlinear elliptic equations. For classical solutions E. Hopf established it as a corollary of the strong maximum principle (see e.g. [11]). For viscosity solutions Trudinger [13] proved the strong comparison principle for locally strictly elliptic equations with Lipschitz cotinuity assumptions on subsolutions and supersolutions. He only state results in [13, Remark 3.2] without the proof. For definitions and the theory of viscosity solutions we refer to the review paper [4] and a nice introductory book [9].

After this work was completed, we were informed of a recent work of Ishii and Yoshimura [8] who proved the strong comparison principle for semicontinous viscosity solutions of uniformly elliptic equations. Their proof is very similar to ours [5].

2. Assumptions on $F = F(p, X)$

We list the basic assumptions on $F = F(p, X)$.

(F1) $$F : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R} \quad \text{is continuous,}$$

where $\mathbf{S}^n$ denotes the space of all real $n \times n$ symmetric matrices.
We will use the following notations;

$$USC(\Omega) = \{\text{upper semicontinuous functions } u : \Omega \to \mathbf{R}\},$$
$$LSC(\Omega) = \{\text{lower semicontinuous functions } u : \Omega \to \mathbf{R}\}.$$

We next describe of a class of equations for which we shall establish a strong comparison principle. We shall introduce a notion called *coercive*.

Definition 2.1 We say that a function $f : \mathbf{R} \times \mathbf{S}^n \to \mathbf{R}$ is *coercive* if for each $M > 0$ there exists a function $\beta = \beta_M : [0, \infty) \to \mathbf{R}$ satisfying

(i) $$\beta \text{ is continuous on } [0, \infty) \text{ and } \lim_{\sigma \to +\infty} \beta(\sigma) = +\infty,$$

(ii) $$f(p, S) \geq b\beta(N)$$

for all $S \in \mathbf{S}^n$, $b > 0$, $N > 0$ and $p \in \mathbf{R}^n$ satisfying

$$S \leq bI, \quad {}^t\mu S \mu \leq -bN, \quad |p| \leq M$$

for some $\mu \in S^{n-1}$. Here I denotes the identity matrix, μ is a row vector, ${}^t\mu$ is the transposed vector of μ and S^{n-1} denotes the set of unit vectors in $\mathbf{R}^n$. The function β is called a *bound* for f.

We shall assume a kind of ellipticity and a Lipschitz continuity of derivative variables p for $F = F(p, X)$.

(F2) There exists a coercive function f such that

$$F(p, X) - F(p, -Y) \geq f(p, X + Y)$$

for all $p \in \mathbf{R}^n$ and for all $X, Y \in \mathbf{S}^n$.

(F3) Let M and K be positive. There exists a positive constant $L_{M,K}$ such that

$$|F(q, X) - F(\tilde{q}, X)| \leq L_{M,K}|q - \tilde{q}|$$

for all $q, \tilde{q} \in \mathbf{R}^n$ satisfying $|q|, |\tilde{q}| \leq M$ and for all $X \in \mathbf{S}^n$ satisfying $||X|| \leq K$, where $||X||$ denotes the operator norm of X as a self-adjoint operator on $\mathbf{R}^n$.

We shall see that the locally strictly ellipticity implies (F2). Let us recall a definition of locally strictly elliptic equations. Let M be positive. If there exists constant $0 < \lambda_M \leq \Lambda_M$ such that

$$\lambda_M \text{ trace } Y \leq F(p, X - Y) - F(p, X) \leq \Lambda_M \text{ trace } Y \tag{2.1}$$

for all $p \in \mathbf{R}^n$ satisfying $|p| \leq M$, $X, Y \in \mathbf{S}^n$ and $Y \geq 0$, then we call $F = F(p, X)$ is *locally strictly elliptic*. It turns out that (F2) is fulfilled if $F = F(p, X)$ is locally strictly elliptic (Proposition 2.4).

Remark 2.2 Of course (F2) is fulfilled if $F = F(p, X)$ is uniformly elliptic. The definition of uniformly ellipic is the following. If there exists constant $0 < \lambda \leq \Lambda$ such that

$$\lambda \text{ trace } Y \leq F(p, X - Y) - F(p, X) \leq \Lambda \text{ trace } Y$$

for all $p \in \mathbf{R}^n$ and for all $X, Y \in \mathbf{S}^n$ and $Y \geq 0$, then we call $F = F(p, X)$ is *uniformly elliptic.*

Let λ_j $(1 \leq j \leq n)$ be the set of eigenvalues of X including multiplicity. Let e_j be eigenvectors of λ_j. We may assume that $\{e_j\}_{j=1}^n$ is an orthogonal basis of $\mathbf{R}^n$. Thus we have a spectral decomposition

$$X = \sum_{j=1}^n \lambda_j e_j \otimes e_j.$$

We define the plus part X_+ and minus part X_- by

$$X_+ := \sum_{j=1}^n (\lambda_j)_+ e_j \otimes e_j, \qquad X_- := \sum_{j=1}^n (\lambda_j)_- e_j \otimes e_j,$$

where $(\lambda_j)_+ := \max\{0, \lambda_j\}$ and $(\lambda_j)_- := \min\{0, \lambda_j\}$.

Proposition 2.3 *Let F be locally strictly elliptic. Then we have*

$$F(p,X) - F(p,-Y) \geq -\Lambda_M \text{ trace } (X+Y)_+ - \lambda_M \text{ trace } (X+Y)_-.$$

As we prove later (see Section 4) $-\Lambda_M$ trace $(X+Y)_+ - \lambda_M$ trace $(X+Y)_-$ is a coercive function for locally strictly elliptic equations. Thus by Proposition 2.3 we have

Proposition 2.4 *Let F be locally strictly elliptic. Then F satisfies (F2).*

Remark 2.5 After this conference Professor Hitoshi Ishii pointed out that for $S \in \mathbf{S}^n$ the function $-\Lambda_M$ trace $S_+ - \lambda_M$ trace S_- is a Pucci operator. For the definition of Pucci operators we refer to the book [3]. Moreover, if F satisfies (F1) and (F2) then F is locally strictly elliptic. Of course under the same assumptions F is uniformly elliptic.

Remark 2.6 (i) For the strong comparison principle one cannot remove (F2) completely. In fact the strong comparison principle fails for a first order equation $|\frac{du}{dx}| = 1$ on $(-1,1)$ which does not fulfill (F2). Indeed there are solutions $u_1(x) = x+1$ and $u_2(x) = -|x|+1$. We observe that $u_1(x) \geq u_2(x)$ on $(-1,1)$ and $u_1(x) \equiv u_2(x)$ on $(-1,0)$. However, $u_1(x) > u_2(x)$ on $(0,1)$. This means that the strong comparison principle is not fulfilled.

(ii) One would like to weaken the Lipschitz condition of $F(p,X)$ in p. For example, we consider

$$|F(q,X) - F(\tilde{q},X)| \leq L_{M,K}|q-\tilde{q}|^m$$

for some m $(0 < m < 1)$. However, for such F we have a counterexample (cf. [2]). Let $0 < m < 1, R > 0$,

$$F(p,X) = -\text{trace } X - |p|^m, \quad \Omega = B(0,R) \subset \mathbf{R}^n.$$

For this F equation (1.1) becomes

$$-\Delta u - |Du|^m = 0 \quad \text{in} \quad B(0,R). \tag{2.2}$$

In [1] there is a comment to (2.2). For (2.2) the strong minimum principle holds, however the strong maximum principle does not hold. In fact, $u(x) = C(R^k - |x|^k)$ with $k = (2-m)/(1-m)$, $C = k^{-1}(n+k-2)^{1/(m-1)}$ is a non constant solution to (2.2) (cf. [2]). This means for (2.2) the strong comparison principle does not hold. So we cannot remove the Lipschitz continuity assumption completely. If we would like to weaken the assumption (F3), we have to consider another way.

Remark 2.7 A typical example is the minimal surface equation

$$-\sqrt{1+|Du|^2}\text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \qquad \text{in} \quad \Omega. \tag{2.3}$$

For this equation $F = F(p, X)$ is given by

$$F(p,X) = -\mathrm{trace}\left\{\left(I - \frac{p \otimes p}{1+|p|^2}\right) X\right\}. \tag{2.4}$$

This $F = F(p, X)$ is locally strictly elliptic. Indeed, for (2.4) elliptic constants are taken by $\lambda_M = 1/(1+M^2)$, $\Lambda_M = 1$. An extended equation of (2.3) is the following.

(2.5)

$$-\mathrm{trace}\left\{A(Du)\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right) D^2u \left(I - \frac{Du \otimes Du}{1+|Du|^2}\right)\right\} = 0 \quad \text{in} \quad \Omega,$$

where $A(p) \in \mathbf{S}^n$ satisfies $A(p) \geq 0$ for all $p \in \mathbf{R}^n$. We shall assume that for each $M > 0$ there exists a constant $C = C(M) > 0$ such that $A(p) \leq CI$ for all $p \in \mathbf{R}^n$ satisfying $|p| \leq M$. We also assume a lower bound such that there exists $c > 0$ satisfying $cI \leq A(p)$ for all $p \in \mathbf{R}^n$. For (2.5) $F = F(p, X)$ is given by

$$F(p,X) = -\mathrm{trace}\left\{A(p)R_pXR_p\right\}, \quad R_p := I - \frac{p \otimes p}{1+|p|^2}. \tag{2.6}$$

This $F = F(p, X)$ is also locally strictly elliptic. Elliptic constants are taken by $\lambda_M = c/(1+M^2)^2$, $\Lambda_M = C$.

3. Main results

Our main theorem is an extension of the strong comparison theorem to viscosity subsolutions and supersolutions to (1.1). In this note we simplified our original proof [5] according to advice of Professor Hitoshi Ishii. Exactly we simplified our proofs from Lemma 3.5.

Theorem 3.1(Strong comparison principle) *Suppose that Ω is a domain in $\mathbf{R}^n$. Assume that F satisfies (F1)–(F3). Let $u \in USC(\Omega)$ and $v \in LSC(\Omega)$ be, respectively, viscosity sub- and supersolutions of (1.1). Assume that $u \leq v$ in Ω and that there exists a point $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$. Then $u \equiv v$ in Ω.*

If v is a constant function in Ω and a constant function is a viscosity solution then Theorem 3.1 gives a strong maximum principle.

We shall prove Theorem 3.1 in several steps. Our proof reflects that of the maximum principle to uniformly elliptic equations in classical sense. Choice of an auxiliary function and some domains in Ω near the point x_0 are very similar to the classical work [12], [7].

Let $a \in \Omega$, $R > 0$,

$$B_0 := (a, R) \subset\subset \Omega, \quad x_0 \in \partial B_0,$$
$$B_1 := B(x_0, \frac{R}{2}) \subset\subset \Omega,$$

where $B(a, R)$ denotes the open ball in $\mathbf{R}^n$ of radius R centered at a. Let for $\gamma > 0$ and $x \in \mathbf{R}^n$

$$z(x) := e^{-\gamma R^2} - e^{-\gamma|x-a|^2}.$$

By definition one observes that

$$\begin{array}{rl} -1 < z(x) < 0 & \text{in } \ B_0, \\ z(x) = 0 & \text{on } \ \partial B_0, \\ 0 < z(x) < 1 & \text{outside } \ \overline{B_0}. \end{array} \tag{3.1}$$

Let $w(x, y)$ be a function on $\Omega \times \Omega$. We set for $(x, y) \in \Omega \times \Omega$ and $\varepsilon, \alpha > 0$,

$$\Phi(x, y) := \varepsilon z(x) + \alpha|x - y|^2,$$
$$\Psi(x, y) := w(x, y) - \Phi(x, y).$$

For proof of Theorem 3.1 we have to study maximum points of $\Psi(x, y)$ on $\overline{B_1} \times \overline{B_1}$ and their values. First we shall consider the value of $\Psi(x, x)$ for $x \in \partial B_1$.

Proposition 3.2 *Let B_0, B_1 and $z(x)$ as stated above. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then*

$$w(x, x) - \varepsilon z(x) < 0 \qquad \text{on} \quad \partial B_1$$

for all $\gamma > 0$ provided that w is upper semicontinuous on $\Omega \times \Omega$, $w(x, x) \leq 0$ for all $x \in \Omega$ and

$$\begin{cases} w(x, x) < 0 & \text{if} \quad x \in \overline{B_0} \setminus \{x_0\}, \\ w(x_0, x_0) = 0. \end{cases}$$

We next study properties of maximum points of $\Psi(x, y)$ on $\overline{B_1} \times \overline{B_1}$.

Proposition 3.3 *Suppose that w be upper semicontinuous on $\Omega \times \Omega$ and that*

$$w(x, x) < 0 \qquad \text{if} \quad x \in \overline{B_0} \setminus \{x_0\},$$
$$w(x_0, x_0) = 0.$$

Let B_0, B_1 and Ψ as stated above and let ε_0 be as in Proposition 3.2. Let $\Psi(x, y)$ attain its maximum at $(x_\alpha, y_\alpha) \in \overline{B_1} \times \overline{B_1}$ for all $0 < \varepsilon < \varepsilon_0$. Then $|x_\alpha - y_\alpha| \to 0$ as $\alpha \to +\infty$; this convergence is uniform in $0 < \varepsilon < \varepsilon_0$ and $\gamma > 0$.

In particular, there exists a point $\hat{x} \in \overline{B_1}$ such that $x_\alpha, y_\alpha \to \hat{x}$ as $\alpha \to +\infty$ by taking a subsequence.

Proposition 3.4 *Assume the same hypotheses of Proposition 3.3. Then there exists $\alpha_0 > 0$ such that if $\alpha > \alpha_0$ then Ψ attains its maximum over $\overline{B_1}$ at an interior point $(x_\alpha, y_\alpha) \in B_1 \times B_1$ for all $0 < \varepsilon < \varepsilon_0$ and $\gamma > 0$.*

Proof. We will show $\hat{x} \in B_1$. Suppose that $\hat{x} \in \partial B_1$. By definition of Ψ and $\Psi(x_\alpha, y_\alpha) \geq 0$ we have

$$w(x_\alpha, y_\alpha) - \varepsilon z(x_\alpha) \geq \Psi(x_\alpha, y_\alpha) \geq 0.$$

Letting $\alpha \to +\infty$ by taking a subsequence we observe that

$$w(\hat{x}, \hat{x}) - \varepsilon z(\hat{x}) \geq 0$$

which contradicts to Proposition 3.2. Thus if $\alpha > 0$ is sufficiently large say $\alpha > \alpha_0$, then $x_\alpha, y_\alpha \in B_1$. □

For the proof of Theorem 3.1 we will use a maximum principle for semicontinuous functions due to Crandall and Ishii [4]. In particular, we shall study several properties on matrices which are useful to calculate matrices appeared in their theory.

Let

$$d(x,\gamma) := 2\varepsilon\gamma e^{-\gamma|x-a|^2},$$
$$B := d(x,\gamma)(I - 2\gamma(x-a) \otimes (x-a)).$$

Lemma 3.5 *For all $0 < \varepsilon \leq 1$ and $N_1 > 0$ there exists $\gamma_0 > 0$ such that if $\gamma > \gamma_0$, then*

(i)
$$B \leq d(x,\gamma)I,$$

(ii)
$${}^t\nu B\nu \leq -d(x,\gamma)|\nu|^2 N_1 \qquad \text{for all} \quad x \in B_1,$$

where ν is an outward normal vector on ∂B_0 at $x_0 \in \partial B_0$ such that $\nu = x_0 - a$.

Proof. (i) This is obvious. (ii) By direct calculation we have

$${}^t\nu B\nu = d(x,\gamma)\{|\nu|^2 - 2\gamma\langle \nu, x-a\rangle^2\}.$$

Note that $\langle \nu, x-a\rangle > 0$ for all $x \in B_1$. For all $N_1 > 0$ there exists $\gamma_0 > 0$ such that if $\gamma > \gamma_0$ then

$$1 - 2\gamma\langle \frac{\nu}{|\nu|}, x-a\rangle^2 \leq -N_1$$

for all $x \in B_1$. Thus for all $N_1 > 0$ there exists $\gamma_0 > 0$ such that if $\gamma > \gamma_0$ then

$${}^t\nu B\nu \leq -d(x,\gamma)|\nu|^2 N_1 \quad \text{for all} \quad x \in B_1$$

□

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We will argue by contradiction. We set $w(x,y) = u(x) - v(y)$ so that w is upper semicontinuous on $\Omega \times \Omega$. Suppose that there would exist a point $x_1 \in \Omega$ such that $u(x_1) < v(x_1)$. By a standard argument there would exist an open ball B_0 with $\overline{B_0} \subset \Omega$ and $x_0' \in \partial B_0$ that satisfies

$$u < v \qquad \text{in} \quad \overline{B_0} \setminus \{x_0'\},$$
$$u(x_0') = v(x_0').$$

We shall replace x_0' with x_0 since $u(x_0) = v(x_0)$. We set $B_0 = B(a, R)$ and $B_1 = B(x_0, \frac{R}{2})$ so that $\overline{B_1} \subset \Omega$. Now we see that all conclusions of Proposition 3.2–3.4 would hold for $\Psi = w - \Phi$ on $\overline{B_1} \times \overline{B_1}$ for sufficiently small ε and sufficiently large α. Proposition 3.4 says that Ψ attains its maximum over $\overline{B_1} \times \overline{B_1}$ at $(x_\alpha, y_\alpha) \in B_1 \times B_1$ for sufficiently small $\varepsilon > 0$ and sufficiently large $\alpha > 0$. In particular,

$$u(x) - v(y) \le u(x_\alpha) - v(y_\alpha) + \Phi(x,y) - \Phi(x_\alpha, y_\alpha)$$

and we observe that

$$u(x) - \varepsilon z(x) - v(y) - \alpha|x-y|^2 \le u(x_\alpha) - \varepsilon z(x_\alpha) - v(y_\alpha) - \alpha|x_\alpha - y_\alpha|^2.$$

Expanding $\alpha|x-y|^2$ at (x_α, y_α) we get

$$\left(\begin{pmatrix} 2\alpha(x_\alpha - y_\alpha) \\ -2\alpha(x_\alpha - y_\alpha) \end{pmatrix}, A \right) \in J^{2,+}((u - \varepsilon z)(x_\alpha) - v(y_\alpha))$$

with

$$A = \begin{pmatrix} 2\alpha I & -2\alpha I \\ -2\alpha I & 2\alpha I \end{pmatrix}.$$

We shall apply the elliptic version of Crandall–Ishii's Lemma [4, Theorem3.2]. We see that for all positive λ, there exists $X, Y \in \mathbf{S}^n$ such that

(i)

$$\begin{aligned} &(2\alpha(x_\alpha - y_\alpha), X) \in \overline{J^{2,+}}((u - \varepsilon z)(x_\alpha)), \\ &(-2\alpha(x_\alpha - y_\alpha), Y) \in \overline{J^{2,+}}(-v(y_\alpha)) \\ &(\Leftrightarrow\ (2\alpha(x_\alpha - y_\alpha), -Y) \in \overline{J^{2,-}}v(y_\alpha)), \end{aligned}$$

(ii)

$$\text{(MI)} \qquad -\left(\frac{1}{\lambda} + \|A\|\right) I_{2n} \le \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le A + \lambda A^2.$$

Here $\overline{J^{2,+}}$ and $\overline{J^{2,-}}$, respectively, denote closure of $J^{2,+}$ and $J^{2,-}$ (cf. [4],[9]). By the definition of elliptic jets $J^{2,+}$ and $\overline{J^{2,+}}$ we see

$$(2\alpha(x_\alpha - y_\alpha) + \varepsilon Dz(x_\alpha), X + \varepsilon D^2 z(x_\alpha)) \in \overline{J^{2,+}}u(x_\alpha).$$

By definition of d and B (see the paragraph just before Lemma 3.5) we obtain identities at $x = x_\alpha$

$$\varepsilon Dz(x_\alpha) = d(x_\alpha, \gamma)(x_\alpha - a), \qquad \varepsilon D^2 z(x_\alpha) = B.$$

Let $\rho(x,\gamma) = d(x,\gamma)(x-a)$ and let $p_\alpha = 2\alpha(x-y)$. Since u is a viscosity subsolution of (1.1), we have

$$F(\rho(x_\alpha, \gamma) + p_\alpha, X + B) \le 0. \tag{3.2}$$

Since v is a viscosity supersolution of (1.1), we have

$$F(p_\alpha, -Y) \ge 0. \tag{3.3}$$

Subtracting (3.3) from (3.2), we get

$$F(\rho(x_\alpha,\gamma)+p_\alpha, X+B) - F(p_\alpha, -Y) \le 0. \tag{3.4}$$

By (F3) we see that

$$\begin{aligned} F(\rho(x_\alpha,\gamma)+p_\alpha, X+B) - F(p_\alpha, X+B) &\ge -L_{M,K}|\rho(x_\alpha,\gamma)| \\ &= -L_{M,K} d(x_\alpha,\gamma)|x_\alpha - a|. \end{aligned}$$

From (MI) we observe that

$$X+Y \le O \quad \text{and} \quad X+Y+B \le B.$$

By (F2) and Lemma 3.5 we observe that

$$F(p_\alpha, X+B) - F(p_\alpha, -Y) \ge f(p_\alpha, X+Y+B) \ge d(x_\alpha,\gamma)\beta(N_1)$$

for all $N_1 > 0$ by taking γ sufficiently large. From (3.4) and $R \le 2|x_\alpha - a| \le 3R$ we see

$$0 \ge d(x_\alpha,\gamma)\beta(N_1) - L_{M,K} d(x_\alpha,\gamma)\frac{3}{2}R.$$

Since $d(x,\gamma) > 0$ we have

$$0 \ge \beta(N_1) - L_{M,K}\frac{3}{2}R.$$

Letting $N_1 \to +\infty$ yields $\beta(N_1) \to +\infty$. This means that there exists N_0 such that if $N_1 > N_0$ then

$$L_{M,K}\frac{3}{2}R < \beta(N_1).$$

We get a contradiction. Now we have completed the proof of Theorem 3.1. □

We also establish the Hopf boundary Lemma.

Theorem 3.6(The Hopf boundary Lemma) *Suppose that Ω is a domain in $\mathbf{R}^n$ and that $x_0 \in \partial\Omega$. Assume that F satisfies (F1), (F2) and (F3). Let $u \in USC(\Omega \cup \{x_0\})$ and $v \in LSC(\Omega \cup \{x_0\})$ be a viscosity subsolution and a supersolution of (1.1), respectively.*

Assume that

$$u \le v \quad \text{in} \quad \Omega \cup \{x_0\}$$

and that there exists a ball $B_0 \subset \Omega$ and a point $x_0 \in \partial B_0$ such that

$$u < v \quad \text{in} \quad \bar{B}_0 \setminus \{x_0\}$$

and $u(x_0) = v(x_0)$.

Then for any $w \in \mathbf{R}^n$ satisfying $\langle w, \nu\rangle < 0$,

$$\limsup_{s\downarrow 0} \frac{(u-v)(x_0+sw) - (u-v)(x_0)}{s} \le c\langle w, \nu\rangle \tag{3.5}$$

with some $c > 0$ independent of w and ν, where ν denotes the outward normal of the boundary ∂B at x_0.

Proof. Let $B_0 = B(a, R)$ and let z be the same function as in (3.1).

To show (3.5) it suffices to prove

$$(3.6) \qquad (u - v - \varepsilon z)(x) \leq 0 \quad \text{in} \quad Z$$

for sufficiently small $\varepsilon > 0$ $(0 < \varepsilon < 1)$ and a domain Z which is neighborhood of x_0 and is contained in B_0. If we have (3.6), we can see

$$(u - v - \varepsilon z)(x_1) \leq (u - v - \varepsilon z)(x_0) \qquad \text{for all} \quad x_1 \in Z.$$

For small $s > 0$ we set $x_1 = x_0 + sw$. Now we observe that

$$\frac{(u - v)(x_0 + sw) - (u - v)(x_0)}{s} \leq \frac{\varepsilon z(x_0 + sw) - \varepsilon z(x_0)}{s}.$$

Since $\langle \nu, w \rangle < 0$, we get

$$\begin{aligned}\limsup_{s \downarrow 0} \frac{(u - v)(x_0 + sw) - (u - v)(x_0)}{s} &\leq \varepsilon \langle Dz(x_0), w \rangle \\ &= 2\varepsilon\gamma e^{-\gamma R^2} \langle \nu, w \rangle < 0.\end{aligned}$$

Thus we obtain (3.5).

It remains to prove (3.6). We argue by contradiction. Let $B_1 = B(x_0, \frac{R}{2})$ and $Z = B_0 \cap B_1$. Suppose that for all ε $(0 < \varepsilon < 1)$ there would exist $\tilde{x} \in \bar{Z}$ such that

$$(u - v - \varepsilon z)(\tilde{x}) = \max_{\bar{Z}} (u - v - \varepsilon z) = \sigma_\varepsilon > 0.$$

On the boundary ∂Z there exits $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ then

$$(3.7) \qquad (u - v - \varepsilon z)(x) \leq 0 \quad \text{on} \quad \partial Z.$$

We see that $\tilde{x} \in Z$ and

$$\max_Z (u - v - \varepsilon z) = \sigma_\varepsilon.$$

Now we set

$$\Phi(x, y) = \varepsilon z(x) + \alpha |x - y|^2,$$

where $\alpha > 0$. We define

$$\Psi(x, y) = u(x) - v(y) - \Phi(x, y).$$

Let Ψ attain its maximum at $(\bar{x}, \bar{y}) \in \bar{Z} \times \bar{Z}$ for all $\varepsilon \in (0, \varepsilon_0)$ and $\alpha > 0$, i.e.,

$$\max_{\bar{Z} \times \bar{Z}} \Psi(x, y) = \Psi(\bar{x}, \bar{y}).$$

We easily see that $\Psi(\bar{x}, \bar{y}) > 0$ since

$$(3.8) \qquad \max_{\bar{Z} \times \bar{Z}} \Psi(x, y) \geq \max_{\bar{Z}} (u - v - \varepsilon)(x) = \sigma_\varepsilon > 0.$$

We observe that

$$M \geq u(\bar{x}) - v(\bar{y}) - \varepsilon z(\bar{x}) > \alpha |\bar{x} - \bar{y}|^2 \geq 0$$

and there exists $\hat{x} \in \bar{Z}$ such that

$$\bar{x}, \bar{y} \to \hat{x} \qquad \text{as} \quad \alpha \to +\infty$$

by taking a subsequence. Note that $\hat{x} \in Z$. Suppose that $\hat{x} \in \partial Z$. By (3.8)

$$u(\bar{x}) - v(\bar{y}) - \varepsilon z(\bar{x}) \geq u(\bar{x}) - v(\bar{y}) - \varepsilon z(\bar{x}) - \alpha|\bar{x} - \bar{y}|^2 \geq \sigma_\varepsilon > 0.$$

Letting $\alpha \to +\infty$ by taking a subsequence we have $(u - v - \varepsilon z)(\hat{x}) > 0$ that contradicts (3.7). Thus if $\alpha > 0$ is sufficiently large say $\alpha > \alpha_0$, then $\bar{x}, \bar{y} \in Z$. Since $u(x) - v(y) \leq u(\bar{x}) - v(\bar{y}) + \Phi(x, y) - \Phi(\bar{x}, \bar{y})$, we argue in the same way as in the proof of Theorem 3.1 with $x_\alpha = \bar{x}$, $y_\alpha = \bar{y}$ to get a contradiction. □

Remark 3.7 Our result roughly speaking that $\partial u/\partial \nu < \partial v/\partial \nu$ at $x = x_0$ if u and v are differentiable at $x = x_0$. For linear elliptic equations the Hopf boundary Lemma implies the strong maximum principle. For some nonlinear degenerate elliptic equations a version of the Hopf boundary Lemma is established by [1, Theorem 1] to prove the strong maximum principle for semicontinuous viscosity solutions. In their situation v is taken a constant.

The proof of Theorem 3.6 is essentialy the same as that of Theorem 3.1. However, u and v may not satisfies the equation (1.1) at $x = x_0$. So we should discuss separately the place where $w - \Phi$ takes maximum values.

4. Key lemma for locally strictly elliptic equations

We give some examples of equation (1.1) and we shall check (F2) holds. Our condition (F2) holds for locally strictly elliptic equations (cf. Proposition 2.3 and 2.4). To verify $-\Lambda \mathrm{trace}(X+Y)_+ - \lambda \mathrm{trace}(X+Y)_-$ which is appeared in Proposition 2.3 is a coercive function, we prepare the following lemma.

Lemma 4.1 *Let $\Lambda \geq \lambda > 0$. Suppose that $b > 0$, $N > 0$, $S \in \mathbf{S}^n$ satisfy*

(4.1) $$S \leq bI,$$

(4.2) $${}^t\mu S \mu \leq -bN \quad \textit{for some} \quad \mu \in S^{n-1},$$

where S^{n-1} denotes the set of unit vector in $\mathbf{R}^n$. Then we have

$$\Lambda \mathit{trace} S_+ + \lambda \mathit{trace} S_- \leq \Lambda(n-1)b - \frac{\lambda N}{n} b.$$

Proof. We may assume that S is a diagonal matrix. Let λ_i $(1 \leq i \leq n)$ be eigenvalues of S. From (4.1) we see $\lambda_i \leq b$ for all i. From (4.2) there exists number ℓ that satisfies $\lambda_\ell \leq -bN/n$. We may assume that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \geq 0 > \lambda_{j+1} \geq \cdots \geq \lambda_{n-1} \geq \lambda_n.$$

From (4.2) at least one eigenvalue is negative. We do not worry about the case all eigenvalues are negative. By the definition of S_+ and S_- we see that

$$\mathrm{trace} S_+ = \sum_{k=1}^{j} \lambda_k, \qquad \mathrm{trace} S_- = \sum_{k=j+1}^{n} \lambda_k.$$

Then we obtain

$$\Lambda \mathrm{trace} S_+ + \lambda \mathrm{trace} S_- = \Lambda \sum_{k=1}^{j} \lambda_k + \lambda \sum_{k=j+1,\ k\neq \ell}^{n} \lambda_k + \lambda\lambda_\ell.$$

By (4.1) and (4.2) we see that

$$\leq \Lambda \sum_{k=1}^{j} b + \lambda \sum_{k=j+1,\ k\neq \ell}^{n} b - \lambda \frac{Nb}{n} \leq \Lambda(n-1)b - \lambda \frac{Nb}{n}.$$

□

Remark 4.2 By Proposition 2.3 and Lemma 4.1 we conclude that to locally strictly elliptic equations coercive function f and a function β which is a bound for f are following; for each $M > 0$ if $|p| \leq M$ then

$$f(p, S) = -\Lambda_M \mathrm{trace} S_+ - \lambda_M \mathrm{trace} S_-,$$
$$\beta(N) = -\Lambda_M(n-1) + \frac{\lambda_M N}{n}.$$

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