A note on asymptotic solutions of Hamilton-Jacobi equations

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This is a survey of my result [10]. In this talk, we consider the viscosity solutions of the Cauchy problem

\[(1) \quad u_t + \alpha x \cdot Du + H(Du) = f \quad \text{for } (x, t) \in \mathbb{R}^N \times (0, \infty), \]

\[(2) \quad u|_{t=0} = \phi \quad \text{for } x \in \mathbb{R}^N, \]

where \(\alpha\) is a positive constant. Our goal is to investigate convergence rates of \(u(t, x)\) to the stationary state as \(t \to \infty\). We assume the following:

(A1) \(H, f, \phi \in C(\mathbb{R}^N)\).

(A2) \(H\) is convex on \(\mathbb{R}^N\).

(A3) \(\lim_{|x| \to \infty} \frac{H(x)}{|x|} = \infty\).

We denote by \(L\) the convex conjugate of \(H\) defined by

\[ L(x) = \sup \{ z \cdot x - H(z) \mid z \in \mathbb{R}^N \}. \]

Then, \(L\) satisfies (A2) and (A3) in place of \(H\). Furthermore, we assume that there is a convex function \(\ell\) on \(\mathbb{R}^N\) satisfying

(A4) \(\lim_{|x| \to \infty} (L(x) - \ell(x)) = \infty\).

(A5) \(\inf \{ f(x) + \ell(-\alpha x) \mid x \in \mathbb{R}^N \} > -\infty\).
\[(A6) \quad \inf \left\{ \phi(x) + \frac{1}{\alpha} \ell(-\alpha x) \mid x \in \mathbb{R}^N \right\} > -\infty.\]

Now, we introduce several notations.

\[c := \min \left\{ f(x) + L(-\alpha x) \mid x \in \mathbb{R}^N \right\}, \quad f_c(x) := f(x) - c,\]

\[Z := \{ x \in \mathbb{R}^N \mid f_c(x) + L(-\alpha x) = 0 \},\]

\[C(x, T) = \{ X \in AC([0, T]) \mid X(0) = x \},\]

\[C(x, y, T) = \{ X \in C(x, T) \mid X(T) = y \},\]

\[d(x, y) = \inf \left\{ \int_0^T [f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))] \, dt \right\} \quad \text{for} \quad T > 0, \quad X \in C(x, y, T),\]

\[\psi(x) = \inf \left\{ \int_0^T [f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))] \, dt + \phi(X(T)) \right\} \quad \text{for} \quad T > 0, \quad X \in C(x, T),\]

\[v(x) = \min_{z \in Z} \left( d(x, z) + \psi(z) \right).\]

The following propositions were proved in [11] (see also the paper of Professor Hitoshi Ishii in this volume).

**Proposition 1.** There is the unique viscosity solution \( u \in C(\mathbb{R}^N \times [0, \infty)) \) of (1)-(2) satisfying for any \( T > 0 \)

\[(3) \quad \lim_{r \to \infty} \inf \left\{ u(x, t) + \frac{1}{\alpha} L(-\alpha x) \mid (x, t) \in (\mathbb{R}^N \setminus B(0, r)) \times [0, T) \right\} = \infty,\]

where \( B(a, r) = \{ x \in \mathbb{R}^N \mid |x - a| \leq r \} \) for \( a \in \mathbb{R}^N \) and \( r > 0 \). \( \square \)

**Proposition 2.** For the unique viscosity solution \( u \in C(\mathbb{R}^N \times [0, \infty)) \) of (1)-(2) satisfying (3), we have

\[(4) \quad \lim_{t \to \infty} \max_{x \in B(0, R)} |u(x, t) - (ct + v(x))| = 0 \quad \text{for} \quad R > 0. \quad \square\]

Note that by the stability theorem of viscosity solutions, \( v \) is a viscosity solution of the equation

\[(5) \quad c + \alpha x \cdot Dv + H(Dv) = f \quad \text{for} \quad x \in \mathbb{R}^N.\]
Next, we consider the convergence rate of (4). First, we consider the case such that the convergence rate of (4) is faster than $e^{-\theta t}$ for some constant $\theta > 0$. Besides (A1) $\sim$ (A6), we assume the following:

(A7) \( H \geq 0 \) in \( \mathbb{R}^N \) with \( H(0) = 0 \).

(A8) \( f \geq 0 \) in \( \mathbb{R}^N \) with \( f(0) = 0 \), and there exists a constant \( \theta > 0 \) such that

\[ \theta \int_0^\infty f(xe^{-\alpha t}) \, dt \leq f(x) \quad \text{for} \ x \in \mathbb{R}^N. \]

(A9) There exists a constant \( m > 0 \) such that

\[ 0 \leq \phi(x) \leq mw(x) \quad \text{for} \ x \in \mathbb{R}^N, \]

where \( w \in C(\mathbb{R}^N) \) is a subsolution of

\[ \alpha x \cdot Dv(x) + H(Dv(x)) = f(x) \quad \text{in} \ \mathbb{R}^N, \]

and satisfies the following inequality for a constant \( \lambda > 0 \):

\[ 0 \leq \lambda w(x) \leq f(x) \quad \text{for} \ x \in \mathbb{R}^N. \]

Example 1. Let \( f \) be a nonnegative and convex function on \( \mathbb{R}^N \) with \( f(0) = 0 \). Then, \( f \) satisfies (A8) for \( \theta = \alpha \).

Example 2. Let \( G \) be a nonnegative and convex function on \( \mathbb{R}^N \) with \( G(0) = 0 \). Assume that there exist constants \( \delta_1, \delta_2 \ (0 < \delta_1 < \delta_2) \) and \( f \in C(\mathbb{R}^N) \) such that

\[ \delta_1 G(x) \leq f(x) \leq \delta_2 G(x) \quad \text{for} \ x \in \mathbb{R}^N. \]

Then, \( f \) satisfies (A8) for \( \theta = \alpha \delta_1 / \delta_2 \).

Example 3. Assume that there are constants \( p \in (1, \infty) \) and \( a \in (0, 1) \) such that

\[ a(\alpha|x|^p + H(|x|^{p-2}x)) \leq f(x) \quad \text{for} \ x \in \mathbb{R}^N. \]

Then, as a function \( \phi \in C(\mathbb{R}^N) \) of (A9), we can take any one satisfying

\[ 0 \leq \phi(x) \leq k|x|^p \quad \text{for} \ x \in \mathbb{R}^N, \]

where \( k > 0 \) is a constant.
Theorem 3. Assume (A1)-(A9). Let $u \in C(\mathbb{R}^N \times [0, \infty))$ be the unique viscosity solution of (1)-(2) satisfying (3). Then, we have $c = 0$, $Z \ni 0$, and,

\begin{equation}
-v(x)e^{-\theta t} \leq u(x, t) - v(x) \leq m v(x)e^{-\theta t} \quad \text{for} \quad (x, t) \in \mathbb{R}^N \times [0, \infty). \tag{6}
\end{equation}

Finally, we give an example, which shows that there is the case such that the convergence rate of (4) is not faster than $t^{-1}$.

Example 4. Let $H(x) = |x|^p/p$ for some constant $p > 1$. Then, $L(x) = |x|^q/q$, where $(1/p) + (1/q) = 1$. For $r > 0$, let

\[ f(x) = -\frac{\alpha^q}{q} \min \{|x|^q, r^q\} \quad \text{for} \quad x \in \mathbb{R}^N. \]

Let $\phi \in C(\mathbb{R}^N)$ be a function satisfying $\phi(x) \geq 0$ for $x \in \mathbb{R}^N$. Then, we have $c = 0$, $Z = B(0, r)$, and,

\begin{equation}
\frac{1}{\alpha} L(-\alpha x)(t + 1)^{-1} \leq u(x, t) - v(x) \quad \text{for} \quad (x, t) \in Z \times [0, \infty). \tag{7}
\end{equation}

References


