Random Point Fields for Para-Particles of order 3

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概要

Random point fields which describe gases consist of para-particles of order three are given by means of the canonical ensemble approach. The analysis for the case of the para-fermion gases is discussed in full detail.

1 Introduction

The purpose of this note is to apply the method which we have developed in [TIa] to statistical mechanics of gases which consist of para-particles of order 3. We begin with quantum mechanical thermal systems of finite fixed numbers of para-bosons and/or para-fermions in the bounded boxes in $\mathbb{R}^d$. Taking the thermodynamic limits, random point fields on $\mathbb{R}^d$ are obtained. We will see that the point fields obtained in this way are those of $\alpha = \pm 1/3$ given in [ShTa03].

We use the representation theory of the symmetric group. (cf. e.g. [JK81, S91, Si96]) Its basic facts are reviewed briefly, in section 2, along the line on which the quantum theory of para-particles are formulated. We state the results in section 3. Section 4 devoted to the full detail of the discussion on the thermodynamic limits for para-fermion's case.

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2 Brief review on Representation of the symmetric group

We say that $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{N}^n$ is a Young frame of length $n$ for the symmetric group $S_N$ if
\[ \sum_{j=1}^{n} \lambda_j = N, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0. \]

We associate the Young frame $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with the diagram of $\lambda_1$-boxes in the first row, $\lambda_2$-boxes in the second row, ..., and $\lambda_n$-boxes in the $n$-th row. A Young tableau on a Young frame is a bijection from the numbers $1, 2, \cdots, N$ to the $N$ boxes of the frame.

Let $M_p^N$ be the set of all the Young frames for $S_N$ which have lengths less than or equal to $p$. For each frame in $M_p^N$, let us choose one tableau from those on the frame. The choices are arbitrary but fixed. $\mathcal{T}_p^N$ denotes the set of all the tableaux chosen in this way. The row stabilizer of tableau $T$ is denoted by $\mathcal{R}(T)$, i.e., the subgroup of $S_N$ consists of those elements that keep all rows of $T$ invariant, and $C(T)$ the column stabilizer whose elements preserve all columns of $T$.

Let us introduce the three elements
\[ a(T) = \frac{1}{\# \mathcal{R}(T)} \sum_{\sigma \in \mathcal{R}(T)} \sigma, \quad b(T) = \frac{1}{\# \mathcal{C}(T)} \sum_{\sigma \in \mathcal{C}(T)} \text{sgn}(\sigma) \sigma \]
and
\[ e(T) = \frac{d_T}{N!} \sum_{\sigma \in \mathcal{R}(T)} \sum_{\tau \in \mathcal{C}(T)} \text{sgn}(\tau) \sigma \tau = c_T a(T)b(T) \]
of the group algebra $\mathbb{C}[S_N]$ for each $T \in \mathcal{T}_p^N$, where $d_T$ is the dimension of the irreducible representation of $S_N$ corresponding to $T$ and $c_T = d_T \# \mathcal{R}(T) \# \mathcal{C}(T)/N!$. As is known,
\[ a(T_1) b(T_2) = b(T_2) a(T_1) = 0 \]  
hold for any $\sigma \in S_N$ if $T_2 \to T_1$. The relations
\[ a(T)^2 = a(T), \quad b(T)^2 = b(T), \quad e(T)^2 = e(T), \quad e(T_1) e(T_2) = 0 \quad (T_1 \neq T_2) \] (2.1)
also hold for $T, T_1, T_2 \in \mathcal{T}_p^N$. For later use, let us introduce
\[ d(T) = e(T) a(T) = c_T a(T)b(T)a(T) \] (2.2)
for $T \in \mathcal{T}_p^N$. They satisfy
\[ d(T)^2 = d(T), \quad d(T_1) d(T_2) = 0 \quad (T_1 \neq T_2) \] (2.4)
which are shown readily from (2.2) and (2.1). The inner product $< \cdot , \cdot >$ of $C[S_N]$ is defined by
\[ < \sigma, \tau > = \delta_{\sigma \tau} \quad \text{for } \sigma, \tau \in S_N \]
and the sesqui-linearity.

The left representation $L$ and the right representation $R$ of $S_N$ on $C[S_N]$ are defined by
\[ L(\sigma)g = L(\sigma) \sum_{\tau \in S_N} g(\tau) \tau = \sum_{\tau \in S_N} g(\sigma^{-1}\tau) \tau \]
and
\[ R(\sigma)g = R(\sigma) \sum_{\tau \in S_N} g(\tau) \tau = \sum_{\tau \in S_N} g(\tau \sigma) \tau, \]
respectively. Here and hereafter we identify $g : S_N \to \mathbb{C}$ and $\sum_{\tau \in S_N} g(\tau) \tau \in C[S_N]$. They are extended to the representation of $C[S_N]$ on $C[S_N]$ as
\[ L(f)g = fg = \sum_{\sigma, \tau} f(\sigma) g(\tau) \sigma \tau = \sum_{\tau} (\sum_{\sigma} f(\sigma \tau^{-1}) g(\tau)) \sigma \]
and
\[ R(f)g = g \hat{f} = \sum_{\sigma, \tau} g(\sigma) f(\tau) \sigma \tau^{-1} = \sum_{\tau} (\sum_{\sigma} g(\sigma \tau) f(\tau)) \sigma, \]
where $\hat{f} = \sum_{\tau} \hat{f}(\tau) \tau = \sum_{\tau} f(\tau^{-1}) \tau = \sum_{\tau} f(\tau) \tau^{-1}$.

The character of the irreducible representation of $S_N$ corresponding to tableau $T \in \mathcal{T}_{p}^{N}$ is obtained by
\[ \chi_T(\sigma) = \sum_{\tau \in S_N} (\tau, L(\sigma) R(e(T)) \tau) = \sum_{\tau, \gamma \in S_N} (\tau, \sigma \tau e(T)) g(\gamma) = \sum_{\tau \in S_N} g(\tau^{-1} \sigma \tau). \]

We introduce a tentative notation
\[ \chi_{\sigma}(g) \equiv \sum_{\tau \in S_N} (\tau, L(\sigma) R(g) \tau) = \sum_{\tau, \gamma \in S_N} (\tau, \sigma \tau e(T)) g(\gamma) = \sum_{\tau \in S_N} g(\tau^{-1} \sigma \tau) \quad (2.5) \]
for $g = \sum_{\tau} g(\tau) \tau \in C[S_N]$. Then $\chi_T = \chi_{e(T)}$ holds.

Now let us consider representations of $S_N$ on Hilbert spaces. Let $\mathcal{H}_L$ be a certain $L^2$ space which will be specified in the next section and $\otimes^N \mathcal{H}_L$ its $N$-fold Hilbert space tensor product. Let $U$ be the representation of $S_N$ on $\otimes^N \mathcal{H}_L$ defined by
\[ U(\sigma)\varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)} \quad \text{for } \varphi_1, \cdots, \varphi_N \in \mathcal{H}_L, \]
or equivalently by
\[ (U(\sigma)f)(x_1, \cdots, x_N) = f(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) \quad \text{for } f \in \otimes^N \mathcal{H}_L. \]
Obviously, $U$ is unitary: $U(\sigma)^* = U(\sigma^{-1}) = U(\sigma)^{-1}$. We extend $U$ for $C[S_N]$ by linearity. Then $U(a(T))$ is an orthogonal projection because of $U(a(T))^* = U(a(T)) = U(a(T))$ and (2.2). So are $U(b(T))'$s, $U(d(T))'$s and $P_{\Delta} = \sum_{T \in \mathcal{T}_{p}^{N}} U(d(T))$. Note that $\text{Ran} \ U(d(T)) = \text{Ran} \ U(e(T))$ because of $d(T)e(T) = e(T)$ and $e(T)d(T) = d(T)$.
3 Para-statistics and Random point fields

3.1 Para-bosons of order 3

Let us consider a quantum system of \( N \) para-bosons of order \( p \) in the box \( \Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d \). We refer the literatures [MeG64, HaT69, StT70] for quantum mechanics of para-particles. (See also [OK69].) The arguments of these literatures indicate that the state space of our system is given by \( \mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^{N} \mathcal{H}_L \), where \( \mathcal{H}_L = L^2(\Lambda_L) \) with Lebesgue measure is the state space of one particle system in \( \Lambda_L \).

We need the heat operator \( G_L = e^{\beta \Delta_L} \) in \( \Lambda_L \), where \( \Delta_L \) is the Laplacian in \( \Lambda_L \) with periodic boundary conditions.

It is obvious that there is a CONS of \( \mathcal{H}_{L,N}^{pB} \) which consists of the vectors of the form \( U(d(T)) \varphi_{k_1}^{(L)} \otimes \cdots \otimes \varphi_{k_N}^{(L)} \), which are the eigenfunctions of \( \otimes^N G_L \). Then, we define the point field \( \mu_{L,N}^{pB} \) of \( N \) free para-bosons of order \( p \) as in section 2 of [TiA] and its generating functional is given by

\[
\int e^{-\langle f, \xi \rangle} d\mu_{L,N}^{pB}(\xi) = \frac{\text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)P_{pB}]}{\text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)P_{pB}]},
\]

where \( f \) is a nonnegative continuous function on \( \Lambda_L \) and \( \tilde{G}_L = G_L^{1/2} e^{-f} G_L^{1/2} \).

Lemma 3.1

\[
\int e^{-\langle f, \xi \rangle} d\mu_{L,N}^{pB}(\xi) = \frac{\sum_{T \in \mathcal{T}_{p}^{N}} \sum_{\sigma \in S_N} \chi_T(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L)U(\sigma)]}{\sum_{T \in \mathcal{T}_{p}^{N}} \sum_{\sigma \in S_N} \chi_T(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]},
\]

(3.1)

\[
= \frac{\sum_{T \in \mathcal{T}_{p}^{N}} \int_{\Lambda_L^N} \det_T \{ \tilde{G}_L(x_i, x_j) \}_{1 \leq i, j \leq N} dx_1 \cdots dx_N}{\sum_{T \in \mathcal{T}_{p}^{N}} \int_{\Lambda_L^N} \det_T \{ G_L(x_i, x_j) \}_{1 \leq i, j \leq N} dx_1 \cdots dx_N}
\]

(3.2)

Remark 1: \( \mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L \) is determined by the choice of the tableaux \( T \)'s. The spaces corresponding to different choices of tableaux are different subspaces of \( \otimes^N \mathcal{H}_L \). However, they are unitarily equivalent and the generating functional given above is not affected by the choice. In fact, \( \chi_T(\sigma) \) depends only on the frame on which the tableau \( T \) is defined.

Remark 2: \( \det T A = \sum_{\sigma \in S_N} \chi_T(\sigma) \prod_{i=1}^{N} A_{i \tau(i)} \) in (3.2) is called immanant.

Proof: Since \( \otimes^N G \) commutes with \( U(\sigma) \) and \( a(T)e(T) = e(T) \), we have

\[
\text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(d(T))] = \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(e(T))U(a(T))]
= \text{Tr}_{\otimes^N \mathcal{H}_L}[U(a(T))(\otimes^N G_L)U(e(T))] = \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(e(T))].
\]

(3.3)
On the other hand, we get from (2.5) that
\[
\sum_{\sigma \in S_N} \chi_{g}(\sigma) \mathrm{Tr}_{\otimes^{N} \mathcal{H}_L}[(\otimes^{N} G_L) U(\sigma)] = \sum_{\tau, \sigma} g(\tau^{-1} \sigma \tau) \mathrm{Tr}_{\otimes^{N} \mathcal{H}_L}[(\otimes^{N} G_L) U(\sigma)]
\]
where we have used the cyclicity of the trace and the commutativity of $U(\tau)$ with $\otimes^{N} G_L$. Putting $g = e(T)$ and using (3.3) and $P_{pB} = \sum_{T \in \mathcal{T}_{pB}} U(d(T))$, we obtain the first equation. The second one is obvious.

Now, let us consider the thermodynamic limit

\[ L, N \rightarrow \infty, \quad N/L^d \rightarrow \rho > 0. \]

We need the heat operator $G = e^{\beta \Delta}$ on $L^2(\mathbb{R}^d)$. In the following, $f$ is a nonnegative continuous function having a compact support. It is supposed to be fixed in the thermodynamic limit. Its support will be contained in $\Lambda_L$ for large enough $L$.

We get the limiting random point field $\mu_{\rho}^{3B}$ on $\mathbb{R}^d$ for the low density region.

**Theorem 3.2** The finite random point field for para-bosons of order 3 defined above converge weakly to the random point field whose Laplace transform is given by

\[
\int e^{-<f, \xi}> d\mu_{\rho}^{3B}(\xi) = \mathrm{Det} [1 + \sqrt{1 - e^{-\beta \Delta}} r_* G(1 - r_* G)^{-1} \sqrt{1 - e^{-\beta \Delta}}]^{-3}
\]

in the thermodynamic limit, where $r_* \in (0, 1)$ is determined by

\[
\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-\beta |p|^2}}{1 - r_* e^{-\beta |p|^2}} = (r_* G(1 - r_* G)^{-1})(x, x),
\]

if

\[
\frac{\rho}{3} < \rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta |p|^2}}{1 - e^{-\beta |p|^2}}.
\]

Remark : The high density region $\rho \geq 3\rho_c$ is related to the Bose-Einstein condensation. We need a different analysis for the region. See [TIb] for the case of $p = 1$ and 2.

### 3.2 Para-fermions of order 3

For Young tableau $T, T'$ denotes the tableau obtained by exchanging the rows and the columns of $T$, i.e., $T'$ is the transpose of $T$. The transpose $\lambda'$ of the frame $\lambda$ can be defined similarly. Then, $T'$ lives on $\lambda'$ if $T$ lives on $\lambda$. It is obvious that

\[
\mathcal{R}(T') = \mathcal{C}(T), \quad \mathcal{C}(T') = \mathcal{R}(T).
\]
The generating functional of the point field $\mu_{L,N}^{pF}$ for $N$ para-fermions of order $p$ in the box $\Lambda_L$ is given by

$$\int e^{-<j,\xi>}d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T\in T_p^N} \text{Tr}_{\otimes^N H_L}}{\sum_{T\in T_p^N} \text{Tr}_{\otimes^N H_L}}\left[\left(\otimes^N G\right)U(d(T'))\right]$$

as in the case of para-bosons of order $p$. And the following expressions also hold.

**Lemma 3.3**

$$\int e^{-<f,\xi>}d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T\in T_p^N} \text{Tr}_{\otimes^N H_L}}{\sum_{T\in T_p^N} \text{Tr}_{\otimes^N H_L}}\left[\left(\otimes^N G\right)U(\sigma)\right]$$

(3.6)

$$= \frac{\sum_{T\in T_p^N} \int_{\Lambda_L^N} \det_{T'}\{G(x_i,x_j)\}dx_1\cdots dx_N}{\sum_{T\in T_p^N} \int_{\Lambda_L^N} \det_{T'}\{G(x_j,x_j)\}dx_1\cdots dx_N}$$

(3.7)

**Theorem 3.4** The finite random point fields for para-fermions of order 3 defined above converge weakly to the point field $\mu_3^{3F}$ whose Laplace transform is given by

$$\int e^{-<j,\xi>}d\mu_3^{3F}(\xi) = \text{Det}\left[1 - \sqrt{1 - e^{-f}} r_* G(1 + r_* G)^{-1} \sqrt{1 - e^{-f}}\right]^3$$

in the thermodynamic limit (3.5), where $r_* \in (0, \infty)$ is determined by

$$\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-|p|^2}}{1 + r_* e^{-|p|^2}} = (r_* G(1 + r_* G)^{-1})(x,x).$$

(3.8)

### 4 Proof of Theorem 3.4

In the rest of this paper, we use results in [TLa] frequently. We refer them e.g., Lemma I.3.2 for Lemma 3.2 of [TLa]. Let $\psi_T$ be the character of the induced representation $\text{Ind}_{\mathcal{R}(T)}^{S_N}[1]$, where 1 is the one dimensional representation $\mathcal{R}(T) \ni \sigma \rightarrow 1$, i.e.,

$$\psi_T(\sigma) = \sum_{\tau \in S_N} <\tau, L(\sigma)R(a(T))\tau> = \chi_\lambda(\sigma).$$

Since the characters $\chi_T$ and $\psi_T$ depend only on the frame on which the tableau $T$ lives, not on $T$ itself, we use the notation $\chi_\lambda$ and $\psi_\lambda$ ( $\lambda \in M_\rho^N$ ) instead of $\chi_T$ and $\psi_T$, respectively.
Let $\delta$ be the frame $(p-1, \cdots, 2, 1, 0) \in M_p^N$. Generalize $\psi_{\mu}$ to those $\mu = (\mu_1, \cdots, \mu_p) \in \mathbb{Z}^p$ which satisfies $\sum_{j=1}^{p} \mu_j = N$ by

$$\psi_{\mu} = 0 \quad \text{for} \quad \mu \in \mathbb{Z}^p - \mathbb{Z}^p_+$$

and

$$\psi_{\mu} = \psi_{\pi\mu} \quad \text{for} \quad \mu \in \mathbb{Z}^p_+ \quad \text{and} \quad \pi \in S_p \quad \text{such that} \quad \pi\mu \in M_p^N,$$

where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. Then the determinantal form [JK81] can be written as

$$\chi_{\lambda} = \sum_{\pi \in S_p} \text{sgn} \pi \psi_{\lambda + \delta - \pi\delta}.$$  \hspace{1cm} (4.1)

Let us recall the relations

$$\chi_{T'}(\sigma) = \text{sgn} \sigma \chi_{T}(\sigma), \quad \varphi_{T'}(\sigma) = \text{sgn} \sigma \psi_{T}(\sigma),$$

where

$$\varphi_{T'}(\sigma) = \sum_{\tau} <\tau, L(\sigma) R(b(T')) \tau > = \chi_{b(T')}(\sigma)$$

denotes the character of the induced representation $\text{Ind}_{C(T')}^{S_N} [\text{sgn}]$, where $\text{sgn}$ is the representation $C(T') = \mathcal{R}(T) \ni \sigma \mapsto \text{sgn} \sigma$. Then we have a variant of (4.1)

$$\chi_{\lambda'} = \sum_{\pi \in S_p} \text{sgn} \pi \varphi_{\lambda' + \delta' - (\pi\delta)'}. \hspace{1cm} (4.2)$$

Now we consider the denominator of (3.6). Let $T \in \mathcal{T}_p^N$ live on $\mu = (\mu_1, \cdots, \mu_p) \in M_p^N$. Thanks to (3.4) for $g = b(T')$, we have

$$\sum_{\sigma \in S_N} \varphi_{T'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G)U(\sigma)) = N! \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G)U(b(T')))$$

$$= N! \prod_{j=1}^{p} \text{Tr}_{\otimes^\mu_j \mathcal{H}_L}((\otimes^\mu G)A_{\mu_j}),$$

where $A_n = \sum_{\tau \in S_n} \text{sgn}(\tau) U(\tau)/n!$ is the anti-symmetrization operator on $\otimes^n \mathcal{H}_L$. In the last step, we have used

$$b(T') = \prod_{j=1}^{p} \sum_{\sigma \in \mathcal{R}_j} \frac{\text{sgn} \sigma}{\# \mathcal{R}_j},$$

where $\mathcal{R}_j$ is the symmetric group of $\mu_j$ numbers which lie on the $j$-th row of the tableau $T$. Then (4.2) yields

$$\sum_{\sigma \in S_N} \chi_{\lambda'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G_L)U(\sigma)) = \sum_{\pi \in S_p} \text{sgn} \pi \sum_{\sigma \in S_N} \varphi_{\lambda' + \delta' - (\pi\delta)'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G_L)U(\sigma))$$
\[= N! \sum_{\pi \in S_p} \prod_{j=1}^{p} \mathrm{Tr}_{\otimes^\lambda_j \pi_j}((\otimes^\lambda_j \pi_j)G_L)A_{\lambda_j \pi_j} \]

Here we understand that \(\mathrm{Tr}_{\otimes^n \mathcal{H}_L}((\otimes^n G)L_n) = 1\) if \(n = 0\) and \(= 0\) if \(n < 0\) in the last expression. Let us recall the defining formula of Fredholm determinant

\[
\mathrm{Det}(1 + J) = \sum_{n=0}^{\infty} \mathrm{Tr}_{\otimes^n \mathcal{H}_L}((\otimes^n J)L_n)
\]

for a trace class operator \(J\). We use it in the form

\[
\mathrm{Tr}_{\otimes^n \mathcal{H}_L}((\otimes^n G)L_n) = \oint_{S_r(0)} \frac{dz}{2\pi iz^n+1} \mathrm{Det}(1 + zG_L),
\]

where \(r > 0\) can be set arbitrary. Note that the right hand side equals to 1 for \(n = 1\) and to 0 for \(n < 0\). Then we have the following expression of the denominator of (3.6)

\[
\sum_{\lambda \in \mathcal{M}_p} \sum_{\sigma \in S_N} \chi_\lambda(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G)L)U(\sigma)
\]

\[
= N! \sum_{\lambda \in \mathcal{M}_p} \sum_{\pi \in S_p} \prod_{j=1}^{p} \frac{\mathrm{Det}(1 + z_j G_L) dz_j}{2\pi i z_j^{\lambda_j - j + \pi(j) + 1}}.
\]

The similar formula for the numerator also holds.

Now we concentrate on the case of \(p = 3\). To make the thermodynamic limit procedure explicit, let us take a sequence \(\{L_n\}_{N \in \mathbb{N}}\) which satisfies \(N/L_n \to \rho\) as \(N \to \infty\). In the followings, \(r = r_k \in [0, \infty)\) denotes the unique solution of

\[
\mathrm{Tr}(r G_{L_k} (1 + r G_{L_k})^{-1}) = k
\]

for \(0 \leq k \leq N\). We suppress the \(N\) dependence of \(r_k\). The existence and the uniqueness of the solution follow from the fact that the left-hand side of (4.5) is a continuous and monotone function of \(r\). See Lemma I.3.2, for details. We put

\[
v_k = \mathrm{Tr}(r_k G_{L_n} (1 + r_k G_{L_n})^{-2})
\]

and

\[
D_{k,l,m} = \oint_{S_r(0)^3} \frac{\prod_{j=1}^{3} \mathrm{Det}(1 + z_j G_{L_k}) (z_1 - z_2) (z_2 - z_3)}{(2\pi i)^3 z_1^{k+1} z_2^{l+1} z_3^{m+1}} d z_1 d z_2 d z_3,
\]
for \( k, l, m \in \mathbb{Z} \). Note that \( D_{k,l,m} = 0 \) if at least one of \( k, l, m \) is negative. Summing over \( \lambda_1 \) and \( \lambda_3 \) in (4.4) for \( p = 3 \), we get

\[
\sum_{\lambda \in M^N} \sum_{\sigma \in S_N} \chi_\lambda(\sigma) \mathrm{Tr}_{\mathcal{H}_{L_{N}}} \left[ \left( \otimes^N G_{L_{N}} \right) U(\sigma) \right] = N! \left( \sum_{l=1}^{[N/3]+1} D_{N+3-2l,l,l-1} + \sum_{l=\lceil N/3 \rceil + 2}^{[N/2]+1} D_{l,l,N+2-2l} \right).
\]

Since \( r > 0 \) of the contour \( S_r(0) \) is arbitrary, we may change the complex integral variables \( z_j = r_j \eta_j \) with \( \eta_j \in S_1(0) \) for \( j = 1, 2, 3 \). Thanks to the property of Fredholm determinant, we have

\[
\mathrm{Det}[1 + z_j G_{L_{N}}] = \mathrm{Det}[1 + r_j G_{L_{N}}] \mathrm{Det}[1 + (\eta_j - 1)r_j G_{L_{N}}(1 + r_j G_{L_{N}})^{-1}]
\]

Now, we can put

\[
F_{k,l,m} = \frac{r_0^{3k_0} v_0^{5/2}}{\mathrm{Det}[1 + r_0 G_{L_{N}}]^{3}} D_{k,l,m} = R_{k_1,k_2,k_3} \sqrt{N+2} I_{k_1,k_2,k_3},
\]

where

\[
R_{k_1,k_2,k_3} = \prod_{j=1}^{3} \frac{r_0^{k_j} \mathrm{Det}[1 + r_j G_{L_{N}}]}{r_j^{k_j} \mathrm{Det}[1 + r_0 G_{L_{N}}]},
\]

and

\[
I_{k_1,k_2,k_3} = \oint \oint \oint_{S_{1}(0)^{3}} \left( \prod_{j=1}^{3} \mathrm{Det}[1 + (\eta_j - 1)r_j G_{L_{N}}(1 + r_j G_{L_{N}})^{-1}] \right)
\times (r_1 \eta_1 - r_2 \eta_2)(r_2 \eta_2 - r_3 \eta_3) \frac{d\eta_1 d\eta_2 d\eta_3}{(2\pi i)^{3} \eta_1^{k_1+1} \eta_2^{k_2+1} \eta_3^{k_3+1}}.
\]

Here \( k_0 = (N+2)/3 \) and \( k_1, k_2, k_3 \in \mathbb{Z}_+ \) satisfy \( k_1 \geq k_2 \geq k_3 \) and \( k_1 + k_2 + k_3 = 3k_0 \). We use the abbreviation \( r_\nu \) and \( v_\nu \) for \( r_{k_\nu} \) and \( v_{k_\nu} (\nu = 0, 1, 2, 3) \), respectively. Here, let us recall that \( r_0 \rightarrow r_* \) in the thermodynamic limit because of \( k_0/L^{d} \rightarrow \rho/3 \), (3.8) and Lemma I.3.5.

Define a sequence \( \{f_N\}_{N \in \mathbb{N}} \) of nonnegative functions on \( \mathbb{R} \) by

\[
f_N(x) = \begin{cases} 
F_{l,l,N+2-2l} & \text{for } \sqrt{N+2} x \in [l-1-(N+2)/3, l-(N+2)/3) \\
F_{N+3-2l,l,l-1} & \text{for } \sqrt{N+2} x \in [l-1-(N+2)/3, l-(N+2)/3) \\
0 & \text{otherwise.}
\end{cases}
\]

Then the denominator of (3.6) becomes

\[
N! \sqrt{N+2} \frac{\mathrm{Det}[1 + r_0 G_{L_{N}}]^{3}}{r_0^{3k_0} v_0^{5/2}} \int_{-\infty}^{\infty} f_N(x) \, dx
\]
We introduce \( \tilde{D}_{k,l,m}, \tilde{\mathcal{F}}_{k,l,m} \) and \( \tilde{f}_{N} \) using \( \tilde{G}_{L_N} \) instead of \( G_{L_N} \) in \( D_{k,l,m}, \mathcal{F}_{k,l,m} \) and \( f_{N} \) and so on, to get the expression

\[
\mathbb{E}_{L,N}^{3F} [e^{-\langle f, \xi \rangle}] = \frac{\text{Det} [1 + \tilde{r}_{0} \tilde{G}_{L_N}]^{3} \tilde{r}_{0}^{5/2} \int_{\infty}^{\infty} \tilde{f}_{N}(x) \, dx}{\text{Det} [1 + r_{0} G_{L_N}]^{3} r_{0}^{5/2} \int_{\infty}^{\infty} f_{N}(x) \, dx}.
\]

From Lemma I.3.6, we have

\[
\frac{\tilde{v}_{0}}{v_{0}} \rightarrow 1
\]

(4.7)
in the thermodynamic limit. Similarly, we obtain

\[
\frac{r_{0}^{k_{0}} \text{Det} [1 + \tilde{r}_{0} \tilde{G}_{L_N}]}{\tilde{r}_{0}^{k_{0}} \text{Det} [1 + r_{0} G_{L_N}]} \rightarrow \text{Det} [1 - \sqrt{1 - e^{-J}} r_{*} G (1 + r_{*} G)^{-1} \sqrt{1 - e^{-J}}]
\]

from the proof of Theorem I.3.1 (see Eq. (a–c), where we should read \( N \) as \( k_0, z_N \) as \( r_0 \) and \( \alpha = -1 \)). Thus Theorem 3.4 is proved, if we get the following lemma:

**Lemma 4.1** Under the thermodynamic limit,

\[
\int_{-\infty}^{\infty} \tilde{f}_{N}(x) \, dx, \int_{-\infty}^{\infty} f_{N}(x) \, dx \rightarrow \int_{-\infty}^{\infty} e^{-2\rho x^{2}/\kappa_{3/2}} \frac{dx}{(2\pi)^{3/2}}
\]

hold, where

\[
\kappa = \int \frac{dp}{(2\pi)^{d}} \frac{r_{*} e^{-\beta |p|^{2}}}{1 + r_{*} e^{-\beta |p|^{2}}}.
\]

**Proof:** Let \( k, r, v \in [0, \infty) \) satisfy the relations

\[
k = \text{Tr} [r G_{L_N} (1 + r G_{L_N})^{-1}], \quad v = \text{Tr} [r G_{L_N} (1 + r G_{L_N})^{-2}].
\]

1° There exist positive constants \( c_1 \) and \( c_2 \) which depend only on the density \( \rho \) such that

\[
r_{j} \leq c_{1}, \quad r_{j} - r_{l} \leq c_{1} \frac{k_{j} - k_{l}}{k_{l}}, \quad c_{2} k_{j} \leq v_{j} \leq k_{j},
\]

hold for \( k_{j}, k_{l} > 0 \) satisfying \( k_{j} > k_{l} \).

We have \( v \leq k \) and \( r \leq r_{N} \) for \( k \leq N \). Recall \( r_{N} \) converges to the constant \( r_{*} \) which determined by

\[
\int \frac{dp}{(2\pi)^{d}} \frac{r_{*} e^{-\beta |p|^{2}}}{1 + r_{*} e^{-\beta |p|^{2}}} = \rho.
\]

Then \( \{r_{N}\} \) is bounded from above. Hence we have \( r \leq r_{N} \leq c_{1} \) and \( v \geq k/(1 + r_{N}) \geq k/(1 + c_{1}) \) since \( 0 \leq G_{L_N} \leq 1 \). Thanks to \( dk/dr = v/r \geq k/c_{1} \), we get \( c_{1} \int_{r_{k}}^{r_{j}} dk/k \geq \int_{r_{k}}^{r_{j}} dr \), which yields the second inequality. \( \Box \)
There exist positive constants $c'_0, c'_1$ and $c'_2$ which depend only on $\rho$ such that

$$A_{k,n} = \oint_{S_t(0)} \text{Det}\left[1 + (\eta - 1)rG_{L_N}(1 + rG_{L_N})^{-1}\right] \frac{(\eta - 1)^n \, d\eta}{2\pi i \eta^{k+1}} \quad (n = 0, 1, 2, \ k = 0, 1, \cdots, N)$$

satisfy

$$A_{k,0} = (1 + o(1))/\sqrt{2\pi v}, \quad A_{k,2} = (-1 + o(1))/\sqrt{2\pi v^3} \quad \text{for large } k \leq N$$

and

$$|A_{k,0}| \leq c'_0/\sqrt{1 + k}, \quad |A_{k,1}| \leq c'_1/\sqrt{1 + k^3}, \quad |A_{k,2}| \leq c'_2/\sqrt{1 + k^3} \quad \text{for all } k = 0, 1, \cdots, N.$$
The integral of the first term of the right hand side is 0. While the second term is bounded by $|x|\delta|h(x)|$, since $|e^\delta - 1| \leq |\delta|e^{\delta_0}$. For the third term, we use (4.9). Then we get the bound $|\int xh_k(x)\,dx| \leq c''/\sqrt{v}$ for $k \geq 1$. Together with $A_{0,1} = 0$, the bounds for $A_{k,1}$ are derived. Similarly, we get the formulae for $A_{k,2}$.

3° Let $(k_1, k_2, k_3) \in \mathbb{Z}_+$ satisfies

$$k_1 \geq k_2 \geq k_3, \quad k_1 + k_2 + k_3 = 3k_0 = N + 2$$

and

$$k_1 = k_2 \text{ or } k_2 = k_3 + 1.$$ 

Then the estimates

$$|v_0^{5/2}I_{k_1,k_2,k_3}| \leq c\left(\frac{k_0}{1+k_3}\right)^{5/2} \leq c' e^{(k_0-k_3)^2/4k_0}$$

hold for all such $(k_1, k_2, k_3)$ and

$$v_0^{5/2} I_{k_1,k_2,k_3} = \frac{v_0^{5/2}(1+o(1))}{(2\pi)^{3/2}v_1^{1/2}v_2^{3/2}v_3^{1/2}}$$

holds for large $N$ and $(k_1, k_2, k_3)$, where $c$, $c'$ are positive constants depending only on $\rho$.

In fact, expanding

$$(r_1\eta_1-r_2\eta_2)(r_2\eta_2-r_3\eta_3) = (r_1(\eta_1-1)-r_2(\eta_2-1)+r_1-r_2)(r_2(\eta_2-1)-r_3(\eta_3-1)+r_2-r_3)$$

in the integrand of $I_{k_1,k_2,k_3}$, we get the first inequality from 1° and 2°. The second inequality is obvious. Similarly, the asymptotic behavior follows.

4°

$$R_{k_1,k_2,k_3} = e^{-\sum_{j=1}^3(k_0-k_j)^2/2v_j}$$

holds where $v_j' = \text{Tr} [r_j'G_{LN}(1+r_j'G_{LN})^{-2}]$ for a certain middle point $r_j'$ between $r_0$ and $r_j$. Especially, we have the bound

$$R_{k_1,k_2,k_3} \leq e^{-(k_0-k_3)^2/2k_0}.$$

Recall that $G_{LN}$ is a non-negative trace class self-adjoint operator. If we put

$$\psi(t) = \log \text{Det}[1+e^t G_{LN}] = \text{Tr} [\log(1+e^t G_{LN})],$$

we have

$$\psi'(t) = \text{Tr} [e^t G_{LN} (1+e^t G_{LN})^{-1}], \quad \psi''(t) = \text{Tr} [e^t G_{LN} (1+e^t G_{LN})^{-2}].$$
In the equality
\[ \psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0) = \int_t^{t_0} (s - t_0)\psi''(s) \, ds + t_0(\psi'(t_0) - \psi'(t)), \]
apply
\[ \int_t^{t_0} (s - t_0)\psi''(s) \, ds = \int_t^{t_0} ds \int_{t_0}^{s} du \psi''(s), \frac{\psi''(u)}{\psi'(u)} = -\frac{(\psi'(t) - \psi'(t_0))^2}{2\psi''(u_c)} \]
where \( u_c \) is a middle point of \( t \) and \( t_0 \). Then we obtain
\[ \frac{e^{t_0\psi'(t_0)}}{e^{t\psi(t)}} \]
\[ \frac{\text{Det}[1 + e^{t}G_{L_{N}}]}{\text{Det}[1 + e^{t_0}G_{L_{N}}]} = e^{\psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0)} = e^{t_0(\psi'(t_0) - \psi'(t)) - (\psi'(t) - \psi'(t_0))^2/2\psi''(u_c)}. \]
Set \( e^t = r_j \) and \( e^{t_0} = r_0 \). Then \( \psi'(t) = k_j, \psi'(t_0) = k_0, \psi''(t) = v_j \) and \( \psi''(t_0) = v_0 \) hold.
Taking the product of those equalities for \( j = 1, 2 \) and \( 3 \), we get the desired expression, since \( 3k_0 = k_1 + k_2 + k_3 \).

5° Recall that the functions \( \varphi_k^{(L)}(x) = L^{-d/2} \exp(i2\pi k \cdot x/L) \) \( (k \in \mathbb{Z}^d) \) constitute a C.O.N.S. of \( L^2(\Lambda_{L}) \), where \( G_{L} \varphi_k^{(L)} = e^{-\beta|2\pi k/L|^2} \varphi_k^{(L)} \) holds for all \( k \in \mathbb{Z}^d \). Then, we obtain
\[ \frac{v_0}{L^d} = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \left( \frac{2\pi}{L} \right)^d \frac{r_0 e^{-\beta|2\pi k/L|^2}}{1 + r_0 e^{-\beta|2\pi k/L|^2}} \to \kappa, \]
in the thermodynamic limit, since \( k_0/L^d \to \rho/3 \) and \( r_0 \to r_* \) hold.
From 3° and 4°, we have a bound
\[ |F_{k_1,k_2,k_3}| \leq c'e^{-(k_0-k_3)^2/4k_0} \] (4.11)
and
\[ F_{k_1,k_2,k_3} = \frac{v_0^{5/2}(1 + o(1))}{(2\pi)^3/2v_1^{1/2}v_2^{3/2}v_3^{1/2}e^{-\Sigma_j(k_0-k_j)/2v'_j}} \] (4.12)
for large \( N, k_1, k_2, k_3 \), where \( v'_j \) is a mean value which we have written \( \psi''(u_c) \) in 4°.
For \( l = 1, 2, \cdots, [N/3] + 1, \sqrt{N + 2x} \in [l - 1 - (N + 2)/3, l - (N + 2)/3) \) implies \( |l - 1 - (N + 2)/3| \geq \sqrt{N + 2}|x| \), hence we get the bound
\[ f_N(x) = F_{N+3-2l,l-1} \leq c'e^{-(N+2)x^2/4k_0} \leq c'e^{-3x^2/4}. \]
We also get \( f_N(x) \leq c' \exp(-3x^2/4) \) for the other cases, similarly.
For fixed \( x \in \mathbb{R} \), we choose \( l \in \mathbb{Z} \) such that \( \sqrt{N + 2x} \in [l - 1 - (N + 2)/3, l - (N + 2)/3) \).
Then we have \( v_j/v_0 \to 1 \) \( (j = 1, 2, 3) \) and
\[ \sum_{j=1}^{3} \frac{(k_0 - k_j)^2}{v'_j} = \frac{4N}{v_0}x^2 + o(1). \]
Hence, we obtain $f_N(x) \rightarrow (2\pi)^{-3/2} \exp(-2\rho x^2/\kappa)$ in the thermodynamic limit. Thus the dominated convergence theorem yields the desired result for $f_N$. Because of (4.7), the one for $\tilde{f}_N$ can be proved similarly.

参考文献


