A SURVEY ON FIXED POINT THEOREMS
IN GENERALIZED CONVEX SPACES

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ABSTRACT. We review some fixed point theorems which have appeared in our previous works [P1-11] on generalized convex spaces.

The concept of generalized convex spaces is a common generalization of various abstract convexities with or without linear structure and includes those of convex subsets of topological vector spaces, convex spaces of Lassonde, C-spaces due to Horvath, and many others. In the present paper, we review some fixed point theorems which have appeared mainly in our previous works [P1-11] on generalized convex spaces. Most of them are generalizations of well-known corresponding ones for topological vector spaces (t.v.s.).

1. Generalized convex spaces

A generalized convex space or a G-convex space $(Y, D; \Gamma)$ consists of a topological space $Y$, a nonempty set $D$, and a multimap $\Gamma : \langle D \rangle \to Y$ such that for each $A \in \langle D \rangle$ with cardinality $|A| = n + 1$, there exists a continuous function $\phi_{A} : \Delta_{n} \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_{A}(\Delta_{J}) \subset \Gamma(J)$, where $\langle D \rangle$ is the class of all nonempty finite subsets of $D$, $\Delta_{n}$ denotes the standard $n$-simplex with vertices $\{e_{i}\}_{i=0}^{n}$, and $\Delta_{J}$ the face of $\Delta_{n}$ corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_{0}, a_{1}, \ldots, a_{n}\}$ and $J = \{a_{i_{0}}, a_{i_{1}}, \ldots, a_{i_{k}}\} \subset A$, then $\Delta_{J} = \text{co}\{e_{i_{0}}, e_{i_{1}}, \ldots, e_{i_{k}}\}$.

We may write $\Gamma_{A} = \Gamma(A)$ and it is possible to assume $\Gamma_{A} = \phi_{A}(\Delta_{n})$ for each $A \in \langle D \rangle$. A G-convex space $(X, D; \Gamma)$ with $X \supset D$ is denoted by $(X \supset D; \Gamma)$ and $(X; \Gamma) := (X, X; \Gamma)$. For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be G-convex if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_{N} \subset Y$. For details on G-convex spaces and examples, see [P1,4,5, PK1-6], where basic theory was extensively developed.

A G-convex space $(X, D; \Gamma)$ is called a C-space if each $\Gamma_{A}$ is contractible (or more generally, $n$-connected for all $n \geq 0$) and, for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_{A} \subset \Gamma_{B}$. For $X = D$, this concept reduces to the one due to Horvath [H1,2].

We give here only a few examples of G-convex spaces:

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Examples 1. [PM] Let $X = D = [0, 1]$ and $Y = D' = S^1 = \{e^{2\pi it} : t \in [0, 1]\}$ in the complex plane $\mathbb{C}$. Let $f : X \to Y$ be a continuous function defined by $f(t) = e^{2\pi it}$. Define $\Gamma : \langle D' \rangle \to Y$ by

$$
\Gamma_A = f(\text{co}(f^{-1}(A))) \quad \text{for} \quad A \in \langle D' \rangle.
$$

Then $(Y \supset D'; \Gamma)$ is a compact $G$-convex space. (More generally, it is known that any continuous image of a $G$-convex space is a $G$-convex space.) We note the following:

1. $S^1$ lacks the fixed point property. Moreover, $S^1$ is an example of a compact $C$-space since each $\Gamma_A$ is contractible. Therefore, it shows that the Schauder conjecture (that is, any compact convex subset of a t.v.s. has the fixed point property) does not hold for $G$-convex spaces.

2. Note that, in $(Y \supset D'; \Gamma)$, singletons are $\Gamma$-convex; that is, $\Gamma_{\{y\}} = \{y\}$ for each $y \in D'$.

3. $(Y, D; \Gamma)$ with $\Gamma : \langle D \rangle \to Y$ defined by

$$
\Gamma_A = f(\text{co}A) \quad \text{for} \quad A \in \langle D \rangle
$$

is an example of a $G$-convex space satisfying $D \not\subset Y$.

Examples 2. Let $X = D = [0, 1]$ and $Y = D' = S^1 = \{e^{2\pi it} : t \in [0, 1]\}$. Define $f$ and $\Gamma_A$ as in Examples 1. Then $(Y \supset D'; \Gamma)$ is a compact $G$-convex space.

1. Note that $1 \in S^1$ and that $\Gamma_{\{1\}} = S^1$ is not contractible. Hence, $(Y \supset D'; \Gamma)$ is not a $C$-space.

2. Moreover $\Gamma_{\{1\}} \neq \{1\}$. Therefore, in general, $\Gamma_{\{y\}} \neq \{y\}$ in a $G$-convex space.

Examples 3. Similarly, for $X = [0, 1] \times [0, 1]$ or $X = [0, 1] \times [0, 1]$, we can made the torus, the Möbius band, and the Klein bottle into compact $G$-convex spaces, as was noted by Horvath [H1].

Several authors modified our definition of $G$-convex spaces and claimed that theirs are general than ours. All of them failed to give any proper meaningful example justifying their claims.

The following is known:

Theorem 1. [PM] Let $X$ be a compact Hausdorff uniform space with a basis $\mathcal{U}$ of the uniformity and $f : X \to X$ a continuous map. Then $f$ has a fixed point if and only if for any $V \in \mathcal{U}$, $\text{Gr}(f) \cap \overline{V} \neq \emptyset$.

2. Fan-Browder maps

A multimap (simply, a map) $T : X \to Y$ is a function from $X$ into the power set $2^Y$ of $Y$. $T(x)$ is called the value of $T$ at $x \in X$ and $T^-(y) := \{x \in X : y \in T(x)\}$ the fiber of $T$ at $y \in Y$. Let $T(A) := \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

For topological spaces $X$ and $Y$, a map $T : X \to Y$ is said to be closed if its graph $\text{Gr}(T) := \{(x, y) : x \in X, \ y \in T(x)\}$ is closed in $X \times Y$, and compact if its range $T(X)$ is contained in a compact subset of $Y$. 

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A map $T : X \rightarrow Y$ is said to be upper semicontinuous (u.s.c.) if for each closed set $B \subseteq Y$, the set $T^{-}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of $X$; lower semicontinuous (l.s.c.) if for each open set $B \subseteq Y$, the set $T^{-}(B)$ is open; and continuous if it is u.s.c. and l.s.c. Note that a compact closed multimap is u.s.c. and compact-valued; and that every u.s.c. map with closed values is closed.

A multimap with nonempty convex values and open fibers is called a Browder map. The well-known Fan-Browder fixed point theorem states that a Browder map $T$ from a compact convex subset $X$ of a t.v.s. into itself has a fixed point [Br].

From the celebrated KKM theorem, we obtained the following general form of the Fan-Browder fixed point theorem:

**Theorem 2.** [P4,8] Let $(X, D; \Gamma)$ be a $G$-convex space, and $S : D \rightarrow X$, $T : X \rightarrow X$ multimaps. Suppose that

1. $S(z)$ is open [resp. closed] for each $z \in D$;
2. for each $y \in X$, $M \in \langle S^{-}(y) \rangle$ implies $\Gamma_{M} \subset T^{-}(y)$; and
3. $X = S(N)$ for some $N \in \langle D \rangle$.

Then $T$ has a fixed point $x_{0} \in X$; that is, $x_{0} \in T(x_{0})$.

In [P8], this is applied to obtain various forms of known Fan-Browder type theorems, the Ky Fan intersection theorem, and the Nash equilibrium theorem.

The following is the dual form of Theorem 2:

**Theorem 3.** [P7] Let $(X, D; \Gamma)$ be a $G$-convex space and $S : X \rightarrow D$, $T : X \rightarrow X$ maps such that

1. for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_{M} \subset T(x)$;
2. $S^{-}(z)$ is open [resp. closed] for each $z \in D$; and
3. $X = \bigcup \{S^{-}(z) : z \in N\}$ for some $N \in \langle D \rangle$.

Then $T$ has a fixed point $x_{0} \in X$.

From Theorem 3, we have the following:

**Theorem 4.** [P10] Let $(X \supset D; \Gamma)$ be a $G$-convex space and $A : X \rightarrow X$ a multimap such that $A(z)$ is $G$-convex for each $z \in X$. If there exist $z_{1}, z_{2}, \cdots, z_{n} \in D$ and nonempty open [resp. closed] subsets $G_{i} \subset A^{-}(z_{i})$ for $i = 1, 2, \cdots, n$ such that $X = \bigcup_{i=1}^{n} G_{i}$, then $A$ has a fixed point.

**Theorem 5.** [P7] Let $(X, D; \Gamma)$ be a $G$-convex space and $S : X \rightarrow D$, $T : X \rightarrow X$ maps such that

1. for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_{M} \subset T(x)$; and
2. $X = \bigcup \{\text{Int} S^{-}(z) : z \in N\}$ for some $N \in \langle D \rangle$.

Then $T$ has a fixed point.

From Theorems 2-5, most of popular variations or generalizations of the Fan-Browder theorem (in the forms of the compact or so-called non-compact versions) can be deduced; see [P7,8,10].
3. **Φ-spaces and compact Φ-maps**

For a topological space $X$ and a $G$-convex space $(Y, D; \Gamma)$, a multimap $T : X \rightarrow Y$ is called a **Φ-map** provided that there exists a multimap $S : X \rightarrow D$ satisfying

(a) for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subseteq T(x)$; and

(b) $X = \bigcup \{\text{Int } S^{-1}(y) : y \in D\}$.

A $G$-convex space $(Y, D; \Gamma)$ is called a **Φ-space** if $Y$ is a Hausdorff uniform space and for each entourage $V$ there exists a $Φ$-map $T : Y \rightarrow Y$ such that $\text{Gr}(T) \subseteq V$. This concept is originated from Horvath [H1], where a number of examples are given.

**Theorem 6.** [P1] If $(Y, D; \Gamma)$ is a $Φ$-space, then any compact continuous function $g : Y \rightarrow Y$ has a fixed point.

Recall that a nonempty convex subset $X$ of a t.v.s. $E$ is said to be **locally convex** (in the sense of Krauthausen) if for every $x \in X$ there exists a basis $V(x)$ of neighborhoods of $x$ such that every $V \in V(x)$ is convex.

It is easily checked that every locally convex subset $X$ is a $Φ$-space $(X; \Gamma)$ with $\Gamma_A = \text{co } A$ for $A \in \langle X \rangle$. Therefore, Theorem 6 works when $X$ is a locally convex subset of a Hausdorff t.v.s. or $X$ is a convex subset of a locally convex Hausdorff t.v.s.

For $C$-spaces, Theorem 6 reduces to Horvath [H1, Theorem 4.4], where some examples of $Φ$-spaces and applications of Theorem 6 were given.

A **$G$-convex uniform space** $(X \supset D; \Gamma)$ is a $G$-convex space such that $D$ is dense in $X$ and $(X, \mathcal{U})$ is a Hausdorff uniform space, where $\mathcal{U}$ is a basis of the uniformity consisting of symmetric entourages.

A **locally $G$-convex space** is a $G$-convex uniform space $(X \supset D; \Gamma)$ with a basis $\mathcal{U}$ such that for each $U \in \mathcal{U}$ and each $x \in X$,

$$U[x] = \{x' \in X : (x, x') \in U\}$$

is $\Gamma$-convex.

**Lemma 1.** [P6] A locally $G$-convex space $(X \supset D; \Gamma)$ is a $Φ$-space.

An **$LG$-space** is a $G$-convex uniform space $(X \supset D; \Gamma)$ with a basis $\mathcal{U}$ such that for each $U \in \mathcal{U}$, $U[C] := \{x \in X : C \cap U[x] \neq \emptyset\}$ is $Γ$-convex whenever $C \subset X$ is $Γ$-convex.

For a $C$-space $(X; \Gamma)$, the concept of $LG$-spaces reduces to that of $LC$-spaces due to Horvath [H1,2].

**Lemma 2.** [P6] Every $LG$-space $(X \supset D; \Gamma)$ is a locally $G$-convex space if $\Gamma\{x\} = \{x\}$ for each $x \in D$.

A $C$-space $(X; \Gamma)$ is an **$LC$-metric space** if $X$ is equipped with a metric $d$ such that for any $\varepsilon > 0$, the set $\{x \in X : d(x, A) < \varepsilon\}$ is $Γ$-convex whenever $A$ is $Γ$-convex in $X$ and open balls in $(X, d)$ are $Γ$-convex.

**Examples 4.** The $G$-convex spaces $(Y \supset D'; \Gamma)$ in Examples 1 and 2 are not $Φ$-spaces because of Theorem 6. Moreover, in view of Lemmas 1 and 2, they are neither locally $G$-convex nor an $LG$-space. Note that in these examples, a neighborhood of $1 \in \text{S}^1$ is not $Γ$-convex.
In 1990, Ben-El-Mechaiekh raised the following problem (see [P2]): Does the Fan-Browder fixed point theorem hold if we assume the map $T$ is compact instead of the compactness of its domain $X$?

This is still open. The following are general forms of partial solutions:

**Theorem 7.** [P2] Let $E$ be a Hausdorff t.v.s. whose nonempty convex subsets have the fixed point property for compact continuous single-valued selfmaps. Let $X$ be a nonempty convex subset of $E$ and $T : X \to X$ a $\Phi$-map. If $T$ is compact, then $T$ has a fixed point.

**Theorem 8.** [P1,2] Let $(Y, D; \Gamma)$ be a paracompact $C$-space. If it is also a $\Phi$-space, then any compact $\Phi$-map $T : Y \to Y$ has a fixed point.

**Theorem 9.** [P2] Let $(X; \Gamma)$ be a Hausdorff $G$-convex space, and $T : X \to X$ a $\Phi$-map. If $T$ is compact, then $T^n$ has a fixed point for $n \geq 2$.

**Theorem 10.** [P2,9] Let $(X \supset D; \Gamma')$ be a paracompact $LC$-space such that $\Gamma_{\{x\}} = \{x\}$ for all $x \in D$. Then any compact $\Phi$-map $T : X \to X$ has a fixed point.

**Theorem 11.** [P9] Let $(X; \Gamma)$ be a paracompact $LC$-space such that $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, $Y$ a compact $LC$-metric subset of $X$, and $Z \subset X$ with $\dim_X Z \leq 0$. Let $T : X \to Y$ be a l.s.c. map with closed values such that $T(x)$ is $\Gamma$-convex for $x \notin Z$. Then $T$ has a fixed point.

In [P3], further fixed point theorems for l.s.c. multimaps in $LC$-metric spaces are given.

### 4. Kakutani maps

Usually, an u.s.c. multimap with nonempty closed convex values is called a Kakutani map within the category of t.v.s.

We have the following fixed point theorem for general Kakutani type maps defined on particular types of $G$-convex spaces:

**Theorem 12.** [P9] Let $(X \supset D; \Gamma)$ be an $LG$-space and $T : X \to X$ a compact u.s.c. multimap with closed $\Gamma$-convex values. Then $T$ has a fixed point $x_0 \in X$.

For a single-valued map, Theorem 12 reduces to the following:

**Corollary 12.1.** [P9] Let $(X \supset D; \Gamma)$ be an $LG$-space such that $\Gamma_{\{x\}} = \{x\}$ for all $x \in D$. Then any compact continuous function $f : X \to X$ has a fixed point.

In view of Lemmas 1 and 2, this is also a simple consequence of Theorem 6.

Let $(X \supset D; \Gamma)$ be a $G$-convex uniform space with a basis $U$ and $K$ a nonempty subset of $X$. We say that $K$ is of the Zima type [PK6] whenever for every $V \in U$ there exists a $U \in U$ such that for every $A \in \langle D \rangle$ and every $\Gamma$-convex subset $M$ of $K$ the following implication holds:

$$M \cap U[z] \neq \emptyset, \forall z \in A \Rightarrow M \cap V[u] \neq \emptyset, \forall u \in \Gamma_A,$$

where $U[z] = \{ x \in X : (z, x) \in U \}$. 
Theorem 13. [PK6] Let \((X \supset D; \Gamma)\) be a \(G\)-convex uniform space and \(T : X \rightarrow X\) a compact u.s.c. map with nonempty closed \(\Gamma\)-convex values. If \(T(X)\) is of the Zima type, then \(T\) has a fixed point \(x_\ast \in X\).

5. Better admissible maps

Let \((X, D; \Gamma)\) be a \(G\)-convex space and \(Y\) a topological space. We define the better admissible class \(\mathfrak{B}\) of multimaps from \(X\) into \(Y\) as follows [P4]:

\[ F \in \mathfrak{B}(X, Y) \iff F : X \rightarrow Y \text{ is a map such that for any } N \in \langle D \rangle \text{ with } |N| = n + 1 \]

and any continuous function \(p : F(\Gamma_N) \rightarrow \Delta_n\), the composition

\[ \Delta_n \xrightarrow{\phi_N, \Gamma_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} F(\Gamma_N) \xrightarrow{p} \Delta_n \]

has a fixed point. Note that \(\Gamma_N\) can be replaced by the compact set \(\phi_N(\Delta_n)\).

We give some subclasses of \(\mathfrak{B}\) as follows [P4, PK1,3]:

For topological spaces \(X\) and \(Y\), an admissible class \(\mathfrak{A}_c(X, Y)\) of maps \(F : X \rightarrow Y\) is one such that, for each nonempty compact subset \(K\) of \(X\), there exists a map \(G \in \mathfrak{A}_c(K, Y)\) satisfying \(G(x) \subset F(x)\) for all \(x \in K\); where \(\mathfrak{A}_c\) consists of finite compositions of maps in a class \(\mathfrak{A}\) of maps satisfying the following properties:

(i) \(\mathfrak{A}\) contains the class \(\mathfrak{C}\) of (single-valued) continuous functions;
(ii) each \(T \in \mathfrak{A}_c\) is u.s.c. with nonempty compact values; and
(iii) for any polytope \(P\), each \(T \in \mathfrak{A}_c(P, P)\) has a fixed point, where the intermediate spaces are suitably chosen.

Here, a polytope \(P\) is a homeomorphic image of a standard simplex. There are lots of examples of \(\mathfrak{A}\) and \(\mathfrak{A}_c^\ast\).

Subclasses of the admissible class \(\mathfrak{A}_c^\ast\) are classes of continuous functions \(\mathfrak{C}\), the Kakutani maps \(K\) (with convex values and codomains are convex spaces), Browder maps, \(\Phi\)-maps, selectionable maps, locally selectionable maps having convex values, the Aronszajn maps \(M\) (with \(R_4\) values), the acyclic maps \(V\) (with acyclic values), the Powers maps \(V_c\) (finite compositions of acyclic maps), the O'Neill maps \(N\) (continuous with values of one or \(m\) acyclic components, where \(m\) is fixed), the u.s.c. approachable maps \(A\) (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, \(\sigma\)-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class \(K_c^+\) of Lassonde, the class \(V_c^+\) of Park et al., u.s.c. approximable maps of Ben-El-Mechaiekh and Idizk, and many others.

Note that for a subset \(X\) of a t.v.s. and any space \(Y\), an admissible class \(\mathfrak{A}_c^\ast(X, Y)\) is a subclass of \(\mathfrak{B}(X, Y)\). Some examples of maps in \(\mathfrak{B}\) not belonging to \(\mathfrak{A}_c^\ast\) were known. Note that the connectivity map due to Nash and Girolo is such an example.
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For a particular type of $G$-convex spaces, we established fixed point theorems for the class $\mathfrak{B}$:

**Theorem 14.** [P4] Let $(X, D; \Gamma)$ be a $\Phi$-space and $F \in \mathfrak{B}(X, X)$. If $F$ is closed and compact, then $F$ has a fixed point.

Note that Theorem 14 generalizes Theorem 6.

For a non-closed map, we have the following:

**Corollary 14.1.** Let $(X, D; \Gamma)$ be a compact $\Phi$-space and $F \in \mathfrak{B}(X, X)$. Then $F$ has a fixed point.

Since a locally $G$-convex space is a $\Phi$-space by Lemma 1, we have

**Corollary 14.2.** Let $(X \supset D; \Gamma)$ be a locally $G$-convex space. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.

Similarly, by Lemma 2, we have

**Corollary 14.3.** Let $(X \supset D; \Gamma)$ be an $LG$-space such that $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.

For topological spaces $X$ and $Y$, we adopt the following [PK1]:

$F \in \mathcal{V}(X, Y) \iff F : X \to Y$ is an acyclic map; that is, an u.s.c. multimap with compact acyclic values.

$F \in \mathcal{V}_{c}(X, Y) \iff F : X \to Y$ is a finite composition of acyclic maps where the intermediate spaces are topological.

It is known that $\mathcal{V}_{c}(X, Y) \subset \mathfrak{B}(X, Y)$ whenever $X$ is a $G$-convex space, and that any map in $\mathcal{V}_{c}$ is closed.

**Corollary 14.4.** Let $(X, D; \Gamma)$ be a $\Phi$-space. Then any compact map $F \in \mathcal{V}_{c}(X, X)$ has a fixed point.

6. Approximable maps

In this section, all spaces are assumed to be Hausdorff.

Recently, Ben-El-Mechaiekh et al. [B, BC] introduced the class $A$ of approachable multimaps as follows:

Let $X$ and $Y$ be uniform spaces (with respective bases $\mathcal{U}$ and $\mathcal{V}$ of symmetric entourages). A multimap $T : X \to Y$ is said to be approachable whenever $T$ admits a continuous $W$-approximative selection $s : X \to Y$ for each $W$ in the basis $\mathcal{W}$ of the product uniformity on $X \times Y$; that is, $\text{Gr}(s) \subset W[\text{Gr}(F)]$, where

$$W[A] := \bigcup_{z \in A} W[z] = \{z' \in X \times Y : W[z'] \cap A \neq \emptyset\}$$

for any $A \subset X \times Y$, and

$$W[z] := \{z' \in X \times Y : (z, z') \in W\}$$
for $z \in X \times Y$.

A multimap $T : X \rightarrow Y$ is said to be approximable if its restriction $T|_K$ to any compact subset $K$ of $X$ is approachable.

Note that an approachable map is always approximable. Recall that Ben-El-Mechaiekh et al. [B, BC] established a large number of properties and examples of approachable or approximable maps.

We denote $F \in \mathcal{A}(X, Y)$ if $F : X \rightarrow Y$ is approachable.

The following two lemmas are [BC, Lemmas 2.4 and 4.1], respectively.

**Lemma 4.** Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$, $(Z, \mathcal{W})$ be three uniform spaces, with $Z$ compact, and let $\Psi : Z \rightarrow X$, $\Phi : X \rightarrow Y$ be two u.s.c. closed-valued approachable maps. Then so is their composition $\Phi \circ \Psi$.

**Lemma 5.** If $X$ is a nonempty convex subset of a locally convex t.v.s. and if $\Phi \in \mathcal{A}(X, X)$ is closed and compact, then $\Phi$ has a fixed point.

From Lemmas 4 and 5, we show that certain approachable maps are better admissible if their domains are G-convex spaces as follows:

**Lemma 6.** [P6] Let $(X : D; \Gamma)$ be a G-convex uniform space and $(Y, \mathcal{V})$ a uniform space. If $F \in \mathcal{A}(X, Y)$ is closed and compact, then $F \in \mathcal{B}(X, Y)$.

From Theorem 14 and Lemma 6, we have

**Theorem 15.** [P6] Let $(X : D; \Gamma)$ be a $\Phi$-space and $F \in \mathcal{A}(X, X)$. If $F$ is closed and compact, then $F$ has a fixed point.

**Examples 5.** We give some examples of approachable maps $T : X \rightarrow Y$ as follows:

1. Any selectable multimap is approximable.
2. A locally selectable map $T$ with convex values is approximable whenever $Y$ is a convex subset of a t.v.s.
3. An u.s.c. map $T$ with nonempty convex values is approachable whenever $X$ is paracompact and $Y$ is a convex subset of a locally convex t.v.s.
4. An u.s.c. map $T$ with nonempty compact contractible values is approachable whenever $X$ is a finite polyhedron.
5. An u.s.c. map $T$ with nonempty compact values having trivial shape (that is, contractible in each neighborhood in $Y$) is approachable whenever $X$ is a finite polyhedron.

For (1) and (2), see [P11]; and for (3)-(5), see [B].

The following is due to Ben-El-Mechaiekh et al. [BC, Proposition 3.9]:

**Lemma 7.** Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniform spaces. If either

(i) $X$ is paracompact and $(Y, \Gamma)$ is an LC-space; or
(ii) $X$ is compact and $(Y, \Gamma)$ is an LG-space,

then every u.s.c. map $F : X \rightarrow Y$ with nonempty $\Gamma$-convex values is approachable; that is, $F \in \mathcal{A}(X, Y)$.

Note that Lemma 7(i) generalizes Examples 5(3).
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In our previous work [P6], (ii) is incorrectly stated and causes some incorrect statements. For example, [P6, Theorem 4] should be stated for $LG$-spaces as in Theorem 12.

From Lemmas 6 and 7, we have the following correction of [P4, Lemma 4.5]:

**Lemma 8.** Let $(X \supset D; \Gamma)$ be a compact $LG$-space. Then any u.s.c. map $F : X \rightarrow X$ with nonempty closed $\Gamma$-convex values belongs to $\mathfrak{B}(X, X)$.

Consequently, correct forms of [P4, Corollary 4.7 and Theorem 4.8] are Theorem 12 and Corollary 14.1, respectively, in the present paper.

We add two types of new multimaps in the class $\mathfrak{B}$:

**Lemma 9.** Let $(X, D; \Gamma)$ be a $G$-convex space and $F : X \rightarrow X$ be an u.s.c. map such that either

(i) $F$ has nonempty compact contractible values; or

(ii) $F$ has nonempty compact values having trivial shape,

then $F \in \mathfrak{B}(X, X)$.

**Proof.** For any $N \in \langle D \rangle$ with $|N| = n + 1$ and any continuous function $p : F(\Gamma_N) \rightarrow \Delta_n$, consider the composition

$$
\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} F(\Gamma_N) \xrightarrow{p} \Delta_n.
$$

Note that $(F|_{\Gamma_N}) \circ \phi_N$ is an u.s.c. multimap having values of the type (i) or (ii) and defined on a finite polyhedron $\Delta_n$. Therefore $p \circ (F|_{\Gamma_N}) \circ \phi_N$ is approachable by Lemma 4, and has a fixed point by Lemma 5. This completes our proof.

From Theorem 14 and Lemma 9, we have

**Theorem 16.** [P4] Let $(X, D; \Gamma)$ be a $\Phi$-space and $F : X \rightarrow X$ be a map such that all of its values are either (i) nonempty contractible or (ii) nonempty and of trivial shape. If $F$ is closed and compact, then $F$ has a fixed point.

Note that Case (i) of Theorem 16 is a consequence of Corollary 14.4. In the category of t.v.s., Theorem 16(i) holds for Kakutani maps since convex values are contractible. But, for $G$-convex spaces, $\Gamma$-convex values are only known to be connected and that is why we need Lemma 7.

**References**


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