数理解析研究所講究録
1484 巻 2006 年 5-15

On Dividends for Cooperative Games

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1 Introduction

All cooperative games (transferable utility games) on a finite set of palyers form a finite dimensional vector space (linear space). The set of unanimity games is a basis in this vector space and the Harsanyi dividends are coefficients in the representatin of a game as linear comibination of the unanimity games. Thus they play a very important role in cooperative game theory. In this paper we discuss two practical situations of games, restrictions on coalitions and fuzzy extensions of games, and show some useful results based on the dividends.

2 Dividends for cooperative games

Let N be a finite set of n elements, i.e., $N=\{1,2,\ldots,n\}$. Elements of N are called players. Any subset of N is called a coalition. A (transferable utility) game on N is a set funcion $v:2^{N}\rightarrow \mathbf{R}$ with $v(\emptyset)=0$. The function v is usually called a characteristic function and each value $v(S)$ is called the worth of the coalition S. The set of all games on N is denoted by Γ^{N}. In the following, we use abbreviated notations such as $v(\{i\})=v(i)$, $S\cup\{i\}=S\cup i,\ \ S\setminus\{i\}=S\setminus i$ and so on. We also denote the set $2^{N}\setminus\emptyset$ as Ω.

Definition 1 *A game $v\in\Gamma^{N}$ is said to be*

1. *monotonic if $v(S)\leq v(T)$ for all $S,T\subseteq N$ such that $S\subseteq T$,*
2. *superadditive if $v(S)+v(T)\leq v(S\cup T)$ for all $S,T\subseteq N$, such that $S\cap T=\emptyset$,*
3. *convex if $v(S)+v(T)\leq v(S\cup T)+v(S\cap T)$, for all $S,T\subseteq N$, or equivalently if $v(S\cup i)-v(S)\leq v(T\cup i)-v(T)$, for all $i\in N$, and $S\subseteq T\subseteq N\setminus i$.*

Definition 2 *A game $v\in\Gamma^{N}$ is said to be symmetric if the condition $|S|=|T|$ for all $S,T\subseteq N$ implies that $v(S)=v(T)$.*

The sum of two games $v,w\in\Gamma^{N}$ is defined by $(v+w)(S)=v(S)+w(S)$ for all $S\subseteq N$, and the scalar multiplication of $v\in\Gamma^{N}$ by a scalar $\alpha\in\mathbf{R}$ is defined by $(\alpha v)(S)=\alpha v(S)$ for all $S\subseteq N$. Thus the space Γ^{N} of all games on N is a vector space and its dimension is clearly $2^{n}-1$, since each game is specified by the worths $v(S)$ for all $S\subseteq N$ with $S\neq\emptyset$. As a basis in this space we may consider unanimity games u_{T} defined by

$$u_{T}(S)=\begin{cases}1 & \text{if } S\supseteq T,\\ 0 & \text{otherwise,}\end{cases}$$

for any $T \in \Omega$. Then each game $v \in \Gamma$ is a linear combination of unanimity games,

$$v = \sum_{T \in \Omega} d_T(v) u_T.$$

The coefficient $d_T(v)$ is given by

$$d_T(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$$

and called the (Harsanyi) dividend of T for the game v. For convenience' sake, we may put $d_\emptyset(v) = 0$. In combinatorics, $d_.(v)$ viewed as a set function on Ω is called the Möbius transform of v. The dividends satisfy the following recursive formula:

$$d_T(v) = \begin{cases} 0, & \text{if } T = \emptyset, \\ v(T) - \sum_{S \subset T} d_S(v), & \text{if } T \neq \emptyset. \end{cases}$$

This formula can be also written as

$$d_T(v) = \begin{cases} v(i), & \text{if } T = \{i\} \text{ for } i \in N, \\ v(T) - \sum_{\emptyset \neq S \subset T} d_S(v), & \text{if } T \subseteq N,\ |T| > 1. \end{cases}$$

It is obvious that $d_T(v+w) = d_T(v) + d_T(w)$, $d_T(\alpha v) = \alpha d_T(v)$. We should also note that

$$v(S) = \sum_{T \in \Omega, T \subseteq S} d_T(v), \quad \forall S \in \Omega.$$

Definition 3 *A game $v \in \Gamma^N$ is said to be*

1. *positive if $d_S(v) \geq 0$ for all $S \in \Omega$.*
2. *almost positive if $d_S(v) \geq 0$ for all $S \in \Omega$ such that $|S| > 1$.*

We may describe some properties of games in terms of the dividends.

Proposition 1 *A game $v \in \Gamma^N$ is symmetric if and only if the following holds:*

$$|S| = |T| \text{ for } S, T \subseteq N \implies d_S(v) = d_T(v).$$

Recalling that $v(S) = \sum_{\emptyset \neq T \subseteq S} d_T(v)$, a game $v \in \Gamma^N$ is monotonic if and only if $\sum_{\emptyset \neq R \subseteq S} d_R(v) \leq \sum_{\emptyset \neq R \subseteq T} d_R(v)$, i.e., $\sum_{S' \subseteq S} \sum_{\emptyset \neq R \subseteq T \setminus S} d_{S' \cup R}(v) \geq 0$ for any $S, T \subseteq N$ with $S \subseteq T$. Similarly, a game $v \in \Gamma^N$ is superadditive if and only if $\sum_{\emptyset \neq S' \subseteq S} \sum_{\emptyset \neq T' \subseteq T} d_{S' \cup T'}(v) \geq 0$. Moreover, for $S \subseteq N$ and $i \notin S$, $v(S \cup i) - v(S) = \sum_{R \in \Omega, R \subseteq S \cup i} d_R(v) - \sum_{R \in \Omega, R \subseteq S} d_R(v) = \sum_{S' \subseteq S} d_{S' \cup i}(v)$. Therefore, a game $v \in \Gamma^N$ is convex if and only if

$$\sum_{S' \subseteq S} d_{S' \cup i}(v) \leq \sum_{T' \subseteq T} d_{T' \cup i}(v), \text{ i.e., } \sum_{S' \subseteq S} \sum_{\emptyset \neq R \subseteq T \setminus S} d_{S' \cup R \cup i}(v) \geq 0$$

for any $i \in N$ and $S \subseteq T \subseteq N \setminus i$. Thus any positive game is convex.

3 Solutions based on the dividends for cooperative games

In a game $v \in \Gamma$, the main issue is the distribution of the worth $v(N)$ among the palyers. A one-point solution of a game is specified by a function $\phi : \Gamma^N \to \mathbf{R}^n$, which asscoaites a payoff vector $\phi(v) = (\phi_i(v))_{i\in N}$ called the value with each game $v \in \Gamma^N$. Since this function ϕ is usually assumed to be linear (with respect to v), the value is a linear combination of the values for unanimity games, i.e.,

$$\phi(v) = \sum_{T\in\Omega} d_T(v)\phi(u_T).$$

Typical examples of values are the Shapley and Banzhaf values given by

$$\varphi_i(u_T) = \begin{cases} \frac{1}{|T|}, & \text{if } i \in T \\ 0, & \text{otherwise,} \end{cases} \qquad \beta_i'(u_T) = \begin{cases} \frac{1}{2^{|T\setminus i|}}, & \text{if } i \in T \\ 0, & \text{otherwise.} \end{cases}$$

respectively. Thus

$$\varphi_i(v) = \sum_{T\in\Omega, T\ni i} \frac{d_T(v)}{|T|}, \quad \beta_i'(v) = \sum_{T\in\Omega N, T\ni i} \frac{d_T(v)}{2^{|T\setminus i|}}.$$

respectively.

More general value is given by the sharing system $p = (p_i^T)_{T\in\Omega, i\in T}$ satisfying $p \geq 0$ and $\sum_{j\in T} p_j^T = 1$ for each $T \in \Omega$. The set of all sharing systems is denoted by P. The payoff vector $\phi^p(v) \in \mathbf{R}^n$, $p \in P$, given by

$$\phi_i^p(v) = \sum_{T\in\Omega, T\ni i} p_i^T d_T(v), \quad i \in N$$

is called a Harsanyi payoff vector [5] or Möbius value (in a restricted case) [3]. It is obvious that $\sum_{i\in N} \phi_i^p(v) = v(N)$, i.e., the Harsanyi payoff vector is efficient. Strictly speaking, the Shapley value is a Harsanyi payoff vector, but the Banzhaf value is not.

Other solutions are given by set-valued solutions. The most fundamental set-valued solution is the core of v, defined by

$$C(v) = \{x \in \mathbf{R}^n \mid x(N) = v(N),\ x(S) \geq v(S),\ \forall S \in \Omega\},$$

where $x(S) = \sum_{i\in S} x_i$. Another set-valued solution is the Harsanyi set (Derks et al. [5]) or the selectope (Bilbao et al. [2], Derks et al. [4]).

Definition 4 *The Harsanyi set $H(v)$ of a game $v \in \Gamma^N$ is defined as the set of all Harsanyi payoff vectors, i.e.,*

$$H(v) = \{\phi^p(v) \mid p \in P\}.$$

The Harsanyi set can be defined in another way as the selectope (see Derks et al. [4]), which will be discussed later.

Proposition 2 *[5] For each game $v \in \Gamma^N$, $C(v) \subseteq H(v)$. Moreover $C(v) = H(v)$ if and only if v is almost positive.*

4 Dividends for cooperative games under restrictions on coalitions

In this section we deal with restrictions on coalitions.

Definition 5 *A subset of 2^N, i.e., a set of coalitions $\mathcal{F}$ is said to be a feasible coalition system (FCS for short) on N if it satisfies the following:*

$$\emptyset \in \mathcal{F}, \quad \{i\} \in \mathcal{F} \quad \forall i \in N.$$

A maximal coalition that belongs to $\mathcal{F}$ and is contained in S is called an $\mathcal{F}$-component of S. We denote by $C_{\mathcal{F}}(S)$ the set of the $\mathcal{F}$-components of S.

Definition 6 *Let $v \in \Gamma^N$ be a game and $\mathcal{F}$ be an FCS on N. The $\mathcal{F}$-restricted game $v^{\mathcal{F}} \in \Gamma^N$ of v is defined by*

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T).$$

Definition 7 *An FCS $\mathcal{F}$ on N is said to be a partition system (PS for short) on N if, for all $S \subseteq N$, the set $C_{\mathcal{F}}(S)$ of $\mathcal{F}$-components of S is a partition, i.e., if $C_{\mathcal{F}}(S) = \{T_1, \dots, T_l\}$ then $T_i \cap T_j = \emptyset$ $(i \neq j)$ and $\bigcup_{j=1}^{l} T_j = S$.*

An FCS $\mathcal{F}$ is a PS if and only if

$$S, T \in \mathcal{F}, \ S \cap T \neq \emptyset \implies S \cup T \in \mathcal{F}.$$

Now we let $\mathcal{F}$ be an FCS and consider the dividends for the $\mathcal{F}$-restricted game $v^{\mathcal{F}}$ of v. We define the $\mathcal{F}$-restricted dividend $d_T^{\mathcal{F}}(v)$ of $T \in \mathcal{F}$ for v in a recursive manner as follows:

$$d_T^{\mathcal{F}}(v) = \begin{cases} 0, & \text{if } T = \emptyset, \\ v(T) - \displaystyle\sum_{S \subset T, S \in \mathcal{F}} d_S^{\mathcal{F}}(v), & \text{if } T \in \mathcal{F}, \end{cases}$$

or

$$d_T^{\mathcal{F}}(v) = \begin{cases} v(i), & \text{if } T = \{i\} \text{ for } i \in N, \\ v(T) - \displaystyle\sum_{\emptyset \neq S \subset T, S \in \mathcal{F}} d_S^{\mathcal{F}}(v), & \text{if } T \in \mathcal{F}, \ |T| > 1. \end{cases}$$

Theorem 1 *Let $v \in \Gamma^N$ be a game and $\mathcal{F}$ be a PS on N. Then*

$$d_T(v^{\mathcal{F}}) = \begin{cases} d_T^{\mathcal{F}}(v), & \text{if } T \in \mathcal{F}, \\ 0, & \text{if } T \notin \mathcal{F}. \end{cases}$$

(Proof) Note that $d_{\emptyset}(v^{\mathcal{F}}) = d_{\emptyset}^{\mathcal{F}}(v) = 0$. We prove the theorem by induction. First let $T = \{i\}$ for $i \in N$. Then $T \in \mathcal{F}$ and

$$d_T(v^{\mathcal{F}}) = v^{\mathcal{F}}(T) = v(T) = d_T^{\mathcal{F}}(v).$$

Next consider the case $|T| > 1$ and $T \in \mathcal{F}$. Then

$$d_T(v^{\mathcal{F}}) = v^{\mathcal{F}}(T) - \sum_{S \subset T} d_S(v^{\mathcal{F}}) = v(T) - \sum_{S \subset T, S \in \mathcal{F}} d_S^{\mathcal{F}}(v) = d_T^{\mathcal{F}}(v).$$

Finally, consider the case $|T| > 1$ and $T \notin \mathcal{F}$ with $C_{\mathcal{F}}(S) = \{T_1, \ldots, T_l\}$. Then

$$d_T(v^{\mathcal{F}}) = v^{\mathcal{F}}(T) - \sum_{S \subset T} d_S(v^{\mathcal{F}}) = \sum_{j=1}^{l} v(T_j) - \sum_{S \subset T, S \in \mathcal{F}} d_S^{\mathcal{F}}(v).$$

If $S \subset T$ and $S \in \mathcal{F}$, then there exists $k \in \{1, \ldots, l\}$ such that $S \cap T_k \neq \emptyset$. Since $\mathcal{F}$ is a PS, $S \cup T_k \in \mathcal{F}$. In view of maximaility of T_k, $S \cup T_k = T_k$, which implies that $S \subseteq T_k$. Hence

$$d_T(v^{\mathcal{F}}) = \sum_{j=1}^{l} \{v(T_j) - \sum_{S \subseteq T_j, S \in \mathcal{F}} d_S^{\mathcal{F}}(v)\} = 0.$$

This completes the proof of the theorem. □

Thus if $\mathcal{F}$ is a PS, we may consider a one-point solution of a game $v \in \Gamma^N$ under the PS $\mathcal{F}$ by the Harsanyi payoff vector of the restricted game $v^{\mathcal{F}}$, i.e., for $p \in P$,

$$\phi_i^p(v^{\mathcal{F}}) = \sum_{T \in \Omega, T \ni i} p_i^T d_T(v^{\mathcal{F}}) = \sum_{T \in \mathcal{F}, T \ni i} p_i^T d_T^{\mathcal{F}}(v), \quad i \in N.$$

Now we consider set-valued solutions of v under $\mathcal{F}$.

Definition 8 *Let $\mathcal{F}$ be an FCS on N such that $N \in \mathcal{F}$. Then the core of v under $\mathcal{F}$ is defined as follows:*

$$C(\mathcal{F}, v) = \{x \in \mathbf{R}^n \mid x(N) = v(N),\ x(S) \geq v(S)\ \forall S \in \mathcal{F}\}.$$

Proposition 3 *[1] Let $\mathcal{F}$ be a PS on N with $N \in \mathcal{F}$. Then*

$$C(v^{\mathcal{F}}) = C(\mathcal{F}, v).$$

Definition 9 *Let $\mathcal{F}$ be an FCS on N such that $N \in \mathcal{F}$. A selector on $\mathcal{F}$ is a function $\alpha : \mathcal{F} \setminus \{\emptyset\} \to N$ with $\alpha(S) \in S$ for every nonempty coalition $S \in \mathcal{F}$.*

We denote by $\mathcal{A}(\mathcal{F})$ the set of all selectors on $\mathcal{F}$.

Definition 10 *Let $\mathcal{F}$ be an FCS on N such that $N \in \mathcal{F}$ and $\alpha \in \mathcal{A}(\mathcal{F})$ be a selctor. The selection corresponding to α is the vector $m^\alpha(\mathcal{F}, v)$ defined by*

$$m_i^\alpha(\mathcal{F}, v) = \sum_{S \in \mathcal{F}, \alpha(S) = i} d_S^{\mathcal{F}}(v)$$

for every $i \in N$ and $v \in \Gamma^N$.

Definition 11 *Let $\mathcal{F}$ be an FCS on N such that $N \in \mathcal{F}$. The selectope for a game $v \in \Gamma^N$ under $\mathcal{F}$ is given by*

$$S(\mathcal{F}, v) = \text{conv } \{m^\alpha(\mathcal{F}, v) \mid \alpha \in \mathcal{A}(\mathcal{F})\}.$$

For a sharing system $p \in P$, the $\mathcal{F}$-restricted Harsanyi payoff vector $\phi^{\mathcal{F},p}(v) \in \mathbf{R}^n$ is given by

$$\phi_i^{\mathcal{F},p}(v) = \sum_{T \in \mathcal{F}, T \ni i} p_i^T d_T^{\mathcal{F}}(v), \quad i \in N.$$

Definition 12 *Let $\mathcal{F}$ be an FCS on N. The $\mathcal{F}$-restricted Harsanyi set of v is defined by*

$$H(\mathcal{F}, v) = \{\phi^{\mathcal{F},p}(v) \mid p \in P\}.$$

Theorem 2 *Let $v \in \Gamma^N$ and $\mathcal{F}$ be an FCS on N such that $N \in \mathcal{F}$. Then*

$$S(\mathcal{F}, v) = H(\mathcal{F}, v).$$

(Proof) ($\subseteq$) For $(q_\alpha)_{\alpha \in \mathcal{A}(\mathcal{F})}$, $q_\alpha \geq 0$, $\sum_{\alpha \in \mathcal{A}(\mathcal{F})} q_\alpha = 1$, let $p_i^T = \sum_{\alpha \in \mathcal{A}(\mathcal{F}), \alpha(T) = i} q_\alpha$ for $T \in \mathcal{F}$, $T \neq \emptyset$. For $T \notin \mathcal{F}$, let p^T be an arbitrary probability distribution on T (It does not appear in the following discussion). Then $p \geq 0$ and for $T \in \mathcal{F}$

$$\sum_{i \in T} p_i^T = \sum_{i \in T} \sum_{\alpha \in \mathcal{A}(\mathcal{F}), \alpha(T) = i} q_\alpha = \sum_{\alpha \in \mathcal{A}(\mathcal{F})} q_\alpha = 1.$$

Hence $p \in P$. Moreover,

$$\begin{aligned} \phi_i^{\mathcal{F},p}(v) &= \sum_{T \in \mathcal{F}, T \ni i} p_i^T d_T^{\mathcal{F}}(v) = \sum_{T \in \mathcal{F}, T \ni i} \sum_{\alpha \in \mathcal{A}(\mathcal{F}), \alpha(T) = i} q_\alpha d_T^{\mathcal{F}}(v) \\ &= \sum_{\alpha \in \mathcal{A}(\mathcal{F})} q_\alpha \sum_{T \in \mathcal{F}, \alpha(T) = i} d_T^{\mathcal{F}}(v) = \sum_{\alpha \in \mathcal{A}(\mathcal{F})} q_\alpha m_i^\alpha(\mathcal{F}, v). \end{aligned}$$

($\supseteq$) For $p \in P$, let $q_\alpha = \prod_{S\in\mathcal{F}, S\neq\emptyset} p^S_{\alpha(S)}$. Then $q_\alpha \geq 0$ and

$$\sum_{\alpha\in\mathcal{A}(\mathcal{F})} q_\alpha = \sum_{\alpha\in\mathcal{A}(\mathcal{F})} \prod_{S\in\mathcal{F},S\neq\emptyset} p^S_{\alpha(S)} = \prod_{S\in\mathcal{F},S\neq\emptyset} (\sum_{i\in S} p^S_i) = 1.$$

Moreover,

$$\begin{aligned}\sum_{\alpha\in\mathcal{A}(\mathcal{F})} q_\alpha m^\alpha_i(\mathcal{F},v) &= \sum_{\alpha\in\mathcal{A}(\mathcal{F})} q_\alpha \sum_{T\in\mathcal{F},\alpha(T)=i} d^{\mathcal{F}}_T(v) = \sum_{T\in\mathcal{F},T\ni i} \sum_{\alpha\in\mathcal{A}(\mathcal{F}),\alpha(T)=i} (\prod_{S\in\mathcal{F},S\neq\emptyset} p^S_{\alpha(S)}) d^{\mathcal{F}}_T(v) \\ &= \sum_{T\in\mathcal{F},T\ni i} p^T_i [\prod_{S\in\mathcal{F},S\neq\emptyset,T} (\sum_{j\in S} p^S_j)] d^{\mathcal{F}}_T(v) = \sum_{T\in\mathcal{F},T\ni i} p^T_i d^{\mathcal{F}}_T(v) = \phi^{\mathcal{F},p}_i(v).\end{aligned}$$

This completes the proof of the theorem. □

Corollary 1 *If $v \in \Gamma^N$ and $\mathcal{F}$ is a PS on N such that $N \in \mathcal{F}$, then $S(\mathcal{F}, v) = H(v^{\mathcal{F}})$.*

Definition 13 *A game $v \in \Gamma^N$ is said to be almost $\mathcal{F}$-positive if $d^{\mathcal{F}}_S(v) \geq 0$ for all $S \in \mathcal{F}$ with $|S| > 1$.*

Proposition 4 *[1] Let $v \in \Gamma^N$ and $\mathcal{F}$ be an FCS on N such that $N \in \mathcal{F}$. Then $S(\mathcal{F}, v) \subseteq C(\mathcal{F}, v)$ if and only if v is almost $\mathcal{F}$-positive.*

Definition 14 *An FCS $\mathcal{F}$ on N is said to be an intersecting system on N if $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then $S \cap T, S \cup T \in \mathcal{F}$.*

Proposition 5 *[1] Let $v \in \Gamma^N$ and $\mathcal{F}$ be an intersecting system on N. Then $C(\mathcal{F}, v) \subseteq S(\mathcal{F}, v)$.*

Corollary 2 *Let $v \in \Gamma^N$ be an almost $\mathcal{F}$-positive game and $\mathcal{F}$ be an intersecting system on N with $N \in \mathcal{F}$. Then*

$$C(\mathcal{F}, v) = C(v^{\mathcal{F}}) = S(\mathcal{F}, v) = S(v^{\mathcal{F}}) = H(\mathcal{F}, v) = H(v^{\mathcal{F}}).$$

5 Fuzzy extensions of cooperative games

In a cooperative game, each coalition $S \subseteq N$ can be identified with the vector e^S defined by $e^S_i = 1$ if $i \in S$ and $e^S_i = 0$ if $i \notin S$ and the domain of the characteristic function v is identified with $\{0,1\}^n$. Hence extending $\{0,1\}^n$ to $[0,1]^n$ implies extending ordinary (crisp) coalitions to fuzzy coalitions. Thus, given the player set N, a cooperative fuzzy game ξ on N is a function from $[0,1]^n$ to $\mathbf{R}$ with $\xi(0) = 0$. The set of all cooperative fuzzy games on N is denoted by Δ^N.

In this paper we use the following notations. First, the vector $e^{\{i\}}$ is simply denoted by e^i. For $s, t \in [0,1]^n$, vectors $s \vee t$ and $s \wedge t \in [0,1]^n$ are defined by

$$(s \vee t)_i = \max\{s_i, t_i\}, \text{ and } (s \wedge t)_i = \min\{s_i, t_i\}, \ i = 1, 2, \dots, n,$$

respectively. For $s \in [0,1]^n$, let $\operatorname{supp} s = \{i \in N \mid s_i > 0\}$.

Definition 15 *Cooperative fuzzy game $\xi \in \Delta^N$ is said to be*

1. *monotonic if $\xi(s) \le \xi(t)$ for all $s, t \in [0,1]^n$ such that $s \le t$.*
2. *weakly superadditive if $\xi(s) + \xi(t) \le \xi(s \vee t)$ for all $s, t \in [0,1]^n$ such that $s \wedge t = 0$.*
3. *strongly superadditive if $\xi(s) + \xi(t) \le \xi(s+t)$ for all $s, t \in [0,1]^n$ such that $s + t \in [0,1]^n$.*
4. *convex if $\xi(s) + \xi(t) \le \xi(s \vee t) + \xi(s \wedge t)$ for all $s, t \in [0,1]^n$.*

The sum of two games $\xi, \eta \in \Delta^N$, and the scalar multiplication of ξ by $\alpha \in \mathbf{R}$ are defined by

$$\begin{aligned} (\xi + \eta)(s) &= \xi(s) + \eta(s), \quad && \forall s \in [0,1]^n, \\ (\alpha\xi)(s) &= \alpha\xi(s), && \forall s \in [0,1]^n, \end{aligned}$$

respectively. Thus, Δ^N is also a vector space.

A typical fuzzy game can be obtained from an ordinary cooperative game $v \in \Gamma^N$ by extending v in an appropriate manner. This extension is denoted by ξ_v hereafter.

Definition 16 *The extension ξ_v is said to be a U-extension if*

$$\xi_{v+w} = \xi_v + \xi_w, \text{ for all } v, w \in \Gamma^N, \ \ \xi_{\alpha v} = \alpha \xi_v, \text{ for all } v \in \Gamma^N, \ \alpha \in \mathbf{R}.$$

Moerover, a U-extension ξ_v is called a W-extension if it additionally satisfies

$$\begin{aligned} &\xi_{u_T}(s) = \xi_{u_T}(s_{|T}), \text{ for all } s \in [0,1]^n, \\ &\xi_{u_T}(s) \le \xi_{u_T}(t), \text{ for all } s, t \in [0,1]^n \text{ such that } s \le t. \end{aligned}$$

Here $s_{|T} \in [0,1]^n$ is defined by $(s_{|T})_i = s_i$ if $i \in T$ and $(s_{|T})_i = 0$ if $i \notin T$.

Since the space Γ^N of all cooperativce games on N is a linear space and the set of unanimity games forms a basis, a U-extension of any game v is specified by those of unanimity games. Namely, if ξ is a U-extension, then

$$\xi_v(s) = \sum_{T \subseteq N} d_T(v) \xi_{u_T}(s), \ \ \forall s \in [0,1]^n.$$

We obtain a stronger result for a W-extension.

Proposition 6 *If $\xi_v \in \Delta^N$ is a W-extension of $v \in \Gamma^N$, then*

$$\xi_v(s) = \sum_{T \subseteq \mathrm{supp}\ s} d_T(v) \xi_{u_T}(s), \ \forall s \in [0,1]^n.$$

Two well-known examples of W-extensions are the multilinear extension and the Lovász extension. The multilinear extension m_v of v, introduced by Owen [8], is given by $m_{u_T}(s) = \prod_{i \in T} s_i$. The explicit formula of the multilinear extension is given by

$$m_v(s) = \sum_{S \subseteq N} \Big(\prod_{i \in S} s_i \prod_{i \in N \setminus S} (1 - s_i)\Big) v(S), \ \ \forall s \in [0,1]^n.$$

On the other hand, the Lovász extension [6] l_v is given by $l_{u_T}(s) = \min_{i \in T} s_i$. For $s \in [0,1]^n$ and $h \in [0,1]$, let $[s]_h = \{i \in N \mid s_i \geq h\}$. Then the explicit formula for the Lovász extension can be written as

$$l_v(s) = \int_0^1 v([s]_h)dh, \ \ \forall s \in [0,1]^n.$$

Some properties of these extensions are given in Tanino [9].

Proposition 7 *If a cooperative game $v \in \Gamma^N$ is superadditive, then its Lovász extension $l_v \in \Delta^N$ is weakly superadditive.*

Proposition 8 *If a cooperative game $v \in \Gamma^N$ is convex, then its Lovász extension $l_v \in \Delta^N$ is a convex cooperative fuzzy game.*

Proposition 9 *A game $v \in \Gamma^N$ is convex if and only if its Lovász extension $l_v \in \Delta^N$ is a strongly superadditive cooperative fuzzy game.*

6 Fuzzy extensions with restrictions on coalitions

If we deal with fuzzy extensions and restrictions on coalitions together in cooperative games, there may exist two approaches:

1. First obtain the restricted game of the original crisp game and extend it.
2. First extend the original crisp game and obtain its restricted game.

In the first approach, we obtain the game $\xi_{v^{\mathcal{F}}}$. In order to discuss the second approach, we introduce some concepts (see Moritani et al. [7]).

Definition 17 *A set $F \subseteq [0,1]^n$ is called a feasible fuzzy coalition system (FFCS for short) on N if*

1. $\alpha e^i \in F$ for any $\alpha \in [0,1]$.
2. For any $s \in [0,1]^n$ and $t \in F$ satisfying $t \leq s$, there exists $\bar{t} \in C^F(s)$ such that $t \leq \bar{t}$.

Here a vector $r \in [0,1]^n$ is said to be an F-vector of s if

$$r \leq s, \ r \in F, \text{ and } r \leq r' \leq s \text{ with } r' \in F \text{ imply that } r' = r,$$

and $C^F(s)$ is the set of all F-vectors of s.

Definition 18 *Given a cooperative fuzzy game $\xi \in \Delta^N$ and an FFCS F on N, the restricted game $\xi^F \in \Delta^N$ of ξ by F is defined by*

$$\xi^F(s) = \sum_{t \in C^F(s)} \xi(t).$$

Definition 19 *FFCS F on N is said to be a partition fuzzy system (PFS for short) if it satisfies one of the following equivalent conditions:*

1. *$C^F(s)$ is a partition of s for any $s \in [0,1]^n$.*
2. *For each $s \in [0,1]^n$, there esists a partition $\{I_1, \ldots, I_l\}$ of* supp s such that $C^F(s) = \{s_{|I_1}, \ldots, s_{|I_l}\}$, where the vector $s_{|I_j}$ is defined by $(s_{|I_j})_k = s_k$ if $k \in I_j$ and $(s_{|I_j})_k = 0$ otherwise.
3. If $s, y \in F$ and $s \wedge y \neq 0$, then $s \vee y \in F$.

The FFCS $F(\mathcal{F})$ corresponding to an FCS $\mathcal{F}$ is defined as follows:

$$F(\mathcal{F}) = \{s \in [0,1]^n \mid \text{supp } s \in \mathcal{F}\}.$$

Proposition 10 *If $\mathcal{F}$ is an FCS on N, then the corresponding $F(\mathcal{F})$ is an FFCS on N. If $\mathcal{F}$ is a PS, then $F(\mathcal{F})$ is a PFS.*

In the second approach, we obtain the fuzzy game $(\xi_v)^{F(\mathcal{F})}$. Our interest lies in the question whether $\xi_{v^{\mathcal{F}}}$ and $(\xi_v)^{F(\mathcal{F})}$ coincide or not.

Lemma 1 *Let $F(\mathcal{F})$ be the FFCS on N corresponding to an FCS $\mathcal{F}$ on N. For an arbitrary $s \in [0,1]^n$, let $C_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}$. Then the set of all F-vectors of s is given by $C^{F(\mathcal{F})}(s) = \{s_{|I_1}, \ldots, s_{|I_l}\}$.*

Lemma 2 *Let $\mathcal{F}$ be a PS on N and $C_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}$ for $s \in [0,1]^n$. Then*

$$\{T \in \mathcal{F} \mid \emptyset \neq T \subseteq \text{supp } s\} = \{T \in \mathcal{F} \mid T \neq \emptyset,\ \exists_1 k \in \{1, \ldots, l\} \ :\ T \subseteq I_k\}.$$

Let $v \in \Gamma^N$ be a game, $\mathcal{F}$ be a PS on N and ξ be a W-estension. For $s \in [0,1]^n$, let $C_{\mathcal{F}}(\text{supp } s) = \{I_1, \ldots, I_l\}$. Then

$$\xi_{v^{\mathcal{F}}}(s) = \sum_{j=1}^{l} \sum_{T \in \mathcal{F}, T \subseteq I_j} d_T(v^{\mathcal{F}}) \xi_{u_T}(s_{|I_j}).$$

On the other hand,

$$(\xi_v)^{F(\mathcal{F})}(s) = \sum_{j=1}^{l} \sum_{T \subseteq I_j} d_T(v) \xi_{u_T}(s_{|I_j}).$$

Therefore, they are generally different.

Definition 20 *FCS $\mathcal{F}$ on N is said to be subcomplete if the condition $T \in \mathcal{F}$ and $S \subseteq T$ implies that $S \in \mathcal{F}$. A subcomplete FCS is simply written as SCS.*

Theorem 3 *Let ξ_v be a W-extension of a game $v \in \Gamma^N$ and $F(\mathcal{F})$ be the FFCS corresponding to an FCS $\mathcal{F}$ on N. If $\mathcal{F}$ is an SCS and PS, then $\xi_{v^{\mathcal{F}}} = (\xi_v)^{F(\mathcal{F})}$.*

(Proof) If $\mathcal{F}$ is an SCS, then $T \in \mathcal{F}$ for all $T \subseteq I_i \in \mathcal{F}$ and $d_T(v^{\mathcal{F}}) = d_T(v)$ for any $T \in \mathcal{F}$. Therefore the theorem is immediate. □

7 Conclusion

In this paper we have focused on the dividends for cooperative games and discussed some solution concepts for cooperative games with restrictions on coalitions. Moreover we have studied extensions of cooperative games to cooperative fuzzy games under restrictions on coalitions. This research is supported by the Japan Society for the Promotion of Science under the Grant-in-Aid for Scientic Research No. 16510114.

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