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A duality theorem for a three-phase partition problem
三相分割問題に対する双対定理*

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Abstract The three-phase partition problem is to divide a given domain $\Omega \subset \mathbb{R}^2$ into three subdomains with a triple junction having least interfacial area. Recently, we proposed a duality theorem for a three-phase partition problem in [5]. We introduced a notion of separation of three convex sets by triangles to define a dual problem. In this paper, we explain its outline.

1. Introduction

The three-phase partition problem is to divide a given domain $\Omega \subset \mathbb{R}^2$ into three subdomains with a triple junction having least interfacial area (Fig.1.1).

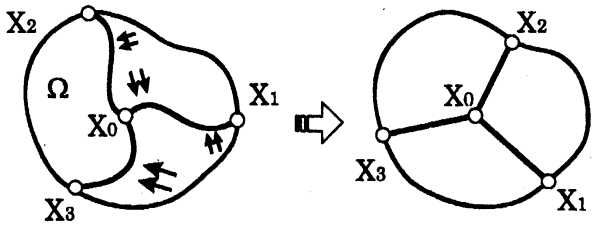

FIGURE 1.1. Three-phase partition problem

Sternberg and Zeimer [7] and Ikota and Yanagida [1] formulated this problem as a variational problem and discussed stability of stationary solutions. However, since the shortest curve joining two points X_0 and X_i is the line segment $[X_0, X_i]$, it can be formulated as an extremal problems in a Euclidean space. From this point of view, we discussed stability and studied its game-theoretic aspect in [2][3]. Further, we gave a duality theorem for an extremal problem (P_0) induced from the three-phase partition problem in [4].

$$(P_0) \qquad \begin{array}{ll} \text{Minimize} & f(X_0,\dots,X_3) := \sum_{i=1}^{3} \|X_i - X_0\| \\ \text{subject to} & X_0 \in \Omega,\ X_i \in C_i\ (i=1,2,3), \end{array}$$

where $\|\cdot\|$ denotes the Euclidean norm and C_i $(i = 1,2,3)$ are closed convex sets with non-empty interior in $\mathbb{R}^2$ such that $\Omega := \mathrm{cl}(\cap_{i=1}^{3} C_i^c)$ is non-empty (Fig. 1.2). Moreover, we improved the duality theorem so that the dual problem does not

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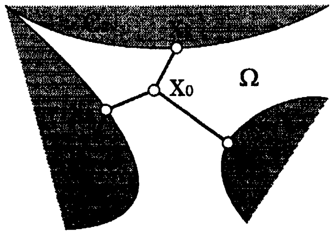

FIGURE 1.2. Primal problem (P_0)

include the variables of the primal problem in [5]. The aim of this paper is to state the outline of [4][5].

In this paper we use the following notations. For any closed convex sets C_1 and C_2, we define $d(C_1, C_2) := \min\{||X_1 - X_2|| \mid X_i \in C_i\ (i = 1, 2)\}$. We denote by $N(X_i; C_i)$ the normal cone of C_i at X_i, that is,

$$N(X_i; C_i) := \{Y \in \mathbb{R}^n \mid Y^T(X - X_i) \le 0\ \forall X \in C_i\}.$$

2. First-order optimality condition

As is easily seen from Fig. 1.2, Ω is not always a convex set. So the primal problem (P_0) is not a convex programming problem in general. We modify it so that it becomes a convex programming problem.

$$(P) \qquad \begin{array}{ll} \text{Minimize} & \sum_{i=1}^{3} ||X_i - X_0|| \\ \text{subject to} & X_0 \in \mathbb{R}^2,\ X_i \in C_i\ (i = 1, 2, 3). \end{array}$$

The only difference is that Ω is replaced by $\mathbb{R}^2$. We say a feasible solution $(X_0, \ldots, X_3)$ for (P_0) (or (P)) non-degenerate if X_0 does not coincide with any X_i $(i = 1, 2, 3)$.

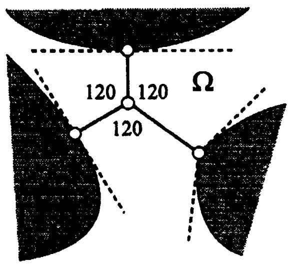

FIGURE 2.1. Young's law and the transversality condition

Theorem 2.1. *Let $(X_0, \ldots, X_3)$ be a non-degenerate minimal solution for (P_0). Then it is a minimum solution for (P). Further, it satisfies Young's law*

$$\angle X_i X_0 X_j = 120^\circ \textit{ for any } i \neq j\ (\in \{1, 2, 3\}) \tag{2.1}$$

and the transversality condition

$$X_0 - X_i \in N(X_i; C_i) \quad (i = 1, 2, 3). \tag{2.2}$$

Proof. There exists an open convex neighborhood C_0 of X_0 such that $(X_0, \dots, X_3)$ is a minimum point of f on $C := C_0 \times C_1 \times C_2 \times C_3$. Since f and C are convex, $(X_0, \dots, X_3)$ is a minimum point of f on $R^n \times C_1 \times C_2 \times C_3$. Hence it is a minimum solution for (P). According to Kuhn-Tucker's theorem, see e.g. Rockafellar [6], there exist multipliers $\lambda_i \geq 0$ $(i = 1, 2, 3)$ such that $0 \in R^{4n}$ belongs to the subdifferential of the Lagrange function

$$L(X_0, \dots, X_3) := \sum_{i=1}^{3} ||X_i - X_0|| + \sum_{i=1}^{3} \lambda_i \delta(X_i|C_i),$$

where $\delta(X_i|C_i)$ denotes the characteristic function of C_i. Picking up X_0-component of the subdifferential ∂L, we have

$$n_1 + n_2 + n_3 = 0 \in R^n, \tag{2.3}$$

where $n_i := (X_0 - X_i)/||X_i - X_0||$, which implies Young's law. Picking up X_i-component $(i = 1, 2, 3)$ of ∂L, we have $0 \in -n_i + \lambda_i N(X_i; C_i)$, which implies the transversality condition. □

Remark 2.1. *In* [1][2][3][7], *smooth cases were studied. Then the transversality condition (2.2) becomes a orthogonality condition, that is, $X_0 - X_i$ touches the boundary $\partial\Omega$ at right angles.*

3. Separation by a triangle

In this section, we first review classical duality theorems in brief. Next, we introduce separation of three convex sets by a triangle.

One of the simplest duality theorems is the following. Let C_1 be a non-empty convex set in $\mathbb{R}^2$ and $A \notin C_1$ a point. Then the primal problem is

$$(P_1) \qquad \begin{array}{ll} \text{Minimize} & ||X_1 - A|| \\ \text{subject to} & X_1 \in C_1. \end{array}$$

Its dual problem (D_1) is to maximize the distance from A to a hyperplane H that separates A and C_1. We can rephrase it as maximizing the width of a river that separates A and C_1 (Fig. 3.1), where a river stands for the area sandwiched between two parallel lines.

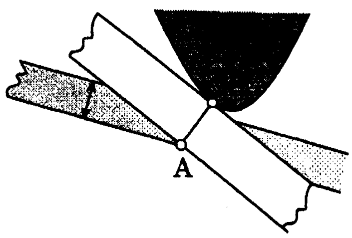

FIGURE 3.1. Dual problem (D_1) is to maximize the width of a river that separates A and C_1.

If we replace A with a convex set C_2 such that $C_1 \cap C_2 = \phi$, then the primal problem becomes as follows.

$$(P_2) \qquad \begin{array}{ll} \text{Minimize} & ||X_1 - X_2|| \\ \text{subject to} & X_i \in C_i \ (i = 1, 2). \end{array}$$

Its dual problem (D_2) is to minimize the width of a river that separates C_1 and C_2 (Fig. 3.2).

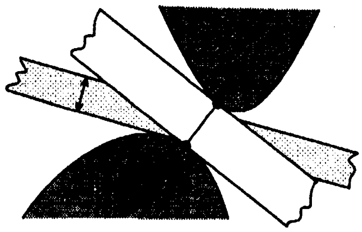

FIGURE 3.2. Dual problem (D_2) is to maximize the width of a river that separates C_1 and C_2.

If we take the epigraph epi$f := \{(x, r) \mid f(x) \le r\}$ of a convex function f and the hypograph hyp$g := \{(x, r) \mid r \le g(x)\}$ of a concave function g as C_1 and C_2, respectively, and measure the width of the river in the vertical direction, then duality between (P_2) and (D_2) becomes to Fenchel's duality, see e.g. [6, Theorem 31.1].

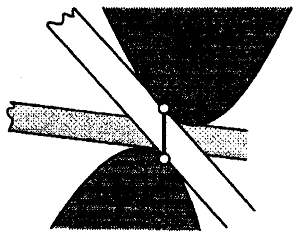

FIGURE 3.3. Fenchel's duality theorem

Therefore, classical dual problems can be described in terms of rivers or hyperplanes separating two convex sets. In this paper, we introduce the notion of triangles separating three convex sets in order to define the dual problem for the three-phase partition problem (P).

Definition 3.1. *We say that a triangle $\Delta \subset \Omega$ separates $\{C_i\}_{i=1}^3$ if there are three closed half spaces $\{H_i^-\}_{i=1}^3$ such that $C_i \subset H_i^-$ for every i and $\Delta = \cap_{i=1}^3 H_i^+$, where H_i^+ denotes the closed half space opposite to H_i^- (Fig. 3.4).*

Before defining the dual problem, let us consider the special case that Ω is a triangle determined by three closed half spaces.

Lemma 3.1. ([4]) *When Ω is a triangle in $\mathbb{R}^2$, it holds that*

$$\min(P) = \min(P_0) = \textit{the smallest height of } \Omega.$$

Duality Theorem

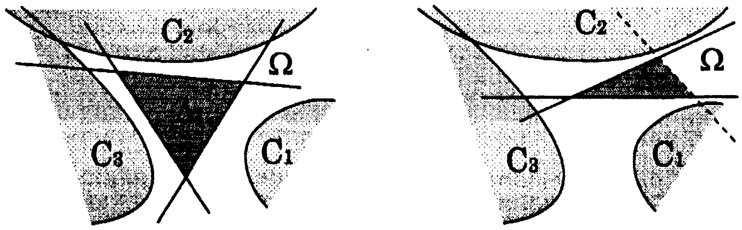

FIGURE 3.4. Δ_1 separates $\{C_i\}_{i=1}^3$, and Δ_2 does not separate $\{C_i\}_{i=1}^3$.

So we define the dual problem as follows.

(D) Maximize the smallest height of a triangle that separates $\{C_i\}_{i=1}^3$.

The following is the main result.

Theorem 3.1. *([5]) Let $(X_0, \ldots, X_3)$ be a non-degenerate minimal solution for (P_0). Then it is a minimum solution for (P) and the strong duality relationship holds.*

$$\sum_{i=1}^{3} ||X_i - X_0|| = \min(P) = \max(D). \tag{3.1}$$

Remark 3.1. *Since the maximum value for (D) is attained by a regular triangle, we may restrict triangles to regular triangles in (D). However, it is clear that regular triangles are not enough when Ω is a (general) triangle. That's why we defined the dual problem with (general) triangles.*

Corollary 3.1. *When Ω is bounded, the dual problem can be simplified as follows.*

(D) *Maximize the smallest height of a triangle contained in Ω.*

Indeed, let Δ be an arbitrary triangle contained in Ω. Then, by separation theorem, there exists a closed half space $H_i^+ \supset \Delta$ such that $C_i \subset H_i^-$ for each $i = 1, 2, 3$. Since $\Delta_1 := \cap_{i=1}^3 H_i^+$ is contained in the bounded set Ω, Δ_1 is a triangle. Further, since $\Delta \subset \Delta_1$, the smallest height of Δ is bounded from above by the smallest height of Δ_1 (Fig. 3.5).

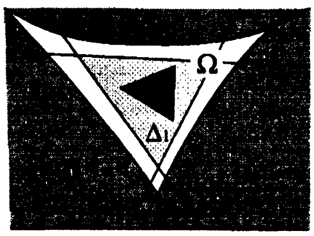

FIGURE 3.5. Although Δ does not separate $\{C_i\}_{i=1}^3$, Δ_1 separates them.

Remark 3.2. *In* [1][7], *they dealt with a weighted objective function. It is not hard to extend the present results to the weighted objective function*

$$\sum_{i=1}^{3} \sigma_i ||X_i - X_0||,$$

where $\sigma_i > 0$ $(i = 1, 2, 3)$ *can be interpreted as interface tension (Fig. 3.6).*

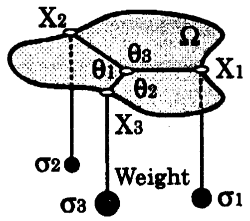

FIGURE 3.6. $\sigma_i > 0$ $(i = 1, 2, 3)$ can be regarded as interface tensions.

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