数理解析研究所講究録
1484 巻 2006 年 40-51

CONVERGENCE THEOREMS OF IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

J. K. KIM, K. S. KIM AND S. M. KIM

ABSTRACT. We prove that an implicit iteration process with errors which is generated by a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the mappings in convex metric spaces. Our main theorems extend and improve the recent results of Sun, Wittmann and Xu-Ori.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that X is a metric space and let $F(T_i)$ $(i \in \mathcal{N})$ be the set of all fixed points of mappings T_i respectively, that is, $F(T_i) = \{x \in X : T_i x = x\}$, where $\mathcal{N} = \{1, 2, 3, \cdots, N\}$. The set of common fixed points of T_i $(i \in \mathcal{N})$ denotes by $\mathcal{F}$, that is, $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$.

Definition 1.1. ([2],[4],[5]) Let $T : X \to X$ be a mapping.

(1) T is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y)$$

2000 Mathematics Subject Classification: 46B25, 47H05, 47H09, 49H10.
All correspondence should be sent to J. K. Kim.
Keywords: implicit iteration process, finite family of asymptotically quasi-nonexpansive mappings, common fixed point, convex metric spaces.
This work was supported by Kyungnam University Foundation Grant 2006.

for all $x, y \in X$.

(2) T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p)$$

for all $x \in X$ and $p \in F(T)$.

(3) T is said to be *asymptotically nonexpansive* if there exists a sequence $h_n \in [1, \infty)$ with $\lim_{n \to \infty} h_n = 1$ such that

$$d(T^n x, T^n y) \leq h_n d(x, y)$$

for all $x, y \in X$ and $n \geq 0$.

(4) T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $h_n \in [1, \infty)$ with $\lim_{n \to \infty} h_n = 1$ such that

$$d(T^n x, p) \leq h_n d(x, p) \tag{1.1}$$

for all $x \in X$, $p \in F(T)$ and $n \geq 0$.

Remark 1.1. From the Definition 1.1, we know that the following implications hold:

$$\begin{array}{ccc} (1) & \Longrightarrow & (3) \\ \Downarrow F(T) \neq \emptyset & & \Downarrow F(T) \neq \emptyset \\ (2) & \Longrightarrow & (4) \end{array}$$

In 2001, Xu-Ori [16] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \cdots, T_N$ be self-mappings of C and suppose that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of T_i, $i = 1, 2, \cdots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0,1)$, $x_0 \in C$:

$$\begin{aligned}
x_1 &= t_1 x_0 + (1-t_1)T_1 x_1,\\
x_2 &= t_2 x_1 + (1-t_2)T_2 x_2,\\
&\vdots\\
x_N &= t_N x_{N-1} + (1-t_N)T_N x_N,\\
x_{N+1} &= t_{N+1} x_N + (1-t_{N+1})T_1 x_{N+1},\\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1-t_n)T_n x_n, \quad n \geq 1, \tag{1.2}$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in $\mathcal{N}$.) And they proved the weak convergence of the process (1.2).

In 2003, Sun [12] extend the process (1.2) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0,1)$ and an initial point $x_0 \in C$, which is defined as follows :

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1},\\
&\vdots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1},\\
&\vdots
\end{aligned}$$

which can be written in the following compact form :

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_n, \quad n \geq 1, \tag{1.3}$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$.

Sun [12] proved the strong convergence of the process (1.3) to a common fixed point, requiring only one member T in the family $\{T_i : i \in \mathcal{N}\}$ to be semi-compact. The result of Sun [12] generalized and extended the corresponding main results of Wittmann [15] and Xu-Ori [16].

The purpose of this paper is to introduce and study the convergence problem of an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. The main result of this paper is also, an extension and improvement of the well-known corresponding results in [1]–[11].

For the sake of convenience, we recall some definitions and notations.

In 1970, Takahashi [13] introduced the concept of convexity in a metric space and the properties of the space.

Definition 1.2. ([13]) Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y).$$

X together with a convex structure W is called a *convex metric space*, denoted it by (X, d, W). A nonempty subset K of X is said to be *convex* if $W(x, y, \lambda) \in K$ for all $(x, y, \lambda) \in K \times K \times I$.

Remark 1.2. Every normed space is a convex metric space, where a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. In fact,

$$\begin{aligned} d(u, W(x, y, z; \alpha, \beta, \gamma)) &= \|u - (\alpha x + \beta y + \gamma z)\| \\ &\leq \alpha \|u - x\| + \beta \|u - y\| + \gamma \|u - z\| \\ &= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z), \quad \forall\, u \in X. \end{aligned}$$

But there exists some convex metric spaces which can not be embedded into normed space.

Example 1.1. Let $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$. For $x = (x_1,\ x_2,\ x_3)$, $y = (y_1, y_2, y_3) \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$, we define a mapping $W : X^3 \times I^3 \to X$ by

$$\begin{aligned} &W(x, y, z; \alpha, \beta, \gamma) \\ &= (\alpha x_1 + \beta y_1 + \gamma z_1,\ \alpha x_2 + \beta y_2 + \gamma z_2,\ \alpha x_3 + \beta y_3 + \gamma z_3) \end{aligned}$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \Big| x_1 y_1 + x_2 y_2 + x_3 y_3 \Big|.$$

Then we can show that (X, d, W) is a convex metric space, but it is not a normed space.

Example 1.2. Let $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\}$. For each $x = (x_1,$ $x_2)$, $y = (y_1, y_2) \in Y$ and $\lambda \in I$. we define a mapping $W : Y^2 \times I \to Y$ by

$$W(x, y; \lambda) = \Big(\lambda x_1 + (1 - \lambda) y_1, \frac{\lambda x_1 x_2 + (1 - \lambda) y_1 y_2}{\lambda x_1 + (1 - \lambda) y_1}\Big)$$

and define a metric $d : Y \times Y \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (Y, d, W) is a convex metric space, but it is not a normed space.

Definition 1.3. Let (X, d, W) be a convex metric space with a convex structure W and let $T_i : X \to X$ $(i \in \mathcal{N})$ be asymptotically quasi-nonexpansive mappings. For any given $x_0 \in X$, the iteration process $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= W(x_0, T_1 x_1, u_1; \alpha_1, \beta_1, \gamma_1), \\ &\vdots \\ x_N &= W(x_{N-1}, T_N x_N, u_N; \alpha_N, \beta_N, \gamma_N), \\ x_{N+1} &= W(x_N, T_1^2 x_{N+1}, u_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}), \\ &\vdots \\ x_{2N} &= W(x_{2N-1}, T_N^2 x_{2N}, u_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}), \\ x_{2N+1} &= W(x_{2N}, T_1^3 x_{2N+1}, u_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}) \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 1 \tag{1.4}$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$, $\{u_n\}$ is bounded sequence in X, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for $n = 1, 2, 3, \cdots$. Process (1.4) is called the *implicit iteration process with error* for a finite family of mappings T_i $(i = 1, 2, \cdots, N)$.

If $u_n = 0$ in (1.4) then,

$$x_n = W(x_{n-1}, T_i^k x_n; \alpha_n, \beta_n), \quad n \geq 1 \tag{1.5}$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$, $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0,1]$ such that $\alpha_n + \beta_n = 1$ for $n = 1, 2, 3, \cdots$. Process (1.5) is called the *implicit iteration process* for a finite family of mappings T_i $(i = 1, 2, \cdots, N)$.

2. Main Results

In order to prove the main theorems of this paper, we need the following lemma:

Lemma 2.1. ([14]) *Let $\{\rho_n\}, \{\lambda_n\}$ and $\{\delta_n\}$ be the nonnegative sequences satisfying*

$$\rho_{n+1} \leq (1 + \lambda_n)\rho_n + \mu_n, \quad \forall\, n \geq n_0,$$

and

$$\sum_{n=n_0}^{\infty} \lambda_n < \infty, \quad \sum_{n=n_0}^{\infty} \mu_n < \infty.$$

Then $\lim_{n\to\infty} \rho_n$ exists.

Now we state and prove the following main theorems of this paper.

Theorem 2.1. *Let (X, d, W) be a complete convex metric space. Let $\{T_i : i \in \mathcal{N}\}$ be a finite family of asymptotically quasi-nonexpansive mappings from X into X, that is,*

$$d(T_i^n x, p_i) \leq (1 + h_{n(i)})d(x, p_i)$$

for all $x \in X$, $p_i \in F(T_i)$, $i \in \mathcal{N}$. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} h_{n(i)} < \infty \quad (i \in \mathcal{N})$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{u_n\}$ is arbitrary bounded sequence in X. Then the implicit iteration process with error $\{x_n\}$ generated by (1.4) converges to a common fixed point of $\{T_i : i \in \mathcal{N}\}$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, \mathcal{F}) = 0,$$

where $D_d(x, \mathcal{F})$ denotes the distance from x to the set $\mathcal{F}$, i.e., $D_d(x, \mathcal{F}) = \inf_{y \in \mathcal{F}} d(x, y)$.

Proof. The necessity is obvious. Thus we will only prove the sufficiency. For any $p \in \mathcal{F}$, from (1.4), where $n = (k-1)N + i$, $T_n = T_{n(\bmod N)} = T_i$, $i \in \mathcal{N}$, it follows that

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), p) \\ &\leq \alpha_n d(x_{n-1}, p) + \beta_n d(T_i^k x_n, p) + \gamma_n d(u_n, p) \\ &\leq \alpha_n d(x_{n-1}, p) + \beta_n (1 + h_{k(i)}) d(x_n, p) + \gamma_n d(u_n, p) \\ &\leq \alpha_n d(x_{n-1}, p) + (\beta_n + h_{k(i)}) d(x_n, p) + \gamma_n d(u_n, p) \\ &\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n + h_{k(i)}) d(x_n, p) + \gamma_n d(u_n, p), \end{aligned} \tag{2.1}$$

for all $p \in \mathcal{F}$. Since $\lim_{n\to\infty} \gamma_n = 0$, there exists a natural number n_1, such that for $n > n_1$, $\gamma_n \leq \frac{s}{2}$. Hence

$$\alpha_n = 1 - \beta_n - \gamma_n \geq 1 - (1 - s) - \frac{s}{2} = \frac{s}{2}$$

for $n > n_1$. Thus, we have by (2.1) that

$$\alpha_n d(x_n, p) \leq \alpha_n d(x_{n-1}, p) + h_{k(i)} d(x_n, p) + \gamma_n d(u_n, p)$$

and

$$\begin{aligned} d(x_n, p) &\leq d(x_{n-1}, p) + \frac{h_{k(i)}}{\alpha_n} d(x_n, p) + \frac{\gamma_n}{\alpha_n} d(u_n, p) \\ &\leq d(x_{n-1}, p) + \frac{2}{s} h_{k(i)} d(x_n, p) + \frac{2}{s} \gamma_n d(u_n, p). \end{aligned} \tag{2.2}$$

Since $\sum_{n=1}^{\infty} h_{k(i)} < \infty$ for all $i \in \mathcal{N}$, $\lim_{n\to\infty} h_{n(i)} = 0$ for each $i \in \mathcal{N}$. Hence there exists a natural number n_2, as $n > \frac{n_2}{N} + 1$ i.e., $n > n_2$ such that

$$h_{n(i)} \leq \frac{s}{4}, \quad \forall\, i \in \mathcal{N}.$$

Then (2.2) becomes

$$d(x_n, p) \leq \frac{s}{s - 2h_{k(i)}} d(x_{n-1}, p) + \frac{2\gamma_n}{s - 2h_{k(i)}} d(u_n, p). \tag{2.3}$$

Let

$$1 + \Delta_{k(i)} = \frac{s}{s - 2h_{k(i)}} = 1 + \frac{2h_{k(i)}}{s - 2h_{k(i)}}.$$

Then

$$\Delta_{k(i)} = \frac{2h_{k(i)}}{s - 2h_{k(i)}} < \frac{4}{s} h_{k(i)}.$$

Therefore

$$\sum_{k=1}^{\infty} \Delta_{k(i)} < \frac{4}{s} \sum_{k=1}^{\infty} h_{k(i)} < \infty, \quad \forall\, i \in \mathcal{N}$$

and (2.3) becomes

$$\begin{aligned} d(x_n, p) &\leq (1 + \Delta_{k(i)}) d(x_{n-1}, p) + \frac{2}{s - 2h_{k(i)}} \gamma_n d(u_n, p) \\ &\leq (1 + \Delta_{k(i)}) d(x_{n-1}, p) + \frac{4}{s} \gamma_n M, \quad \forall\, p \in \mathcal{F}, \end{aligned} \tag{2.4}$$

where, $M = \sup_{n \geq 1} d(u_n, p)$. This implies that

$$D_d(x_n, \mathcal{F}) \leq (1 + \Delta_{k(i)}) d(x_{n-1}, \mathcal{F}) + \frac{4M}{s} \gamma_n.$$

Since $\sum_{k=1}^{\infty} \Delta_{k(i)} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, from Lemma 2.1, we have

$$\lim_{n\to\infty} D_d(x_n, \mathcal{F}) = 0.$$

Next, we will prove that the process $\{x_n\}$ is Cauchy. Note that when $a > 0$, $1 + a \le e^a$, from (2.4) we have

$$\begin{aligned} d(x_{n+m}, p) &\le (1 + \Delta_{k(i)}) d(x_{n+m-1}, p) + \frac{4M}{s}\gamma_{n+m} \\ &\le (1 + \Delta_{k(i)}) \Big[(1 + \Delta_{k(i)}) d(x_{n+m-2}, p) + \frac{4M}{s}\gamma_{n+m-1}\Big] \\ &\quad + \frac{4M}{s}\gamma_{n+m} \\ &\le (1 + \Delta_{k(i)})^2 \Big[(1 + \Delta_{k(i)}) d(x_{n+m-3}, p) + \frac{4M}{s}\gamma_{n+m-2}\Big] \\ &\quad + \frac{4M}{s}(1 + \Delta_{k(i)})(\gamma_{n+m-1} + \gamma_{n+m}) \\ &\le (1 + \Delta_{k(i)})^3 d(x_{n+m-3}, p) \\ &\quad + \frac{4M}{s}(1 + \Delta_{k(i)})^3(\gamma_{n+m-2} + \gamma_{n+m-1} + \gamma_{n+m}) \\ &\le \cdots \\ &\le \exp\Big\{\sum_{i=1}^{N}\sum_{k=1}^{\infty}\Delta_{k(i)}\Big\} d(x_n, p) \\ &\quad + \frac{4M}{s}\exp\Big\{\sum_{i=1}^{N}\sum_{k=1}^{\infty}\Delta_{k(i)}\Big\}\sum_{j=n+1}^{n+m}\gamma_j \\ &\le M' d(x_n, p) + \frac{4MM'}{s}\sum_{j=n+1}^{n+m}\gamma_j, \end{aligned} \tag{2.5}$$

for all $p \in \mathcal{F}$ and $n, m \in \mathbb{N}$, where $M' = \exp\Big\{\sum_{i=1}^{N}\sum_{k=1}^{\infty}\Delta_{k(i)}\Big\} < \infty$. Since $\lim_{n\to\infty} D_d(x_n, \mathcal{F}) = 0$ and $\sum_{n=1}^{\infty} h_{k(i)} < \infty$ $(i \in \mathcal{N})$, there exists a natural number n_1 such that for $n \ge n_1$,

$$D_d(x_n, \mathcal{F}) < \frac{\varepsilon}{4M'} \quad \text{and} \quad \sum_{j=n_1+1}^{\infty}\gamma_j \le \frac{s\cdot\varepsilon}{16MM'}.$$

Thus there exists a point $p_1 \in \mathcal{F}$ such that $d(x_{n_1}, p_1) \leq \frac{\varepsilon}{4M'}$ by the definition of $D_d(x_n, \mathcal{F})$. It follows, from (2.5) that for all $n \geq n_1$ and $m \geq 0$,

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(x_n, p_1) \\ &\leq M'd(x_{n_1}, p_1) + \frac{4MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j + M'd(x_{n_1}, p_1) \\ &\quad + \frac{4MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j \\ &< M' \cdot \frac{\varepsilon}{4M'} + \frac{4MM'}{s} \cdot \frac{s \cdot \varepsilon}{16MM'} + M' \cdot \frac{\varepsilon}{4M'} \\ &\quad + \frac{4MM'}{s} \cdot \frac{s \cdot \varepsilon}{16MM'} \\ &= \varepsilon. \end{aligned}$$

This implies that $\{x_n\}$ is Cauchy. Because the space is complete, the process $\{x_n\}$ is convergent. Let $\lim_{n\to\infty} x_n = p$. Moreover, since the set of fixed points of asymptotically quasi-nonexpansive mapping is closed, so is $\mathcal{F}$, thus $p \in \mathcal{F}$ from $\lim_{n\to\infty} D_d(x_n, \mathcal{F}) = 0$, i.e., p is a common fixed point of $\{T_i : i \in \mathcal{N}\}$. This completes the proof. □

If $u_n = 0$, in Theorem 2.1, we can easily obtain the following theorem.

Theorem 2.2. *Let (X, d, W) be a complete convex metric space. Let $\{T_i : i \in \mathcal{N}\}$ be a finite family of asymptotically quasi-nonexpansive mappings from X into X, that is,*

$$d(T_i^n x, p_i) \leq (1 + h_{n(i)})d(x, p_i)$$

for all $x \in X$, $p_i \in F(T_i)$, $i \in \mathcal{N}$. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$, $\sum_{n=1}^{\infty} h_{n(i)} < \infty$ $(i \in \mathcal{N})$. Then the implicit iteration process $\{x_n\}$ generated by (1.5) converges to a common fixed point of $\{T_i : i \in \mathcal{N}\}$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, \mathcal{F}) = 0.$$

From Theorem 2.1, we can also easily obtain the following theorem.

Theorem 2.3. *Let (X, d, W) be a complete convex metric space. Let $\{T_i : i \in \mathcal{N}\}$ be a finite family of quasi-nonexpansive mappings from X into X, that is,*

$$d(T_i x, p_i) \leq d(x, p_i)$$

for all $x \in X$, $p_i \in F(T_i)$, $i \in \mathcal{N}$. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{u_n\}$ is arbitrary bounded sequence in X. Then the implicit iteration process with error $\{x_n\}$ generated by (1.4) converges to a common fixed point $\{T_i : i \in \mathcal{N}\}$ if and only if

$$\liminf_{n \to \infty} D_d(x_n, \mathcal{F}) = 0.$$

Remark 2.1. The results presented in this chapter are extensions and improvements of the corresponding results in Wittmann [15], Xu-Ori [16] and Sun [12].

References

1. S. S. Chang and J. K. Kim, *Convergence theorems of the Ishikawa type iterarive sequences with errors for generalized quasi-contractive mappings in convex metric spaces*, Applied Mathematics Letters **16(4)** (2003), 535-542.
2. S. S. Chang, J. K. Kim and D. S. Jin, *Iterative sequences with errors for asymptotically quasi-nonexpansive type mappings in convex metric spaces*, Archives Ineq. and Appl. **2(4)** (2004), 365–374.
3. S. S. Chang, *On the approximating problem of fixed points for asymptotically nonexpansive mappings*, Indian J. Pure and Appl. **32, No 9** (2001), 1–11.
4. J. K. Kim, K. H. Kim and K. S. Kim, *Convergence theorems of modified three-step iterative sequences with mixed errors for asymptotically quasi-nonexpansive mappings in Banach spaces*, PanAmerican Math. Jour. **14** (2004), 45–54.
5. J. K. Kim, K. H. Kim and K. S. Kim, *Three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in convex metric spaces*, Nonlinear Analysis and Convex Analysis, Research Institute for Mathematical Sciences Kyoto University, Kyoto, Japan, 1365 (2004), 156–165.
6. Q. H. Liu, *Iteration sequences for asymptotically quasi-nonexpansive mappings with error member of uniformly convex Banach spaces*, J. Math. Anal. Appl. **266** (2002), 468–471.
7. Q. H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl. **259** (2001), 1–7.
8. Q. H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl. **259** (2001), 18–24.

9. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
10. W. V. Petryshyn and T. E. Williamson, *Strong and weak convergence of the sequence of successive approximations for asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl. **43** (1973), 459–497.
11. J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), 407–413.
12. Z. H. Sun, *Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl. **286** (2003), 351–358.
13. W. Takahashi, *A convexity in metric space and nonexpansive mappings I*, Kodai Math. Sem. Rep. **22** (1970), 142–149.
14. K. K. Tan and H. K. Xu, *Approximating fixed point of nonexpansive mappings by the Ishikawa iterative process*, J. Math. Anal. Appl. **178** (1993), 301–308.
15. R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486–491.
16. H. K. Xu and R. G. Ori, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim. **22** (2001), 767–773.

J. K. Kim
Department of Mathematics,
Kyungnam University,
Masan, Kyungnam, 631-701, Korea
E-mail address: `jongkyuk@kyungnam.ac.kr`

K. S. Kim
Department of Mathematics,
Kyungnam University,
Masan, Kyungnam, 631-701, Korea
E-mail address: `kksmj@mail.kyungnam.ac.kr`

S. M. Kim
Department of Mathematics,
Kyungnam University,
Masan, Kyungnam, 631-701, Korea