数理解析研究所講究録
1485 巻 2006 年 42-49

1次元連続体上の力学系

Dynamical Systems on 1-dimensional Continua

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1次元の力学系理論では、現在まで主に閉区間、円周、グラフ上の力学系が詳しく研究されてきました (例えば [10] 参照)。近年、少し位相的に複雑な集合であるフラクタル集合が、数学だけでなく色々な分野の学問に登場し、その重要性が認識されてきている。ここでは、特に、樹木＝dendrite, regular curve, Menger curve (=メンガー・スポンジ) などのフラクタル集合上の力学系を考察する。

1 Introduction

Recently, many geometric and dynamical properties of fractal sets have been studied. In this note, we study dynamical properties of maps on regular curves and Menger maniforlds, which are contained in the class of fractal sets. It is well known that in the dynamics of a piecewise strictly monotone (= piecewise embedding) map f on an interval, the topological entropy can be expressed in terms of the growth of the number (= the lap number) of strictly monotone intervals for f^n (see the papers of M. Misiurewicz, W. Szlenko [11] and L. S. Young [16], and also see [10, Theorem 7.1]). We generalize the theorem of M. Misiurewicz, W. Szlenko and L. S. Young to the cases of regular curves and dendrites.

All spaces considered in this note are assumed to be separable metric spaces. *Maps* are continuous functions. For a space X, let $\mathrm{Comp}(X)$ be the set of all components of X. By a *compactum* X we mean a compact metric space. A *continuum* is a nonempty connected compactum. For a set A, $|A|$ denotes the cardinarity of the set A. A map $f: X \to Y$ of compacta is an *embedding map* if $f: X \to f(X)$ is a homeomorphism. A map $f: X \to Y$ of compacta is *monotone* if for each $y \in f(X)$, $f^{-1}(y)$ is connected.

A continuum X is a *regular continuum* (=*regular curve*) if for each $x \in X$ and each open neighborhood V of x in X, there is an open neighborhood U of x in X such that $U \subset V$ and the boundary set $Bd(U)$ of U is a finite set. Clearly, each regular curve is a *Peano curve* (= 1-dimensional locally connected continuum). For each $p \in X$, we define the cardinal number $ls_X(p)$ of p as follows: $ls_X(p) \leq \alpha$ (α is a cardinal number) if and only if for any neighborhood V of p there is a neighborhood $U \subset V$ of p in X such that $|\mathrm{Comp}(U - \{p\})| \leq \alpha$, and $ls_X(p) = \alpha$ if and only if $ls_X(p) \leq \alpha$ and the inequality $ls_X(p) \leq \beta$ for $\beta < \alpha$ does not hold. We define $ls(X) < \infty$ if $ls_X(p) < \infty$ for each $p \in X$.

A continuum X is a *dendrite* (= 1-dimensional compact AR) if X is a locally connected continuum which contains no simple closed curve. It is well known that each local dendrite (= 1-dimensional compact ANR) is a regular curve. Note that each graph (= 1-dimensional finite polyhedron) is a local dendrite. There are many regular curves which are not local dendrites. Many *fractal sets* (see [2] and [4]) are regular curves which are not local dendrites. For example, the Sierpinski triangle S is a well-known regular curve with $ls_S(p) \leq 2$ for each $p \in S$. The Menger curve and the Sierpinski carpet are not regular curves.

2 Depth of Birkhoff centers of dendrites

We say that a point $x \in X$ is a *nonwandering* point of a map $f : X \to X$ if for each neighborhood U of x in X, there exists a natural number $n \geq 1$ such that $f^n(U) \cap U \neq \phi$. The set of nonwandering points of f is denoted by $\Omega(f)$. To introduce the notion of Birkhoff center, we put $f_1 = f|_{\Omega(f)} : \Omega_1(f) = \Omega(f) \to \Omega(f)$ and $\Omega_2(f) = \Omega(f_1) = \Omega(f|_{\Omega(f)})$. We continue this process. Then $X = \Omega_0(f) \supset \Omega_1(f) \supset \Omega_2(f) \supset \cdots, \Omega_{\alpha+1}(f) = \Omega(f_\alpha) = \Omega(f|\Omega_{\alpha(f)})$ and $\Omega_\lambda(f) = \cap_{\alpha<\lambda}\Omega_\alpha(f)$, where λ is a limit ordinal number. We say that $\Omega_\alpha(f)$ is the *Birkhoff center* of f if $\Omega_\alpha(f) = \Omega_{\alpha+1}(f)$, and put *depth*$(f)$ =min$\{\alpha \mid \Omega_\alpha(f) = \Omega_{\alpha+1}(f)\}$. Note that *depth*$(f) < w_1$, where w_1 is the first uncountable ordinal number. It is well known that for any map $f : I = [0,1] \to I$, *depth*$(f) \leq 2$ and for any map $f : G \to G$ of any graph G, *depth*$(f) \leq 3$. For dendrites, we have the following.

Theorem 2.1. *There is a dendrite D such that for any countable ordinal number α there is a map $f : D \to D$ such that depth$(f) = \alpha$. In particular, there is a map $f : I^2 \to I^2$ such that depth$(f) = \alpha$, and there is a homeomorphism $f : I^3 \to I^3$ such that depth$(f) = \alpha$, where $I = [0,1]$.*

3 Topological Entropy of Piecewise Embedding Maps on Regular Curves

Let $f : X \to X$ be a map of a compactum X and let $K \subset X$ be a closed subset of X. We define the topological entropy $h(f, K)$ of f with respect to K as follows (see [1, 10 and 15]). Let n be a natural number and $\epsilon > 0$. A subset F of K is an (n, ϵ)-*spanning set* for f with respect to K if for each $x \in K$, there is $y \in F$ such that

$$\max\{d(f^i(x), f^i(y)) \mid 0 \leq i \leq n-1\} < \epsilon.$$

A subset E of K is an (n, ϵ)-*separated set* for f with respect to K if for each

$x, y \in E$ with $x \neq y$, there is $0 \leq j \leq n-1$ such that

$$d(f^j(x), f^j(y)) > \epsilon.$$

Let $r_n(\epsilon, K)$ be the smallest cardinality of all (n, ϵ)-spanning sets for f with respect to K. Also, let $s_n(\epsilon, K)$ be the maximal cardinality of all (n, ϵ)-separated sets for f with respect to K. Put

$$r(\epsilon, K) = \limsup_{n \to \infty} \ (1/n) \log r_n(\epsilon, K)$$

and

$$s(\epsilon, K) = \limsup_{n \to \infty} \ (1/n) \log s_n(\epsilon, K).$$

Also, put

$$h(f, K) = \lim_{\epsilon \to 0} r(\epsilon, K).$$

Then it is well known that $h(f, K) = \lim_{\epsilon \to 0} s(\epsilon, K)$. Finally, put

$$h(f) = h(f, X).$$

It is well known that $h(f)$ is equal to the topological entropy which was defined by Adler, Konheim and McAndrew (see [1]).

Let X be a regular continuum. A finite closed covering $\mathcal{A}$ of a regular curve X is a *regular partition* of X provided that if $A, A' \in \mathcal{A}$ and $A \neq A'$, then $\mathrm{Int}(A) \neq \phi$, $A \cap A' = Bd(A) \cap Bd(A')$, and $Bd(A)$ is a finit set. We can easily see that if X is a regular curve and $\epsilon > 0$, then there is a regular partition $\mathcal{A}$ of X such that mesh $\mathcal{A} < \epsilon$, that is, diam $A < \epsilon$ for each $A \in \mathcal{A}$.

For a regular partition $\mathcal{A}$ of X, moreover, $\mathcal{A}$ is called a *strongly regular partition* if $ls_X(a) < \infty$ for each $a \in \bigcup\{Bd(A) | \ A \in \mathcal{A}\}$.

A map $f : X \to X$ is a *piecewise embedding map* with respect to a regular partition $\mathcal{A}$ if the restriction $f|A : A \to X$ is an embedding (= injective) map for each $A \in \mathcal{A}$. A map $f : X \to X$ is a *piecewise monotone map* with respect to $\mathcal{A}$ if the restriction $f|A : A \to f(A)$ is a monotone map for each $A \in \mathcal{A}$.

The following theorem of M. Misiurewicz, W. Szlenko [11] and L. S. Young [16] is well known.

Theorem 3.1. (Misiurewicz-Szlenko and Young) *If $f : I = [0, 1] \to I$ is a piecewise embedding map (i.e., there is a finite sequence $c_1, c_2, ..., c_k$ of I such that $c_0 = 0 < c_1 < c_2 < ... < c_k = 1$, each restriction $f|[c_i, c_{i+1}] : [c_i, c_{i+1}] \to I$ is an embedding (=strictly monotone) map and each c_i $(i = 1, 2, .., k-1)$ is a turning point of f, then*

$$h(f) = \lim_{n \to \infty} (1/n) \log l(f^n),$$

where $l(f^n)$ denotes the lap number of f^n.

Let $f : X \to X$ be a map of a regular curve X and let $\mathcal{A} = \{A_1, A_2, ..., A_m\}$ be a regular partition of X. For each $n \geq 0$, consider the itinerary set $It(f, n; \mathcal{A})$ for f and n defined by

$$It(f, n; \mathcal{A}) = \{(x_0, x_1, ..., x_{n-1}) \mid x_i \in \{1, 2, ..., m\} \text{ and } \bigcap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(A_{x_i})) \neq \phi\}.$$

Put $I(f, n; \mathcal{A}) = |It(f, n; \mathcal{A})|$. Note that $I(f, n+m; \mathcal{A}) \leq I(f, n; \mathcal{A}) \cdot I(f, m; \mathcal{A})$. Hence we see that the limit $\lim_{n\to\infty}(1/n) \log I(f, n; \mathcal{A})$ exists. Note that if $f : I \to I$ is a piecewise embedding map of the unit interval I, then $l(f^{n-1}) = I(f, n; \mathcal{A})$, where $\mathcal{A} = \{[c_i, c_{i+1}] \mid i = 0, 1, ..., k-1\}$.

We can generalize the theorem of Misiurewicz-Szlenko and Young to the case of piecewise embedding maps with respect to strongly regular partitions of regular curves.

Theorem 3.2. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$h(f) = \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

For the proof of the above theorem, we need the following Bowen's result.

Proposition 3.3. (Bowen) *Let X and Y be compacta, and let $f : X \to X$, g: $Y \to Y$ be maps. If $\pi : X \to Y$ is an onto map such that $\pi \cdot f = g \cdot \pi$, then*

$$h(g) \leq h(f) \leq h(g) + \sup_{y\in Y} h(f, \pi^{-1}(y)).$$

Theorem 3.4. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a regular partition $\mathcal{A}$ of X, then*

$$h(f) \leq \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

Let $f : X \to X$ be a piecewise embedding map of a regular curve X with respect to a regular partition $\mathcal{A} = \{A_1, A_2, ..., A_m\}$ of X. Note that $m = |\mathcal{A}|$. Define an $m \times m$ matrix $M_f = (a_{ij})$ by the following; $a_{ij} = 1$ if $f(\mathrm{Int}(A_i)) \supset \mathrm{Int}(A_j)$, and $a_{ij} = 0$ otherwise. Also, define an $m \times m$ matrix $N_f = (b_{ij})$ by the following; $b_{ij} = 1$ if $f(\mathrm{Int}(A_i)) \cap \mathrm{Int}(A_j) \neq \phi$, and $b_{ij} = 0$ otherwise. Let $\lambda(M_f)$ be the real eigenvalue of M_f such that $\lambda(M_f) \geq |\lambda|$ for all the other eigenvalue λ of M_f. Then we have the following corollary.

Corollary 3.5. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$\lambda(M_f) \leq h(f) \leq \lambda(N_f).$$

Remark. (1) The assertion of Theorem 3.2 is not true for piecewise embedding maps on Peano curves. Let $X = \mu^1$ be the Menger curve. We can choose a homeomorphism $f : X \to X$ such that $h(f) \neq 0$. Then f is also a piecewise embedding map with respect to $\mathcal{A} = \{X\}$ and

$$h(f) > 0 = \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

(2) There is a piecewise embedding map $f : X \to X$ of a dendrite X with respect to a regular partition $\mathcal{A}$ of X such that

$$h(f) < \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

The assertion of Theorem 3.2 is not true for piecewise embedding maps with respect to regular partitions of regular curves.
(3) Moreover, there is a homeomorphism $f : X \to X$ of a dendrite X such that

$$h(f) < \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A})$$

for some regular partition $\mathcal{A}$ of X.

For a map $f : X \to X$ of a regular curve X and a regular partition $\mathcal{A} = \{A_i|\ i = 1, 2, ..., m\}$ of X, we put

$$\sum(f, \mathcal{A}) = \{(x_i)_{i=0}^{\infty}|\ A_{x_i} \in \mathcal{A} \text{ and } \bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i})) \neq \phi \text{ for all } n = 0, 1, 2, ...\}.$$

Also, let $\sigma_{(f,\mathcal{A})} : \sum(f, \mathcal{A}) \to \sum(f, \mathcal{A})$ be the shift map defined by

$$\sigma_{(f,\mathcal{A})}((x_i)_{i=0}^{\infty}) = (x_{i+1})_{i=0}^{\infty}.$$

Then we have

Theorem 3.6. *Let X be a dendrite. If a map $f : X \to X$ is a piecewise monotone map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$h(f) = h(\sigma_{(f,\mathcal{A})}).$$

For each map $f : X \to X$ of a compactum X and a natural number n, put

$$\varphi(f, n) = \sup\{|Comp(f^{-n}(y))|\ |\ y \in X\}.$$

Then we have the following theorem.

Theorem 3.7. *If $f : X \to X$ is a map of a regular curve X, then*

$$h(f) \leq \limsup_{n\to\infty}\ (1/n) \log \varphi(f, n).$$

4 Mesures and topological dynamics on Menger manifolds

The theory of Menger manifolds was founded by Anderson and Bestvina (see [17] and [18]) and has been studied by many authors. We study Menger manifolds from the viewpoint of dynamical systems. Anderson and Bestvina gave a characterization of Menger manifolds as follows. For a compactum M, M is a n-dimensional Menger manifold if and only if (1) $\dim M = n$, (2) M is locally $(n-1)$-connected, (3) M has disjoint n-cell property, i.e., for any $\epsilon > 0$ and any maps $f, g : I^n \to M$, there are maps $f', g' : I^n \to M$ such that $d(f, f') < \epsilon$, $d(g, g') < \epsilon$ and $f'(I^n) \cap g'(I^n) = \phi$.

Note that 0-dimensional Menger manifold = Cantor set, and 1-dimensional Menger manifold = Menger curve.

A homeomorphism $f : X \to X$ of a compactum X with a measure μ is *ergodic* if f is μ-measure-preserving, and for any measurable set E of X such that $f^{-1}(E) = E$, we have either $\mu(E) = 0$ or $\mu(E) = 1$. Let $H(X, \mu)$ be the set of all μ-measure preserving homeomorphisms of X and $E(X, \mu)$ the set of all ergodic homeomorphisms of $H(X, \mu)$.

Then we have the following results ([9]).

Theorem 4.1. *Let μ_1, μ_2 be nonatomic locally positive Lebesgue-Stieltjes measures on Menger n-manifolds M $(n \geq 1)$. Then there is a homeomorphism $h : M \to M$ such that $\mu_1 = h^*\mu_2$.*

Theorem 4.2. *Let μ be nonatomic locally positive Lebesgue-Stieltjes measure on Menger n-manifolds M $(n \geq 1)$. Then $E(M, \mu)$ is a dense G_δ-subset of $H(M, \mu)$.*

Corollary 4.3. *There are many chaotic homeomorphisms of Devaney and Li-Yorke on each Menger manifold.*

5 Problems

Finally, we have the following problems.

Problem 5.1. *Is it true that for any countable ordinal number α, there is a homeomorphism $f : I^2 \to I^2$ such that depth$(f) = \alpha$?*

Problem 5.2. *In the statement of Theorem 3.6, is the following equality true*

$$h(\sigma_{(f,\mathcal{A})}) = \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}) \ ?$$

Problem 5.3. *Let X be a regular curve. Is it true that if a map $f : X \to X$ is a piecewise monotone map with respect to a strongly regular partition $\mathcal{A}$ of X, then $h(f) = \lim_{n\to\infty}(1/n) \log I(f, n; \mathcal{A})$?*

In particular, the next problems are interesting.

Problem 5.4. *Does there exist a minimal homeomorphism of an n-dimensional Menger manifold* $(n \geq 1)$ *?*

Problem 5.5. *Does there exist an expansive homeomorphism of an n-dimensional Menger manifold* $(n \geq 1)$ *?*

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