Projective embeddings and Lagrangian fibrations of Kummer varieties

Yuichi Nohara*
Graduate School of Mathematics, Nagoya University

1 Introduction

Let \((X, \omega)\) be a compact Kähler manifold and \((L, h) \to X\) a Hermitian line bundle with \(c_1(M, h) = \omega\). Then for sufficiently large integer \(k\), \(X\) can be embedded into a projective space by basis \(s_0, \ldots, s_{N_k}\) of \(H^0(X, L^k)\):

\[ \iota_k : X \hookrightarrow \mathbb{C}P^{N_k} = \mathbb{P}H^0(X, L^k)^* \]

Set \(\omega_k = \frac{1}{k} \iota_k^* \omega_{\text{FS}}\), where \(\omega_{\text{FS}}\) is the Fubini-Study metric on \(\mathbb{C}P^{N_k}\). Then Tian [10] and Zelditch [12] proved that \(\omega_k\) converge to \(\omega\) under appropriate choices of basis of \(H^0(X, L^k)\). More precisely,

**Theorem 1.1 (Zelditch [12]).** Suppose that the basis \(s_0, \ldots, s_{N_k} \in H^0(X, L^k)\) are orthonormal with respect to the \(L^2\)-inner product for each \(k \gg 1\). Then there exist constants \(C_q > 0\) independent of \(k\) such that

\[ \| \omega - \omega_k \|_{C^q} \leq \frac{C_q}{k} . \]

In this article, we study asymptotic behavior of projective embeddings and the amoebas of abelian varieties and Kummer varieties. We can think of this as a Lagrangian fibration version of the above theorem.

We consider a natural torus action on \(\mathbb{C}P^{N_k}\). Then we have a moment map

\[ \mu_k : \mathbb{C}P^{N_k} \longrightarrow \Delta_k \subset \text{Lie}(T^{N_k})^* \]

of the \(T^{N_k}\)-action. Note that \(\mu_k\) is a Lagrangian fibration of \(\mathbb{C}P^{N_k}\) with respect to the Fubini-Study metric \(\omega_{\text{FS}}\). We denote \(B_k = \mu_k(\iota_k(X))\). \(B_k\) is called a compactified amoeba. We are interested in the asymptotic behavior of the restriction \(\pi_k : X \to B_k\) of the moment map \(\mu_k\). Amoebas heavily depend on the choice of projective embeddings. Thus the choice of basis of holomorphic sections is an important problem. Of course, there is not a natural choice of basis in general. However, we have some natural choice of basis in special cases

*e-mail: m98014i@math.nagoya-u.ac.jp*
such as the case of toric varieties and abelian varieties. In these cases, the basis are related to Lagrangian fibrations $\pi : (X, \omega) \to B$ of $X$. We compare $\pi$ and $\pi_k$.

First we consider the simplest case, i.e. the case of toric varieties. Let $(X, L)$ be a polarized toric variety. In this case, $H^0(X, L^k)$ is spanned by (Laurent) monomials $z^I = z_1^{i_1} \cdots z_n^{i_n}$. Let $\pi : X \to \Delta$ be a moment map of a natural torus action, where $\Delta$ is the moment polytope of $X$. Then each monomial corresponds to a lattice point in $\Delta$:

$$I = (i_1, \ldots, i_n) \in k\Delta \cap \mathbb{Z}^n \mapsto z^I \in H^0(X, L^k).$$

We consider the projective embedding $\iota_k : X \hookrightarrow \mathbb{CP}^{N_k}$ defined by the monomials. Then $\pi_k : X \to \Delta_k$ is invariant under the $T^n$-action. Hence we have the following commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\iota_k} & \mathbb{CP}^{N_k} \\
\downarrow \pi & & \downarrow \pi_k \\
\Delta & \xrightarrow{\mu_k} & \Delta_k
\end{array}$$

In particular, $B_k$ is the image of the $n$-dimensional polytope $\Delta$.

**Remark 1.2.** Note that $\dim \mathbb{R} B_k = 2n = \dim \mathbb{R} X$ in general.

The case of abelian varieties is less trivial. Let $A = \mathbb{C}^n/\Omega \mathbb{Z}^n + \mathbb{Z}^n$ be an abelian variety and $L \to A$ a principally polarization. Then holomorphic sections of $L^k$ are essentially given by the theta functions. There are some natural choices of basis of theta functions. For example,

$$\vartheta \begin{bmatrix} 0 \\ -b \end{bmatrix} (k^{-1} \Omega, z), \quad b \in \frac{1}{k} \mathbb{Z}^n/\mathbb{Z}^n$$

give a basis of $H^0(A, L^k)$, where

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega, z) = \sum_{l \in \mathbb{Z}^n} \exp \left( \pi \sqrt{-1}^{t}(l + a)\Omega(l + a) + 2\pi \sqrt{-1}^{t}(l + a)(z + b) \right).$$

In particular, we have the following isomorphism

$$H^0(A, L^k) \cong \bigoplus_{b \in \frac{1}{k} \mathbb{Z}^n/\mathbb{Z}^n} \mathbb{C} \cdot b.$$

This isomorphism can be given by the Lagrangian fibration

$$\pi : A \to T^n, \quad z = \Omega x + y \mapsto y,$$

and this is interpreted in terms of geometric quantization ([11]) or mirror symmetry ([7], [4]). We consider the projective embeddings defined by the above basis:

$$\iota_k : A \hookrightarrow \mathbb{CP}^{k^n-1}, \quad z \mapsto \left( \vartheta \begin{bmatrix} 0 \\ -b_1 \end{bmatrix} (k^{-1} \Omega, z); \ldots; \vartheta \begin{bmatrix} 0 \\ -b_{k^n} \end{bmatrix} (k^{-1} \Omega, z) \right).$$
In this case, the restriction

$$\pi_k = \mu_k \circ \iota_k : A \rightarrow B_k$$

is not the same as the Lagrangian fibration $\pi$. However, we can easily see that $\pi_k$ is invariant under the translations

$$\Omega x + y \mapsto \Omega(x + a) + y, \quad a \in \frac{1}{k} \mathbb{Z}^n / \mathbb{Z}^n.$$ 

Therefore, $\pi_k$ looks "close" to $\pi$ for large $k$. In fact, this can be justified by using the notion of Gromov-Hausdorff distance.

We discuss this more precisely in the next section. The case of Kummer varieties is discussed in Section 3.

2 The case of abelian varieties

Let $A = \mathbb{C}^n / \Omega \mathbb{Z}^n + \mathbb{Z}^n$ be an $n$-dimensional abelian variety as in the previous section. We take a principal polarization $L \rightarrow A$ defined by

$$L = (\mathbb{C}^n \times \mathbb{C}) / \sim,$$

where

$$(z, \zeta) \sim (z + \lambda, e^{\pi t \lambda (\im \Omega)^{-1} z + \frac{\pi}{2} \mathrm{c}_\lambda (\im \Omega)^{-1} z} \zeta)$$

for $\lambda \in \Omega \mathbb{Z}^n + \mathbb{Z}^n$. Then $L$ is symmetric, i.e.

$$(-1)^*_AL \cong L,$$

where

$$(-1)_A : A \rightarrow A, \quad z \mapsto -z$$

is the inverse morphism.

Remark 2.1. The choice of $L$ is not essential. In fact, any other principal polarization can be obtained as a pull-back of $L$ by some translation. The symmetricity condition is important when we deal with the case of Kummer varieties.

Let $\omega_0$ be the flat Kähler metric in the class $c_1(L)$ and fix a Hermitian metric $h_0$ of $L$ such that $c_1(L, h_0) = \omega_0$.

Let $T^f$ and $T^b$ be $n$-dimensional tori $\mathbb{R}^n / \mathbb{Z}^n$ and identify

$$A \cong T^f \times T^b, \quad \Omega x + y \mapsto (x, y).$$

Then the natural projection

$$\pi : A \rightarrow T^b, \quad \Omega x + y \mapsto y$$

is a Lagrangian fibration with respect to $\omega_0$. 
We denote the subgroups of $T^b$ of $k$-torsion points by
\[ T^b_k = \frac{1}{k} \mathbb{Z}^n / \mathbb{Z}^n = \{ b_i \}_{i=1, \ldots, k^n} \subset T^b. \]

Then
\[ s_{b_i} = C k^{-\frac{n}{4}} \exp \left( \frac{\pi}{2} k^t (\text{Im} \Omega) z \right) \theta(k^{-1} \Omega, z), \quad i = 1, \ldots, k^n \]
give a basis of $H^0(A, L^k)$, where $C$ is a constant determined by $\Omega$ and $h_0$. It is known that $s_{b_i}$ has a peak along the fiber $\pi^{-1}(b_i)$. An important property for our purpose is the following:

**Proposition 2.2.** $s_1, \ldots, s_k^n$ are orthonormal basis of $H^0(X, L^k)$ with respect to the $L^2$-inner product.

We consider the projective embedding defined by these theta functions
\[ \iota_k : A \rightarrow \mathbb{C}\mathrm{P}^{k^n-1}, \quad z \mapsto \left( \vartheta \begin{bmatrix} 0 \\ -b_1 \end{bmatrix} (k^{-1} \Omega, z) : \cdots : \vartheta \begin{bmatrix} 0 \\ -b_{k^n} \end{bmatrix} (k^{-1} \Omega, z) \right). \]

The moment map of $\mathbb{C}\mathrm{P}^{k^n-1}$ is given by
\[ \mu_k : (Z^1 : \cdots : Z^{k^n}) \mapsto \frac{1}{\sum |Z^i|^2} \left( |Z^1|^2, \ldots, |Z^{k^n}|^2 \right), \]
where $(Z^1 : \cdots : Z^{k^n})$ is the homogeneous coordinate of $\mathbb{C}\mathrm{P}^{k^n-1}$. We set
\[ B_k := \mu_k(\iota_k(X)), \]
\[ \pi_k := \mu_k \circ \iota_k : A \rightarrow B_k \]
as before. We also denote the restriction of the Fubini-Study metric to $X$ by
\[ \omega_k := \frac{1}{k} \iota_k^* \omega_{\text{FS}}, \]
here we normalize the Fubini-Study metric in order to $\omega_k$ represents $c_1(L)$.

We compare $\pi : (A, \omega_0) \rightarrow T^b$ and $\pi_k : (A, \omega_k) \rightarrow B_k$ as maps between metric spaces. For that purpose, we need to define distances on $T^b$ and $B_k$. We define a metric on $T^b$ in such a way that $\pi : (A, \omega_0) \rightarrow T^b$ is a Riemannian submersion. The distance on $B_k$ is induced from a metric on the moment polytope $\Delta_k$. The metric on $\Delta_k$ is also defined in such a way that
\[ \mu_k : \left( \mathbb{C}\mathrm{P}^{N_k}, \frac{1}{k} \omega_{\text{FS}} \right) \rightarrow \Delta_k \]
is a Riemannian submersion in the interior of $\Delta_k$.

**Theorem 2.3 ([5]).** $\pi_k : (A, \omega_k) \rightarrow B_k$ converge to $\pi : (A, \omega) \rightarrow T^b$ in the following sense.
(1) $\omega_k$ converge to $\omega$ in $C^\infty$ as $k \to \infty$. In particular, the sequence \{(A, \omega_k)\} of Riemannian manifolds converges to $(A, \omega_0)$ with respect to the Gromov-Hausdorff distance.

(2) $B_k$ converge to $T^b$ as $k \to \infty$ with respect to the Gromov-Hausdorff distance.

(3) $\{\pi_k\}$ converges to $\pi$ as maps between metric spaces.

Before the proof, we recall the notion of Gromov-Hausdorff convergence and convergence of maps.

First we recall the definition of Hausdorff distance. Let $Z$ be a metric space and $X, Y \subset Z$ be two subsets. We denote the $\epsilon$-neighborhood of $X$ in $Z$ by $B(X, \epsilon)$. Then the Hausdorff distance between $X$ and $Y$ is given by

$$d^Z_{\text{H}}(X, Y) = \inf \{\epsilon > 0 \mid X \subset B(Y, \epsilon), Y \subset B(X, \epsilon)\}.$$ 

For metric spaces $X$ and $Y$, the Gromov-Hausdorff distance is defined by

$$d_{\text{GH}}(X, Y) = \inf \{d^Z_{\text{H}}(X, Y) \mid X, Y \leftrightarrow Z \text{ are isometric embeddings.}\}.$$ 

Next we recall the notion of convergence of maps (see also [6]). Let $f_k : X_k \to Y_k$, $f : X \to Y$ be maps between metric spaces. Suppose that $X_k$ and $Y_k$ converge to $X$ and $Y$ respectively with respect to the Gromov-Hausdorff distance. Then by definition, there exist isometric embeddings $X, X_k \hookrightarrow Z$ and $Y, Y_k \hookrightarrow W$ into some metric spaces such that $X_i$ (resp. $Y_k$) converge to $X$ (resp. $Y$) with respect to the Hausdorff topology in $Z$ (resp. $W$). We say that $\{f_k\}$ converges to $f$ if for every sequence $x_k \in X_k$ converging to $x \in X$, $f_k(x_k)$ converges to $f(x)$ in $W$.

**Outline of the proof**

(1) is a direct consequence of Theorem 1.1 and Proposition 2.2.

(2) Decompose $T\mathbb{C}P^{N_k}$ into horizontal and vertical parts:

$$T\mathbb{C}P^{N_k} = T_{\mathbb{C}P^{N_k}/\Delta_k, p} \oplus (T_{\mathbb{C}P^{N_k}/\Delta_k, p})^\perp,$$

where $T_{\mathbb{C}P^{N_k}/\Delta_k, p} = \ker d\mu_k$ is the tangent space to the fiber of $\mu_k$ and $(T_{\mathbb{C}P^{N_k}/\Delta_k, p})^\perp$ is the orthogonal complement with respect to the Fubini-Study metric. Similarly, we decompose the tangent space of $A$:

$$T_xA = T_{A/T^b, z} \oplus (T_{A/T^b, z})^\perp,$$

where $(T_{A/T^b, z})^\perp$ is the orthogonal complement of $T_{A/T^b, z} = \ker d\pi$ with respect to the flat metric $\omega_0$. Then the metrics on $\Delta_k$ and $T^b$ are given by the restriction of $\omega_k$ and $\omega_0$ on the horizontal subspaces respectively. Therefore we need to compare two horizontal and vertical spaces. We can prove that these two decompositions are close in the following sense:
Lemma 2.4. (1) If $\xi \in T_{A/T^{b},z}$, then
\[ |d_{\iota_{k}}(\xi)^{H}| \leq \frac{C}{\sqrt{k}}|\xi|. \]

(2) If $\eta \in (T_{A/T^{b},z})^{\perp}$,
\[ |d_{\iota_{k}}(\eta)^{V}| \leq \frac{C}{\sqrt{k}}|\eta|. \]

This lemma follows from the asymptotic behavior of the theta functions. By using the above estimates, we have
\[ d_{GH}(T^{b}, B_{k}) \leq \frac{C}{\sqrt{k}}. \]

In fact, we can show that the composition
\[ \varphi_{k} = \pi_{k} \circ \sigma_{0} : T^{b} \rightarrow B_{k} \]

of the zero section $\sigma_{0} : T^{b} \rightarrow A$ and $\pi_{k}$ is "almost isometric" (a $C_{\sqrt{k}}$-Hausdorff approximation (see [3] for the definition)).

3 The case of Kummer varieties

Let $(A, L)$ be a polarized abelian variety as in the previous section. The Kummer variety of $A$ is defined by
\[ X = A/(-1)_{A}. \]

We take a line bundle $M \rightarrow X$ satisfying
\[ p^{*}M \cong L^{2}, \]

where $p : A \rightarrow X$ is the natural projection. From the fact that $p^{*} : \text{Pic}(X) \rightarrow \text{Pic}(A)$ is injective, we have
\[ p^{*}M^{k} \cong L^{2k}. \]

It is easy to see that $p^{*} : H^{0}(X, M^{k}) \rightarrow H^{0}(A, L^{2k})$ is injective and the image is spanned by
\[ s_{b_{i}} + s_{-b_{i}}, \quad b_{i} \in T_{2k}^{b}, \]

(see [1] and [8]). Note that
\[ N_{k} + 1 = \dim H^{0}(X, M^{k}) = 2^{n-1}(k^{n} + 1). \]

Let $\omega$ be the orbifold Kähler metric induced from the flat metric $2\omega_{0}$ on $A$. Then $[\omega] = c_{1}(M)$. We also have a Lagrangian fibration
\[ \pi : (X, \omega) \rightarrow B = T^{b}/(-1) \]
induced by $\pi : A \to T^b$. We set 
\[ t_i = \begin{cases} 
\frac{1}{\sqrt{2^n}}(s_{b_i} + s_{-b_i}), & \text{if } b_i \in T_{2k}^b \setminus T_2^b, \\
\frac{1}{\sqrt{2^{n-1}}}s_{b_i}, & \text{if } b_i \in T_2^b.
\end{cases} \]

Then \( \{t_i\} \) is an orthonormal basis of \( H^0(X, M^k) \).

We denote by \( \iota_{k} : X \to \mathbb{C}P^{N_k} \) the projective embedding defined by \( \{t_i\} \), \( \pi_{k} : X \to B_{k} \) the restriction of the moment map, and \( \omega_{k} = \frac{1}{k}\iota_{k}^{*}\omega_{\text{FS}} \) as before.

Then the same theorem holds for \( X \) as well.

**Theorem 3.1.** (1) \( \{(X, \omega_{k})\} \) converges to \( (X, \omega) \) with respect to the Gromov-Hausdorff distance.

(2) \( B_{k} \) converge to \( B \) with respect to the Gromov-Hausdorff distance.

(3) \( \{\pi_{k}\} \) converges to \( \pi \) as maps between metric spaces.

**Outline of the proof**

(1) follows from the fact that \( \{t_i\} \) are orthonormal and an orbifold version of Theorem 1.1:

**Theorem 3.2 (Song [9], Dai-Liu-Ma [2]).** Let \( (X, \omega) \) be a compact Kähler orbifold of dimension \( n \geq 2 \) with only finite isolated singularities \( \text{Sing}(X) = \{e_j\}_{j=1}^{m} \) and \( (M, h) \to X \) be an orbifold Hermitian line bundle with \( c_1(M, h) = \omega \). For \( k \gg 1 \), we consider the projective embedding \( \iota_{k} : X \to \mathbb{C}P^{N_k} \) defined by an orthonormal basis. We put \( \omega_{k} = \frac{1}{k}\iota_{k}^{*}\omega_{\text{FS}} \) as before. Then

\[ ||\omega - \omega_{k}||_{C^q, z} \leq C_q \left( \frac{1}{k} + k^{\frac{q}{2}}e^{-k\delta r(z)^2} \right), \]

where \( || \cdot ||_{C^q, z} \) is the \( C^q \)-norm at \( z \in X \) and \( r(z) \) is the distance between \( z \) and the singular set \( \{e_j\} \).

(2) What we must take care of is the existence of singular fibers. For each \( b \in \text{Sing}(B) = T_{2}^b/(-1) \), we denote the \( \sqrt{\log k}/\delta k \)-neighborhood of the singular fiber \( \pi^{-1}(b) \) by

\[ N_{b,k} = \left\{ z \in X \mid d(z, \pi^{-1}(b)) \leq \sqrt{\frac{\log k}{\delta k}} \right\} \]

and put

\[ X(k) = X \setminus \bigcup_{b \in \text{Sing}(B)} N_{b,k}. \]

Then we can show that \( \pi(N_{b,k}) \) and \( \pi_{k}(N_{b,k}) \) are small for large \( k \) (in fact, their diameters can be bounded by \( O\left( \sqrt{\frac{\log k}{k}} \right) \)). Therefore we may "ignore" these parts. On the other hand, we have the same estimates as in Lemma 2.4 on \( X(k) \). Hence we can apply the same arguments to this situation.
References


