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Projective embeddings and Lagrangian fibrations of Kummer varieties

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1 Introduction

Let (X,ω) be a compact Kähler manifold and $(L,h)\rightarrow X$ a Hermitian line bundle with $c_{1}(M,h)=\omega$. Then for sufficiently large integer k, X can be embedded into a projective space by basis $s_{0},\ldots,s_{N_{k}}$ of $H^{0}(X,L^{k})$:

$$\iota_{k}:X\hookrightarrow\mathbb{CP}^{N_{k}}=\mathbb{P}H^{0}(X,L^{k})^{*}.$$

Set $\omega_{k}=\frac{1}{k}\iota_{k}^{*}\omega_{\mathrm{FS}}$, where ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^{N_{k}}$. Then Tian [10] and Zelditch [12] proved that ω_{k} converge to ω under appropriate choices of basis of $H^{0}(X,L^{k})$. More precisely,

Theorem 1.1 (Zelditch [12]). *Suppose that the basis $s_{0},\ldots,s_{N_{k}}\in H^{0}(X,L^{k})$ are orthonormal orthonormal with respect to the L^{2}-inner product for each $k\gg 1$. Then there exist constants $C_{q}>0$ independent of k such that*

$$\Vert\omega-\omega_{k}\Vert_{C^{q}}\leq\frac{C_{q}}{k}.$$

In this article, we study asymptotic behavior of projective embeddings and the amoebas of abelian varieties and Kummer varieties. We can think of this as a Lagrangian fibration version of the above theorem.

We consider a natural torus action on $\mathbb{CP}^{N_{k}}$. Then we have a moment map

$$\mu_{k}:\mathbb{CP}^{N_{k}}\longrightarrow\Delta_{k}\subset\mathrm{Lie}(T^{N_{k}})^{*}$$

of the $T^{N_{k}}$-action. Note that μ_{k} is a Lagrangian fibration of $\mathbb{CP}^{N_{k}}$ with respect to the Fubini-Study metric ω_{FS}. We denote $B_{k}=\mu_{k}(\iota_{k}(X))$. B_{k} is called a *compactified amoeba.* We are interested in the asymptotic behavior of the restriction $\pi_{k}:X\rightarrow B_{k}$ of the moment map μ_{k}. Amoebas heavily depend on the choice of projective embeddings. Thus the choice of basis of holomorphic sections is an important problem. Of course, there is not a natural choice of basis in general. However, we have some natural choice of basis in special cases

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such as the case of toric varieties and abelian varieties. In these cases, the basis are related to Lagrangian fibrations $\pi : (X, \omega) \to B$ of X. We compare π and π_k.

First we consider the simplest case, i.e. the case of toric varieties. Let (X, L) be a polarized toric variety. In this case, $H^0(X, L^k)$ is spanned by (Laurent) monomials $z^I = z_i^{i_1} \cdots z_n^{i_n}$. Let $\pi : X \to \Delta$ be a moment map of a natural torus action, where Δ is the moment polytope of X. Then each monomial corresponds to a lattice point in Δ:

$$I = (i_1, \dots, i_n) \in k\Delta \cap \mathbb{Z}^n \longleftrightarrow z^I \in H^0(X, L^k).$$

We consider the projective embedding $\iota_k : X \hookrightarrow \mathbb{CP}^{N_k}$ defined by the monomials. Then $\pi_k : X \to \Delta_k$ is invariant under the T^n-action. Hence we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota_k"] \arrow[d, "\pi"'] \arrow[dr, "\pi_k"] & \mathbb{CP}^{N_k} \arrow[d, "\mu_k"] \\ \Delta \arrow[r] & \Delta_k \end{tikzcd}$$

In particular, B_k is the image of the n-dimensional polytope Δ.

Remark 1.2. Note that $\dim_{\mathbb{R}} B_k = 2n = \dim_{\mathbb{R}} X$ in general.

The case of abelian varieties is less trivial. Let $A = \mathbb{C}^n / \Omega\mathbb{Z}^n + \mathbb{Z}^n$ be an abelian variety and $L \to A$ a principally polarization. Then holomorphic sections of L^k are essentially given by the *theta functions*. There are some natural choices of basis of theta functions. For example,

$$\vartheta\begin{bmatrix} 0 \\ -b \end{bmatrix}(k^{-1}\Omega, z), \quad b \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$$

give a basis of $H^0(A, L^k)$, where

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\Omega, z) = \sum_{l \in \mathbb{Z}^n} \exp\Big(\pi\sqrt{-1}{}^t(l+a)\Omega(l+a) + 2\pi\sqrt{-1}{}^t(l+a)(z+b)\Big).$$

In particular, we have the following isomorphism

$$H^0(A, L^k) \cong \bigoplus_{b \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n} \mathbb{C} \cdot b.$$

This isomorphism can be given by the Lagrangian fibration

$$\pi : A \longrightarrow T^n, \quad z = \Omega x + y \longmapsto y,$$

and this is interpreted in terms of geometric quantization ([11]) or mirror symmetry ([7], [4]). We consider the projective embeddings defined by the above basis:

$$\iota_k : A \hookrightarrow \mathbb{CP}^{k^n - 1}, \quad z \mapsto \left(\vartheta\begin{bmatrix} 0 \\ -b_1 \end{bmatrix}(k^{-1}\Omega, z) : \cdots : \vartheta\begin{bmatrix} 0 \\ -b_{k^n} \end{bmatrix}(k^{-1}\Omega, z)\right).$$

In this case, the restriction

$$\pi_k = \mu_k \circ \iota_k : A \longrightarrow B_k$$

is not the same as the Lagrangian fibration π. However, we can easily see that π_k is invariant under the translations

$$\Omega x + y \longmapsto \Omega(x + a) + y\,, \quad a \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n\,.$$

Therefore, π_k looks "close" to π for large k. In fact, this can be justified by using the notion of *Gromov-Hausdorff distance.*

We discuss this more precisely in the next section. The case of Kummer varieties is discussed in Section 3.

2 The case of abelian varieties

Let $A = \mathbb{C}^n/\Omega\mathbb{Z}^n + \mathbb{Z}^n$ be an n-dimensional abelian variety as in the previous section. We take a principal polarization $L \to A$ defined by

$$L = (\mathbb{C}^n \times \mathbb{C})/\sim,$$

where

$$(z, \zeta) \sim (z + \lambda, e^{\pi\,{}^t\lambda(\operatorname{Im}\Omega)^{-1}z + \frac{\pi}{2}{}^t\lambda(\operatorname{Im}\Omega)^{-1}\lambda}\zeta)$$

for $\lambda \in \Omega\mathbb{Z}^n + \mathbb{Z}^n$. Then L is symmetric, i.e.

$$(-1)_A^* L \cong L,$$

where

$$(-1)_A : A \longrightarrow A, \quad z \longmapsto -z$$

is the inverse morphism.

Remark 2.1. The choice of L is not essential. In fact, any other principal polarization can be obtained as a pull-back of L by some translation. The symmetricity condition is important when we deal with the case of Kummer varieties.

Let ω_0 be the flat Kähler metric in the class $c_1(L)$ and fix a Hermitian metric h_0 of L such that $c_1(L, h_0) = \omega_0$.

Let T^f and T^b be n-dimensional tori $\mathbb{R}^n/\mathbb{Z}^n$ and identify

$$A \cong T^f \times T^b\,, \quad \Omega x + y \longleftrightarrow (x, y)\,.$$

Then the natural projection

$$\pi : A \longrightarrow T^b, \quad \Omega x + y \longmapsto y$$

is a Lagrangian fibration with respect to ω_0.

We denote the subgroups of T^b of k-torsion points by

$$T_k^b = \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n = \{b_i\}_{i=1,\dots,k^n} \subset T^b .$$

Then

$$s_{b_i} = Ck^{-\frac{n}{4}} \exp\left(\frac{\pi}{2}k^t z(\operatorname{Im}\Omega)z\right) \vartheta\begin{bmatrix} 0 \\ -b_i \end{bmatrix}(k^{-1}\Omega, z), \quad i = 1, \dots, k^n$$

give a basis of $H^0(A, L^k)$, where C is a constant determined by Ω and h_0. It is known that s_{b_i} has a peak along the fiber $\pi^{-1}(b_i)$. An important property for our purpose is the following:

Proposition 2.2. *$s_1, \dots, s_{k^n}$ are orthonormal basis of $H^0(X, L^k)$ with respect to the L^2-inner product.*

We consider the projective embedding defined by these theta functions

$$\iota_k : A \hookrightarrow \mathbb{C}\mathbb{P}^{k^n-1}, \quad z \mapsto \left(\vartheta\begin{bmatrix} 0 \\ -b_1 \end{bmatrix}(k^{-1}\Omega, z) : \cdots : \vartheta\begin{bmatrix} 0 \\ -b_{k^n} \end{bmatrix}(k^{-1}\Omega, z)\right) .$$

The moment map of $\mathbb{C}\mathbb{P}^{k^n-1}$ is given by

$$\mu_k : (Z^1 : \cdots : Z^{k^n}) \longmapsto \frac{1}{\sum |Z^i|^2}\left(|Z^1|^2, \dots, |Z^{k^n}|^2\right) ,$$

where $(Z^1 : \cdots : Z^{k^n})$ is the homogeneous coordinate of $\mathbb{C}\mathbb{P}^{k^n-1}$. We set

$$B_k := \mu_k(\iota_k(X)),$$
$$\pi_k := \mu_k \circ \iota_k : A \to B_k$$

as before. We also denote the restriction of the Fubini-Study metric to X by

$$\omega_k := \frac{1}{k}\iota_k^* \omega_{\mathrm{FS}} ,$$

here we normalize the Fubini-Study metric in order to ω_k represents $c_1(L)$.

We compare $\pi : (A, \omega_0) \to T^b$ and $\pi_k : (A, \omega_k) \to B_k$ as maps between metric spaces. For that purpose, we need to define distances on T^b and B_k. We define a metric on T^b in such a way that $\pi : (A, \omega_0) \to T^b$ is a Riemannian submersion. The distance on B_k is induced from a metric on the moment polytope Δ_k. The metric on Δ_k is also defined in such a way that

$$\mu_k : \left(\mathbb{C}\mathbb{P}^{N_k}, \frac{1}{k}\omega_{\mathrm{FS}}\right) \longrightarrow \Delta_k$$

is a Riemannian submersion in the interior of Δ_k.

Theorem 2.3 ([5]). *$\pi_k : (A, \omega_k) \to B_k$ converge to $\pi : (A, \omega) \to T^b$ in the following sense.*

(1) *ω_k converge to ω in C^∞ as $k \to \infty$. In particular, the sequence $\{(A, \omega_k)\}$ of Riemannian manifolds converges to (A, ω_0) with respect to the Gromov-Hausdorff distance.*

(2) *B_k converge to T^b as $k \to \infty$ with respect to the Gromov-Hausdorff distance.*

(3) *$\{\pi_k\}$ converges to π as maps between metric spaces.*

Before the proof, we recall the notion of Gromov-Hausdorff convergence and convergence of maps.

First we recall the definition of *Hausdorff distance.* Let Z be a metric space and $X, Y \subset Z$ be two subsets. We denote the ε-neighborhood of X in Z by $B(X, \varepsilon)$. Then the Hausdorff distance between X and Y is given by

$$d_{\mathrm{H}}^{Z}(X,Y) = \inf\{\varepsilon > 0 \mid X \subset B(Y,\varepsilon),\ Y \subset B(X,\varepsilon)\}\,.$$

For metric spaces X and Y, the *Gromov-Hausdorff distance* is defined by

$$d_{\mathrm{GH}}(X,Y) = \inf\{d_{\mathrm{H}}^{Z}(X,Y) \mid X, Y \hookrightarrow Z \text{ are isometric embeddings.}\}.$$

Next we recall the notion of convergence of maps (see also [6]). Let $f_k : X_k \to Y_k$, $f : X \to Y$ be maps between metric spaces. Suppose that X_k and Y_k converge to X and Y respectively with respect to the Gromov-Hausdorff distance. Then by definition, there exist isometric embeddings X, $X_k \hookrightarrow Z$ and Y, $Y_k \hookrightarrow W$ into some metric spaces such that X_i (resp. Y_k) converge to X (resp. Y) with respect to the Hausdorff topology in Z (resp. W). We say that $\{f_i\}$ converges to f if for every sequence $x_k \in X_k$ converging to $x \in X$, $f_k(x_k)$ converges to $f(x)$ in W.

Outline of the proof

(1) is a direct consequence of Theorem 1.1 and Proposition 2.2.

(2) Decompose $T\mathbb{CP}^{N_k}$ into horizontal and vertical parts:

$$\begin{array}{ccccc} T_p\mathbb{CP}^{N_k} & = & T_{\mathbb{CP}^{N_k}/\Delta_k,p} & \oplus & (T_{\mathbb{CP}^{N_k}/\Delta_k,p})^\perp \\ \xi & = & \xi^V & + & \xi^H \end{array}$$

where $T_{\mathbb{CP}^{N_k}/\Delta_k,p} = \ker d\mu_k$ is the tangent space to the fiber of μ_k and $(T_{\mathbb{CP}^{N_k}/\Delta_k,p})^\perp$ is the orthogonal complement with respect to the Fubini-Study metric. Similarly we decompose the tangent space of A:

$$T_zA = T_{A/T^b,z} \oplus (T_{A/T^b,z})^\perp\,,$$

where $(T_{A/T^b,z})^\perp$ is the orthogonal complement of $T_{A/T^b,z} = \ker d\pi$ with respect to the flat metric ω_0. Then the metrics on Δ_k and T^b are given by the restriction of ω_k and ω_0 on the horizontal subspaces respectively. Therefore we need to compare two horizontal and vertical spaces. We can prove that these two decompositions are close in the following sense:

Lemma 2.4. *(1) If $\xi \in T_{A/T^b,z}$, then*

$$\left| d\iota_k(\xi)^H \right| \le \frac{C}{\sqrt{k}} |\xi| \,.$$

(2) If $\eta \in (T_{A/T^b,z})^\perp$,

$$\left| d\iota_k(\eta)^V \right| \le \frac{C}{\sqrt{k}} |\eta| \,.$$

This lemma follows from the asymptotic behavior of the theta functions. By using the above estimates, we have

$$d_{\mathrm{GH}}(T^b, B_k) \le \frac{C}{\sqrt{k}} \,.$$

In fact, we can show that the composition

$$\varphi_k = \pi_k \circ \sigma_0 : T^b \longrightarrow B_k$$

of the zero section $\sigma_0 : T^b \to A$ and π_k is "almost isometric" (a $\frac{C}{\sqrt{k}}$-Hausdorff approximation (see [3] for the definition)).

3 The case of Kummer varieties

Let (A, L) be a polarized abelian variety as in the previous section. The Kummer variety of A is defined by

$$X = A/(-1)_A \,.$$

We take a line bundle $M \to X$ satisfying

$$p^* M \cong L^2 \,,$$

where $p : A \to X$ is the natural projection. From the fact that $p^* : \mathrm{Pic}(X) \to \mathrm{Pic}(A)$ is injective, we have

$$p^* M^k \cong L^{2k} \,.$$

It is easy to see that $p^* : H^0(X, M^k) \to H^0(A, L^{2k})$ is injective and the image is spanned by

$$s_{b_i} + s_{-b_i} \,, \quad b_i \in T^b_{2k}$$

(see [1] and [8]). Note that

$$N_k + 1 = \dim H^0(X, M^k) = 2^{n-1}(k^n + 1) \,.$$

Let ω be the orbifold Kähler metric induced from the flat metric $2\omega_0$ on A. Then $[\omega] = c_1(M)$. We also have a Lagrangian fibration

$$\pi : (X, \omega) \to B = T^b/(-1)$$

induced by $\pi : A \to T^b$. We set

$$t_i = \begin{cases} \frac{1}{\sqrt{2^n}}(s_{b_i} + s_{-b_i}), & \text{if } b_i \in T^b_{2k} \backslash T^b_2, \\ \frac{1}{\sqrt{2^{n-1}}} s_{b_i}, & \text{if } b_i \in T^b_2. \end{cases}$$

Then $\{t_i\}$ is an orthonormal basis of $H^0(X, M^k)$.

We denote by $\iota_k : X \to \mathbb{CP}^{N_k}$ the projective embedding defined by $\{t_i\}$, $\pi_k : X \to B_k$ the restriction of the moment map, and $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ as before. Then the same theorem holds for X as well.

Theorem 3.1. *(1) $\{(X, \omega_k)\}$ converges to (X, ω) with respect to the Gromov-Hausdorff distance.*

(2) B_k converge to B with respect to the Gromov-Hausdorff distance.

(3) $\{\pi_k\}$ converges to π as maps between metric spaces.

Outline of the proof

(1) follows from the fact that $\{t_i\}$ are orthonormal and an orbifold version of Theorem 1.1:

Theorem 3.2 (Song [9], Dai-Liu-Ma [2]). *Let (X, ω) be a compact Kähler orbifold of dimension $n \geq 2$ with only finite isolated singularities* $\mathrm{Sing}(X) = \{e_j\}_{j=1}^m$ *and $(M, h) \to X$ be an orbifold Hermitian line bundle with $c_1(M, h) = \omega$. For $k \gg 1$, we consider the projective embedding $\iota_k : X \to \mathbb{CP}^{N_k}$ defined by an orthonormal basis. We put $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ as before. Then*

$$\|\omega - \omega_k\|_{C^q, z} \leq C_q \left(\frac{1}{k} + k^{\frac{q}{2}} e^{-k\delta r(z)^2} \right),$$

where $\|\cdot\|_{C^q,z}$ is the C^q-norm at $z \in X$ and $r(z)$ is the distance between z and the singular set $\{e_j\}$.

(2) What we must take care of is the existence of singular fibers. For each $b \in \mathrm{Sing}(B) = T^b_2/(-1)$, we denote the $\sqrt{\frac{\log k}{\delta k}}$-neighborhood of the singular fiber $\pi^{-1}(b)$ by

$$N_{b,k} = \left\{ z \in X \;\middle|\; d(z, \pi^{-1}(b)) \leq \sqrt{\frac{\log k}{\delta k}} \right\}$$

and put

$$X(k) = X \backslash \bigcup_{b \in \mathrm{Sing(B)}} N_{b,k}.$$

Then we can show that $\pi(N_{b,k})$ and $\pi_k(N_{b,k})$ are small for large k (in fact, their diameters can be bounded by $O\left(\sqrt{\frac{\log k}{k}}\right)$). Therefore we may "ignore" these parts. On the other hand, we have the same estimates as in Lemma 2.4 on $X(k)$. Hence we can apply the same arguments to this situation.

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