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THE ASYMPTOTIC EXPANSION OF BERGMAN KERNELS ON SYMPLECTIC MANIFOLDS

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1. INTRODUCTION

In this talk, we explain some ideas of our approach to the asymptotic expansion of the Bergman kernel associated to a line bundle. The basic philosophy developed in [8, 15, 18] is that the spectral gap properties for the operators proved in [3, 14] implies the existence of the asymptotic expansion for the corresponding Bergman kernels if the manifold X is compact or not, or singular, or with boundary, by using the analytic localization technique inspired by [2, §11]. The interested readers may find complete references in [8, 15, 17], and in the forthcoming book [18].

We consider a compact complex manifold (X, J) with complex structure J, and holomorphic vector bundles L, E on X, with $\operatorname{rk} L = 1$. Let $\{H^{0,q}(X, L^p \otimes E)\}_{q=0}^{n}$ be the Dolbeault cohomology groups of the Dolbeault complex $(\Omega^{0,\bullet}(X, L^p \otimes E), \overline{\partial}^{L^p\otimes E}) := (\oplus_q \Omega^{0,q}(X, L^p \otimes E), \overline{\partial}^{L^p\otimes E})$.

We fix Hermitian metrics h^L, h^E on L, E. Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature R^L and let g^{TX} be a Riemannian metric on X such that

$$g^{TX}(J\cdot, J\cdot) = g^{TX}(\cdot, \cdot). \tag{1.1}$$

We denote by $\overline{\partial}^{L^p\otimes E,*}$ the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X, L^p \otimes E)$ endowed with the L^2-scalar product associated to the metrics h^L, h^E and g^{TX} and the Riemannian volume form $dv_X(x')$. Set

$$D_p = \sqrt{2}\big(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\big). \tag{1.2}$$

Then $\frac{1}{2}D_p^2$ is the Kodaira-Laplacian acting on $\Omega^{0,\bullet}(X, L^p \otimes E)$ and preserves its $\mathbb{Z}$-grading. By Hodge theory, we know that

$$\operatorname{Ker} D_p|_{\Omega^{0,q}} = \operatorname{Ker} D_p^2|_{\Omega^{0,q}} \simeq H^{0,q}(X, L^p \otimes E). \tag{1.3}$$

We denote by P_p the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ onto $\operatorname{Ker} D_p$. The Bergman kernel $P_p(x, x')$, $(x, x' \in X)$ of $L^p \otimes E$ is the smooth kernel of P_p with respect to the Riemannian volume form $dv_X(x')$.

In this setting, we are interested to understand the asymptotic expansion of $P_p(x, x')$ as $p \to \infty$. If R^L is positive, it is studied in [22, 20, 26, 7, 4, 21, 13, 23, 12] in various generalities. Moreover, the coefficients in the diagonal asymptotic expansion encode geometric information about the underlying complex projective manifolds. This diagonal asymptotic expansion plays a crucial role in the recent work of Donaldson [10] where the

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existence of Kähler metrics with constant scalar curvature is shown to be closely related to Chow-Mumford stability.

In the symplectic setting, Dai, Liu and Ma [8] studied the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on compact symplectic manifolds, and related it to that of the corresponding heat kernel. This approach is inspired by local Index Theory, especially by the analytic localization techniques of Bismut-Lebeau [2, §11]. In [8] they also focused on the full off-diagonal asymptotic expansion [8, Theorem 4.18] which is needed to study the Bergman kernel on orbifolds. By exhibiting the spectral gap properties of the corresponding operators, in [17], we also explained that without changing any step in the proof of [8, Theorem 4.18], the full off-diagonal asymptotic expansion still holds in the complex or symplectic setting if the curvature R^L of L is only non-degenerate. Note that Berman and Sjöstrand [1] recently also studied the asymptotic expansion in the complex setting when the curvauture R^L of L is only non-degenerate.

Along with $\frac{1}{2}D_p^2$ there is another geometrically defined generalization to symplectic manifolds of the Kodaira-Laplace operator, namely the renormalized Bochner-Laplacian. In this talk, we explain the asymptotic expansion of the generalized Bergman kernels of the renormalized Bochner-Laplacian on high tensor powers of a positive line bundle on compact symplectic manifolds. In this situation the operators have small eigenvalues when the power $p \to \infty$ (the only small eigenvalue is zero in [8], thus we have the key equation [8, (3.89)]) and we are interested in obtaining Theorem 3.2, that is, the *near* diagonal expansion of the generalized Bergman kernels.

There are three steps: In Step 1, by using the finite propagation speed of solutions of hyperbolic equations, we can localize our problem if the spectral gap properties holds. In Step 2, we work on $T_{x_0}X \simeq \mathbb{R}^{2n}$, and extend the bundles and connections from a neighborhood of 0 to all of $T_{x_0}X$ such that the curvature of the line bundle L is uniformly non-degenerate on $T_{x_0}X$. We combine the Sobolev norm estimates from [8] and a formal power series method to obtain the asymptotic expansion. In Step 3 we compute the coefficients by using the formal power series method from Step 2. Actually, in [17], we compute also some coefficients in the asymptotic expansion of the Bergman kernel associated to the spin^c Dirac operators in [8] by using the formal power series method here.

2. Main results

Let (X, ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L = \omega$, where $R^L = (\nabla^L)^2$ is the curvature of (L, ∇^L). Let (E, h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E.

Let g^{TX} be a Riemannian metric on X. Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}) with its curvature R^{TX} and its scalar curvature r^X. Let dv_X be the Riemannian volume form of (TX, g^{TX}). The scalar product on the space $\mathscr{C}^\infty(X, L^p \otimes E)$ of smooth sections of $L^p \otimes E$ is given by $\langle s_1, s_2\rangle = \int_X \langle s_1(x), s_2(x)\rangle_{L^p\otimes E}\, dv_X(x)$.

Let $\mathbf{J} : TX \longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u,v) = g^{TX}(\mathbf{J}u,v) \tag{2.1}$$

for $u,v \in TX$. Let J be an almost complex structure which is (separately) compatible with g^{TX} and ω, especially, $\omega(\cdot, J\cdot)$ defines a metric on TX. Then J commutes also with $\mathbf{J}$. Let $\nabla^X J \in T^*X \otimes \operatorname{End}(TX)$ be the covariant derivative of J induced by ∇^{TX}. Let $\nabla^{L^p\otimes E}$ be the connection on $L^p \otimes E$ induced by ∇^L and ∇^E. Let $\{e_i\}_i$ be an orthonormal frame of (TX, g^{TX}). Set $|\nabla^X J|^2 = \sum_{ij} |(\nabla^X_{e_i} J)e_j|^2$. Let $\Delta^{L^p\otimes E} = -\sum_i [(\nabla^{L^p\otimes E}_{e_i})^2 - \nabla^{L^p\otimes E}_{\nabla^{TX}_{e_i} e_i}]$ be the induced Bochner-Laplacian acting on $\mathscr{C}^\infty(X, L^p \otimes E)$. We fix a smooth hermitian section Φ of $\operatorname{End}(E)$ on X. Set $\tau(x) = -\pi \operatorname{Tr}_{|TX}[J\mathbf{J}]$, and

$$\Delta_{p,\Phi} = \Delta^{L^p\otimes E} - p\tau + \Phi. \tag{2.2}$$

By [14, Cor. 1.2] (cf. also [11, Theorem 2]) there exist μ_0, $C_L > 0$ independent of p such that the spectrum of $\Delta_{p,\Phi}$ satisfies

$$\operatorname{Spec} \Delta_{p,\Phi} \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty[\,. \tag{2.3}$$

This is the spectral gap property which plays an essential role in our approach. In the first place, it indicates a natural space of sections which replace the space of holomorphic sections from the complex case.

Let $P_{0,p}$ be the orthogonal projection from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle\cdot,\cdot\rangle)$ onto the eigenspace of $\Delta_{p,\Phi}$ with eigenvalues in $[-C_L, C_L]$. If the complex case (i.e. J is integrable and $\Phi = -\frac{\sqrt{-1}}{2} R^E(e_j, Je_j)$) the interval $[-C_L, C_L]$ contains for p large enough only the eigenvalue 0 whose eigenspace consists of holomorphic sections. For the computation of the spectral density function we need more general kernels. Namely, we define $P_{q,p}(x,x')$, $q \geqslant 0$ as the smooth kernels of the operators $P_{q,p} = (\Delta_{p,\Phi})^q P_{0,p}$ (we set $(\Delta_{p,\Phi})^0 = 1$) with respect to $dv_X(x')$. They are called the generalized Bergman kernels of the renormalized Bochner-Laplacian $\Delta_{p,\Phi}$. Let $\det \mathbf{J}$ be the determinant function of $\mathbf{J}_x \in \operatorname{End}(T_xX)$.

Theorem 2.1. *There exist smooth coefficients $b_{q,r}(x) \in \operatorname{End}(E)_x$ which are polynomials in R^{TX}, R^E (and R^L, Φ) and their derivatives of order $\leqslant 2(r+q)-1$ (resp. $2(r+q)$), and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x, and*

$$b_{0,0} = (\det \mathbf{J})^{1/2} \operatorname{Id}_E, \tag{2.4}$$

such that for any $k,l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x \in X$, $p \in \mathbb{N}$,

$$\Big|\frac{1}{p^n} P_{q,p}(x,x) - \sum_{r=0}^{k} b_{q,r}(x) p^{-r}\Big|_{\mathscr{C}^l} \leqslant C_{k,l}\, p^{-k-1}. \tag{2.5}$$

Moreover, the expansion is uniform in that for any $k,l \in \mathbb{N}$, there is an integer s such that if all data (g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ) run over a bounded set in the $\mathscr{C}^s$-norm and g^{TX} stays bounded below, the constant $C_{k,l}$ is independent of g^{TX}; and the $\mathscr{C}^l$-norm in (2.5) includes also the derivatives on the parameters.

Theorem 2.2. *If $J = \mathbf{J}$, then for $q \geqslant 1$,*

$$b_{0,1} = \frac{1}{8\pi}\Big[r^X + \frac{1}{4}|\nabla^X J|^2 + 2\sqrt{-1}R^E(e_j, Je_j)\Big], \tag{2.6}$$

$$b_{q,0} = \Big(\frac{1}{24}|\nabla^X J|^2 + \frac{\sqrt{-1}}{2}R^E(e_j, Je_j) + \Phi\Big)^q. \tag{2.7}$$

Theorem 2.1 for $q = 0$ and (2.6) generalize the results of [7], [26], [13] and [23] to the symplectic case. The term $r^X + \frac{1}{4}|\nabla^X J|^2$ in (2.6) is called the Hermitian scalar curvature in the literature and is a natural substitute for the Riemannian scalar curvature in the almost-Kähler case. It was used by Donaldson [9] to define the moment map on the space of compatible almost-complex structures. We can view (2.7) as an extension and refinement of the results of [11, §5] about the density of states function of $\Delta_{p,\Phi}$, (2.7) implies also a correction of a formula in [6].

Now, we try to explain the near-diagonal expansion of $P_{q,p}(x, x')$.

Let a^X be the injectivity radius of (X, g^{TX}). We fix $\varepsilon \in]0, a^X/4[$. We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε. We identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ by using the exponential map of (X, g^{TX}).

We fix $x_0 \in X$. For $Z \in B^{T_{x_0}X}(0,\varepsilon)$ we identify L_Z, E_Z and $(L^p\otimes E)_Z$ to L_{x_0}, E_{x_0} and $(L^p\otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p\otimes E}$ along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_0}(uZ)$. Then under our identification, $P_{q,p}(Z,Z')$ is a section of $\mathrm{End}(E)_{x_0}$ on $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leq \varepsilon$, we denote it by $P_{q,p,x_0}(Z,Z')$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Then we can view $P_{q,p,x_0}(Z,Z')$ as a smooth section of $\pi^*\mathrm{End}(E)$ on $TX\times_X TX$ (which is defined for $|Z|, |Z'| \leq \varepsilon$) by identifying a section $S \in \mathscr{C}^\infty(TX\times_X TX, \pi^*\mathrm{End}(E))$ with the family $(S_x)_{x\in X}$. We denote by $|\ \ |_{\mathscr{C}^s(X)}$ a $\mathscr{C}^s$ norm on it for the parameter $x_0 \in X$.

We will define the function $P^N(Z,Z')$ in (4.5).

Theorem 2.3. *There exist $J_{q,r}(Z,Z') \in \mathrm{End}(E)_{x_0}$ polynomials in Z, Z' with the same parity as r and $\deg J_{q,r}(Z,Z') \leq 3r$, whose coefficients are polynomials in R^{TX}, R^E (and R^L, Φ) and their derivatives of order $\leqslant r-1$ (resp. r), and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x_0, such that if we denote by*

$$\mathscr{F}_{q,r}(Z,Z') = J_{q,r}(Z,Z')P^N(Z,Z'), \tag{2.8}$$

then for $k, m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that if $t \in]0,1]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma/\sqrt{p}$,

(2.9)
$$\sup_{|\alpha|+|\alpha'|\leqslant m}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\Big(P_{q,p}(Z,Z') - \sum_{r=2q}^{k}\mathscr{F}_{q,r}(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2}\Big)\Big|_{\mathscr{C}^{m'}(X)} \leqslant Cp^{-(k-m+1)/2}.$$

3. Idea of the proofs

3.1. **Localization.** First, (2.3) and the finite propagation speed for hyperbolic equations, allows us to localize the problem. In particular, the asymptotics of $P_{q,p}(x_0, x')$ as $p\to\infty$ are localized on a neighborhood of x_0. Thus we can translate our analysis from X to the manifold $\mathbb{R}^{2n} \simeq T_{x_0}X =: X_0$.

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that $f(v) = 1$ for $|v| \leqslant \varepsilon/2$, and $f(v) = 0$ for $|v| \geqslant \varepsilon$. Set

$$F(a) = \Big(\int_{-\infty}^{+\infty} f(v)dv\Big)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v)dv. \tag{3.1}$$

Then $F(a)$ is an even function and lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0) = 1$. Let $\widetilde{F}$ be the holomorphic function on $\mathbb{C}$ such that $\widetilde{F}(a^2) = F(a)$. The restriction of $\widetilde{F}$ to $\mathbb{R}$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$. Then there exists $\{c_j\}_{j=1}^{\infty}$ such that for any $k \in \mathbb{N}$, the function

$$F_k(a) = \widetilde{F}(a) - \sum_{j=1}^{k} c_j a^j \widetilde{F}(a), \tag{3.2}$$

verifies

$$F_k^{(i)}(0) = 0 \quad \text{for any } 0 < i \leqslant k. \tag{3.3}$$

Proposition 3.1. *For any $k, m \in \mathbb{N}$, there exists $C_{k,m} > 0$ such that for $p \geqslant 1$*

$$\Big| F_k\big(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\big)(x,x') - P_{0,p}(x,x')\Big|_{\mathscr{C}^m(X\times X)} \leqslant C_{k,m} p^{-\frac{k}{2}+2(2m+2n+1)}. \tag{3.4}$$

Here the $\mathscr{C}^m$ norm is induced by ∇^L, ∇^E, h^L, h^E and g^{TX}.

Using (3.1), (3.2) and the finite propagation speed of solutions of hyperbolic equations, it is clear that for $x, x' \in X$, $F_k(\frac{1}{\sqrt{p}}\Delta_{p,\Phi})(x,\cdot)$ only depends on the restriction of $\Delta_{p,\Phi}$ to $B^X(x, \varepsilon p^{-\frac{1}{4}})$, and $F_k(\frac{1}{\sqrt{p}}\Delta_{p,\Phi})(x,x') = 0$, if $d(x,x') \geqslant \varepsilon p^{-\frac{1}{4}}$. This means that the asymptotic of $\Delta^q_{p,\Phi} P_{\mathcal{H}_p}(x,\cdot)$ when $p \to +\infty$, modulo $\mathscr{O}(p^{-\infty})$ (i.e. terms whose $\mathscr{C}^m$ norm is $\mathscr{O}(p^{-l})$ for any $l, m \in \mathbb{N}$), only depends on the restriction of $\Delta_{p,\Phi}$ to $B^X(x, \varepsilon p^{-\frac{1}{4}})$.

3.2. **Uniform estimate of the generalized Bergman kernels.** We will work on the normal coordinate for $x_0 \in X$. We identify the fibers of (L, h^L), (E, h^E) with $(L_{x_0}, h^{L_{x_0}})$, $(E_{x_0}, h^{E_{x_0}})$ respectively, in a neighborhood of x_0, by using the parallel transport with respect to ∇^L, ∇^E along the radial direction.

We then extend the bundles and connections from a neighborhood of 0 to all of $T_{x_0}X$. In particular, we can extend ∇^L (resp. ∇^E) to a Hermitian connection ∇^{L_0} on $(L_0, h^{L_0}) = (X_0 \times L_{x_0}, h^{L_{x_0}})$ (resp. ∇^{E_0} on $(E_0, h^{E_0}) = (X_0 \times E_{x_0}, h^{E_{x_0}})$) on $T_{x_0}X$ in such a way so that we still have positive curvature R^{L_0}; in addition $R^{L_0} = R^L_{x_0}$ outside a compact set. We also extend the metric g^{TX_0}, the almost complex structure J_0, and the smooth section Φ_0, (resp. the connection ∇^{E_0}) in such a way that they coincide with their values at 0 (resp. the trivial connection) outside a compact set. Moreover, using a fixed unit vector $S_L \in L_{x_0}$ and the above discussion, we construct an isometry $E_0 \otimes L_0^p \simeq E_{x_0}$. Let $\Delta^{X_0}_{p,\Phi_0}$ be the renormalized Bochner-Laplacian on X_0 associated to the above data by a formula analogous to (2.2). Then (2.3) still holds for $\Delta^{X_0}_{p,\Phi_0}$ with μ_0 replaced by $4\mu_0/5$.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$ and $\kappa(Z)$ be the smooth positive function defined by the equation $dv_{X_0}(Z) = \kappa(Z) dv_{TX}(Z)$, with $k(0) = 1$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$, $Z \in \mathbb{R}^{2n}$ and $t = 1/\sqrt{p}$, set $\|s\|_0^2 = \int_{\mathbb{R}^{2n}} |s(Z)|^2_{h^{E_{x_0}}} dv_{TX}(Z)$, and consider

$$\mathscr{L}_t = S_t^{-1} t^2 \kappa^{\frac{1}{2}} \Delta^{X_0}_{p,\Phi_0} \kappa^{-\frac{1}{2}} S_t, \quad \text{where } (S_t s)(Z) = s(Z/t). \tag{3.5}$$

Then $\mathscr{L}_t$ is a family of self-adjoint differential operators with coefficients in $\mathrm{End}(E)_{x_0}$. We denote by $\mathcal{P}_{0,t} : (\mathscr{C}^\infty(X_0, E_{x_0}), \|\ \|_0) \to (\mathscr{C}^\infty(X_0, E_{x_0}), \|\ \|_0)$ the spectral projection of $\mathscr{L}_t$ corresponding to the interval $[-C_{L_0}t^2, C_{L_0}t^2]$. Let $\mathcal{P}_{q,t}(Z,Z') = \mathcal{P}_{q,t,x_0}(Z,Z')$, $(Z, Z' \in X_0,\ q \geqslant 0)$ be the smooth kernel of $\mathcal{P}_{q,t} = (\mathscr{L}_t)^q \mathcal{P}_{0,t}$ with respect to $dv_{TX}(Z')$. We can view $\mathcal{P}_{q,t,x}(Z,Z')$ as a smooth section of $\pi^* \mathrm{End}(E)$ over $TX \times_X TX$, where $\pi : TX \times_X TX \to X$. Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\mu_0/4$. By (2.3),

$$\mathcal{P}_{q,t} = \frac{1}{2\pi i}\int_\delta \lambda^q(\lambda - \mathscr{L}_t)^{-1} d\lambda. \tag{3.6}$$

From (2.3) and (3.6) we can apply the techniques in [8], which are inspired by [2, §11], to get the following key estimate.

Theorem 3.2. *There exist smooth sections $F_{q,r} \in \mathscr{C}^\infty(TX \times_X TX, \pi^*\mathrm{End}(E))$ such that for $k, m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that if $t \in]0,1]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$,*

$$\sup_{|\alpha|,|\alpha'|\leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(\mathcal{P}_{q,t} - \sum_{r=0}^{k} F_{q,r} t^r\Big)(Z,Z')\right|_{\mathscr{C}^{m'}(X)} \leqslant Ct^k. \tag{3.7}$$

Let $P_{0,q,p}(Z,Z') \in \mathrm{End}(E_{x_0})$ $(Z, Z' \in X_0)$ be the analogue of $P_{q,p}(x,x')$. By (3.5), for $Z, Z' \in \mathbb{R}^{2n}$,

$$P_{0,q,p}(Z,Z') = t^{-2n-2q}\, \kappa^{-\frac{1}{2}}(Z)\, \mathcal{P}_{q,t}(Z/t, Z'/t)\, \kappa^{-\frac{1}{2}}(Z'). \tag{3.8}$$

By Proposition 3.1, we know that

$$P_{0,q,p}(Z,Z') = P_{q,p,x_0}(Z,Z') + \mathscr{O}(p^{-\infty}), \tag{3.9}$$

uniformly for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leq \varepsilon/2$.

To complete the proof the Theorem 2.1, we finally prove $F_{q,r} = 0$ for $r < 2q$. In fact, (3.7) and (3.8) yield

$$b_{q,r}(x_0) = F_{q,2r+2q}(0,0). \tag{3.10}$$

4. Evaluation of $F_{q,r}$

The almost complex structure J induces a splitting $T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. We choose $\{w_i\}_{i=1}^n$ to be an orthonormal basis of $T^{(1,0)}_{x_0}X$, such that

$$-2\pi\sqrt{-1}\mathbf{J}_{x_0} = \mathrm{diag}(a_1, \cdots, a_n) \in \mathrm{End}(T^{(1,0)}_{x_0}X). \tag{4.1}$$

We use the orthonormal basis $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \ldots, n$ of $T_{x_0}X$ to introduce the normal coordinates as in Section 3. In what follows we will use the complex coordinates $z = (z_1, \cdots, z_n)$, thus $Z = z + \overline{z}$, and $w_i = \sqrt{2}\frac{\partial}{\partial z_i}$, $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial \overline{z}_i}$. It is very useful to introduce the creation and annihilation operators b_i, b_i^+,

$$b_i = -2\tfrac{\partial}{\partial z_i} + \frac{1}{2}a_i\overline{z}_i\,, \quad b_i^+ = 2\tfrac{\partial}{\partial \overline{z}_i} + \frac{1}{2}a_i z_i\,, \quad b = (b_1, \cdots, b_n)\,. \tag{4.2}$$

Now there are second order differential operators $\mathcal{O}_r$ whose coefficients are polynomials in Z with coefficients being polynomials in R^{TX}, $R^{\det}$, R^E, R^L and their derivatives at x_0, such that

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{\infty} \mathcal{O}_r t^r, \quad \text{with } \mathscr{L}_0 = \sum_i b_i b_i^+ . \tag{4.3}$$

Theorem 4.1. *The spectrum of the restriction of $\mathscr{L}_0$ to $L^2(\mathbb{R}^{2n})$ is given by $\left\{2\sum_{i=1}^n \alpha_i a_i : \alpha_i \in \mathbb{N}\right\}$ and an orthogonal basis of the eigenspace of $2\sum_{i=1}^n \alpha_i a_i$ is given by*

$$b^\alpha\Big(z^\beta \exp\Big(-\frac{1}{4}\sum_i a_i |z_i|^2\Big)\Big), \quad \textit{with } \beta \in \mathbb{N}^n . \tag{4.4}$$

Let $N^\perp$ be the orthogonal space of $N = \operatorname{Ker} \mathscr{L}_0$ in $(L^2(\mathbb{R}^{2n}, E_{x_0}), \|\ \|_0)$. Let P^N, $P^{N^\perp}$ be the orthogonal projections from $L^2(\mathbb{R}^{2n}, E_{x_0})$ onto N, $N^\perp$, respectively. Let $P^N(Z, Z')$ be the smooth kernel of the operator P^N with respect to $dv_{TX}(Z')$. ¿From (4.4), we get

$$P^N(Z, Z') = \frac{1}{(2\pi)^n} \prod_{i=1}^n a_i \; \exp\Big(-\frac{1}{4}\sum_i a_i\big(|z_i|^2 + |z_i'|^2 - 2 z_i \overline{z}_i'\big)\Big). \tag{4.5}$$

Now for $\lambda \in \delta$, we solve for the following formal power series on t, with $g_r(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}), N)$, $f_r^\perp(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}), N^\perp)$,

$$(\lambda - \mathscr{L}_t) \sum_{r=0}^{\infty} \Big(g_r(\lambda) + f_r^\perp(\lambda)\Big) t^r = \operatorname{Id}_{L^2(\mathbb{R}^{2n}, E_{x_0})} . \tag{4.6}$$

¿From (3.6), (4.6), we claim that

$$F_{q,r} = \frac{1}{2\pi i} \int_\delta \lambda^q g_r(\lambda) d\lambda + \frac{1}{2\pi i} \int_\delta \lambda^q f_r^\perp(\lambda) d\lambda. \tag{4.7}$$

¿From Theorem 4.1, (4.7), the key observation that $P^N \mathcal{O}_1 P^N = 0$, and the residue formula, we can get $F_{q,r}$ by using the operators $\mathscr{L}_0^{-1}$, P^N, $P^{N^\perp}$, $\mathcal{O}_i$, $(i \leqslant r)$. This gives us a method to compute $b_{q,r}$ in view of Theorem 4.1 and (3.10). Especially, for $q > 0, r < 2q$,

$$\begin{aligned}
F_{0,0} &= P^N, \quad F_{q,r} = 0, \\
F_{q,2q} &= (P^N \mathcal{O}_2 P^N - P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 P^N)^q P^N, \\
F_{0,2} &= \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 P^N - \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_2 P^N \\
&\quad + P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} - P^N \mathcal{O}_2 \mathscr{L}_0^{-1} P^{N^\perp} \\
&\quad + P^{N^\perp} \mathscr{L}_0^{-1} \mathcal{O}_1 P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} - P^N \mathcal{O}_1 \mathscr{L}_0^{-2} P^{N^\perp} \mathcal{O}_1 P^N.
\end{aligned} \tag{4.8}$$

In fact $\mathscr{L}_0$ and $\mathcal{O}_r$ are formal adjoints with respect to $\|\ \|_0$; thus in $F_{0,2}$ we only need to compute the first two terms, as the last two terms are their adjoints. This simplifies the computation in Theorem 2.2.

5. Generalizations to non-compact manifolds

In this section we come back to the case of complex manifolds, which was briefly discussed in the introduction, but focus on non-compact manifolds. Let (X,Θ) be a Hermitian manifold of dimension n, where Θ is the $(1,1)$ form associated to a hermitian metric on X. Given a Hermitian holomorphic bundles L and E on X with $\operatorname{rk} L = 1$, we consider the space of L^2 holomorphic sections $H^0_{(2)}(X, L^p\otimes E)$. Let P_p be the orthogonal projection from the space $L^2(X, L^p\otimes E)$ of L^2 sections of $L^p\otimes E$ onto $H^0_{(2)}(X, L^p\otimes E)$. By generalizing the definition from Section 1, we define the Bergman kernel $P_p(x,x')$, $(x,x'\in X)$ to be the Schwartz kernel of P_p with respect to the Riemannian volume form $dv_X(x')$ associated to (X,Θ). By the ellipticity of the Kodaira-Laplacian and Schwartz kernel theorem, we know $P_p(x,x')$ is $\mathscr{C}^\infty$. Choose an orthonormal basis $(S^p_i)_{i=1}^{d_p}$ $(d_p\in\mathbb{N}\cup\{\infty\})$ of $H^0_{(2)}(X, L^p\otimes E)$. The Bergman kernel can then be expressed as

$$P_p(x,x') = \sum_{i=1}^{d_p} S^p_i(x)\otimes (S^p_i(x'))^* \in (L^p\otimes E)_x\otimes (L^p\otimes E)^*_{x'}.$$

Let $K_X = \det(T^{*(1,0)}X)$ be the canonical line bundle of X and $R^{\det}$ be the curvature of K^*_X relative to the metric induced by Θ. The line bundle L is supposed to be positive and we set $\omega = \frac{\sqrt{-1}}{2\pi}R^L$.

We denote by g^{TX}_ω the Riemannian metric associated to ω and by r^X_ω the scalar curvature of g^{TX}_ω. Moreover, let $\alpha_1,\ldots,\alpha_n$ be the eigenvalues of ω with respect to Θ $(\alpha_j = a_j/(2\pi)$, $j=1,\ldots,n$ where $a_1,\ldots,a_n$ are defined by (4.1) and (2.1) with g^{TX} the Riemannian metric associated to $\Theta)$. The torsion of Θ is $T = [i(\Theta), \partial\Theta]$, where $i(\Theta) = (\Theta\wedge\cdot)^*$ is the interior multiplication with Θ.

Theorem 5.1 ([15]). *Assume that (X,Θ) is a complete Hermitian manifold of dimension n. Suppose that there exist $\varepsilon>0$, $C>0$ such that*

$$\sqrt{-1}R^L \geqslant \varepsilon\Theta\,,\quad \sqrt{-1}R^{\det}\geqslant -C\Theta\,,\quad \sqrt{-1}R^E\geqslant -C\Theta\,,\qquad |T|\leqslant C\Theta\,. \tag{5.1}$$

Then the kernel $P_p(x,x')$ has a full off-diagonal asymptotic expansion uniformly on compact sets of $X\times X$ and $P_p(x,x)$ has an asymptotic expansion analogous to (2.5) *uniformly on compact sets of X. Moreover, $b_0 = \alpha_1\cdots\alpha_n \operatorname{Id}_E$ and*

$$b_1 = \frac{\alpha_1\cdots\alpha_n}{8\pi}\Big[r^X_\omega \operatorname{Id}_E - 2\Delta_\omega\Big(\log(\alpha_1\cdots\alpha_n)\Big)\operatorname{Id}_E + 4\sum_{j=1}^n R^E(w_{\omega,j},\overline{w}_{\omega,j})\Big]\,,$$

where $\{w_{\omega,j}\}$ is an orthonormal basis of $(T^{(1,0)}X, g^{TX}_\omega)$.

By *full off-diagonal* expansion we mean an expansion analogous to (2.9) where we allow $|Z|,|Z'|\leqslant\sigma$.

Let us remark that if $L=K_X$, the first two conditions in (5.1) are to be replaced by

$$h^L \text{ is induced by } \Theta \text{ and } \sqrt{-1}R^{\det} < -\varepsilon\Theta. \tag{5.2}$$

Moreover, if (X,Θ) is Kähler, the condition on the torsion T is trivially satisfied.

The proof is based on the observation that the Kodaira-Laplacian $\square_p = \frac{1}{2}D^2_p$ acting on $L^2(X, L^p\otimes E)$ has a spectral gap as in (2.3). The proof of Theorem 2.1 applies then and delivers the result.

Theorem 5.1 has several applications e.g. holomorphic Morse inequalities on non-compact manifolds (as the well-known results of Nadel-Tsuji [19], see also [15, 25]) or Berezin-Toeplitz quantization (see [18] or the fothcommimg [16]).

We will emphazise in the sequel the Bergman kernel for a singular metric. Let X be a compact complex manifold. A *singular Kähler metric* on X is a closed, strictly positive $(1,1)$-current ω. If the cohomology class of ω in $H^2(X,\mathbb{R})$ is integral, there exists a holomorphic line bundle (L,h^L), endowed with a singular Hermitian metric, such that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$ in the sense of currents. We call (L,h^L) a *singular polarization* of ω.

If we change the metric h^L, the curvature of the new metric will be in the same cohomology class as ω. In this case we speak of a polarization of $[\omega]\in H^2(X,\mathbb{R})$. Our purpose is to define an appropriate notion of polarized section of L^p, possibly by changing the metric of L, and study the associated Bergman kernel.

Corollary 5.2. *Let (X,ω) be a compact complex manifold with a singular Kähler metric with integral cohomology class. Let (L,h^L) be a singular polarization of $[\omega]$ with strictly positive curvature current having singular support along a proper analytic set Σ. Then the Bergman kernel of the space of polarized sections*

$$H^0_{(2)}(X\smallsetminus\Sigma,L^p)=\left\{u\in L^{0,0}_2(X\smallsetminus\Sigma,L^p\,,\,\Theta_P\,,h^L_\varepsilon):\overline{\partial}^{L^p}u=0\right\}$$

has the asymptotic expansion as in Theorem 5.1 for $X\smallsetminus\Sigma$, where Θ_P is a generalized Poincaré metric on $X\smallsetminus\Sigma$ and h^L_ε is a modified Hermitian metric on L.

Using an idea of Takayama [24], Corollary 5.2 gives a proof of the Shiffman-Ji-Bonavero-Takayama criterion, about the characterization of Moishezon manifolds by $(1,1)$ positive currents.

We mention further the Berezin-Toeplitz quantization. Assume that X is a complex manifold and let $\mathscr{C}^\infty_{const}(X)$ denote the algebra of smooth functions of X which are constant outside a compact set. For any $f\in\mathscr{C}^\infty_{const}(X)$ we denote for simplicity the operator of multiplication with f still by f and consider the linear operator

$$T_{f,p}:L^2(X,L^p)\longrightarrow L^2(X,L^p)\,,\quad T_{f,p}=P_p\,f\,P_p\,. \tag{5.3}$$

The family $(T_{f,p})_{p\geqslant1}$ is called a Toeplitz operator. The following result generalizes [5] to non-compact manifolds.

Corollary 5.3. *We assume that (X,Θ) and (L,h^L) satisfy the same hypothesis as in Theorem 5.1 or* (5.2). *Let $f,g\in\mathscr{C}^\infty_{const}(X)$. The product of the two corresponding Toeplitz operators admits the asymptotic expansion*

$$T_{f,p}T_{g,p}=\sum_{r=0}^{\infty}p^{-r}T_{C_r(f,g),p}+\mathscr{O}(p^{-\infty}) \tag{5.4}$$

where C_r are differential operators. More precisely,

$$C_0(f,g)=fg\,,\quad C_1(f,g)-C_1(g,f)=\frac{1}{\sqrt{-1}}\{f,g\} \tag{5.5}$$

where the Poisson bracket is taken with respect to the metric $2\pi\omega$. Therefore

$$[T_{f,p},T_{g,p}]=p^{-1}T_{\frac{1}{\sqrt{-1}}\{f,g\},p}+\mathscr{O}(p^{-2}). \tag{5.6}$$

Remark 5.4. For any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ we can consider the linear operator

$$T_{f,p} : \boldsymbol{L}^2(X, L^p \otimes E) \longrightarrow \boldsymbol{L}^2(X, L^p \otimes E)\,, \quad T_{f,p} = P_p\, f\, P_p\,. \tag{5.7}$$

Then (5.4) holds for any $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ which are constant outside some compact set. Moreover, (5.5), (5.6) still hold for $f, g \in \mathscr{C}^\infty_{const}(X) \subset \mathscr{C}^\infty(X, \mathrm{End}(E))$.

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