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Nonsynchronously observed diffusions and covariance estimation

Takaki Hayashi*
Columbia University, Department of Statistics
1255 Amsterdam Avenue, New York 10027, U.S.A.
Keio University, Graduate School of Business Administration
2-1-1 Hiyoshi-honcho, Yokohama 223-8523, JAPAN
E-mail: hayashi@stat.columbia.edu
and
Nakahiro Yoshida
University of Tokyo, Graduate School of Mathematical Sciences
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, JAPAN
E-mail: nakahiro@ms.u-tokyo.ac.jp

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Abstract

We consider the problem of estimating the covariance/correlation of two diffusion-type processes when the processes are observed only at discrete times in a *nonsynchronous* manner. The purpose of the paper is to overview the new methodology that the authors have been proposing since 2003, which is free of any 'synchronization' processing of original data. Specifically, it briefly presents major results obtained in [8], [6] and [7], i.e., *consistency* and *asymptotic normality* of the proposed covariance/correlation estimators as the observation interval size shrinks to zero.

Key words: diffusion; discrete sampling; high-frequency data; mathematical finance; nonsynchronicity; quadratic variation; realized volatility
JEL classification: C13, C32

1 Introduction

Consider the case when two continuous diffusion processes are observed only at discrete times in a *nonsynchronous* manner. We are interested in estimating the *covariance/correlation* of the two processes accurately in such a situation. This kind of problem arises typically in *high-frequency finance*. A popular approach to this is to compute

$$V^{k,l}_{\pi(m)} := \sum_{i=1}^{m}(P^k_{t_i} - P^k_{t_{i-1}})(P^l_{t_i} - P^l_{t_{i-1}}), \quad k,l = 1,2, \tag{1.1}$$

which is often called the *realized volatility* estimator (case: $k = l$) or the *realized covariance* estimator (case: $k \neq l$) in the literature; see, e.g., [1]. Here, P^1 and P^2 are continuous semimartingales representing log-prices,

$0 = t_0 < t_1 < \cdots < t_m = T$ are grid points for measuring their respective changes with mesh size $\pi(m) := \max_{1\leq i\leq m} |t_i - t_{i-1}|$. Similarly, the standardized covariance estimator, $R^{k,l}_{\pi(m)} := V^{1,2}_{\pi(m)}/\sqrt{V^{1,1}_{\pi(m)}V^{2,2}_{\pi(m)}}$, is called the *realized correlation* estimator. The popularity of the estimators come from its *consistency*, i.e., as $\pi(m) \to 0$, one has $V^{k,l}_{\pi(m)} \to \langle P^k, P^l\rangle_T$ in probability, not to mention from their ease of implementation. For practical convenience it is standard to take equal spacing, i.e., $t_i - t_{i-1} = T/m$ $(=: h)$, $i \geq 1$.

Actual transaction data are recorded at random times in a irregular manner. This fact requires one who adopts (1.1) to 'synchronize' two time series *a priori*; choose a common interval length h first, then impute missing observations by some interpolation scheme – typically either previous-tick interpolation or linear interpolation. Inevitably, the value of $V^{k,l}_h$ depends heavily on the choice of h as well as an interpolation method adopted, so does that of $R^{k,l}_h$. It can be easily shown that such arbitrary choices would produce biases in $V^{k,l}_h$ or $R^{k,l}_h$; see [8] and the references therein. By and large, most of the existing approaches rely on the 'synchronization' of the original data.

Estimation problems of the diffusion parameter for diffusion processes based on discrete-time samples have been well studied in statistics. See [12], [13], [14], [3], [4], and [10]; however, nonsynchronicity seems to have been rarely treated. To tackle the nonsynchronicity estimation problem we proposed a new estimation procedure in 2003, which is free of any 'synchronization' processing of original data (see [8]). We are going to review the methodology and some theoretical results obtained since then ([6], [7]).

2 The theory

Suppose P^l follows the one-dimensional Itô process

$$dP^l_t = \mu^l_t dt + \sigma^l_t dW^l_t, \quad P^l_0 = p^l,\ 0 \leq t \leq T,\ l = 1, 2, \tag{2.1}$$

with $d\langle W^1, W^2\rangle_t = \rho_t dt$, where $\rho_\cdot \in (-1, 1)$ is an unknown, deterministic function, $p^l > 0$ is a constant, $\mu^l_\cdot$ is a progressively measurable (possibly unknown) function, and $\sigma^l_\cdot > 0$ is a deterministic and bounded (possibly unknown) function. let $T \in (0, \infty)$ be an arbitrary terminal time for observing P^ls.

Let $\Pi^1 := (I^i)_{i=1,2,\ldots}$ and $\Pi^2 := (J^i)_{i=1,2,\ldots}$ be random intervals reading from left to right, each of which partitions $(0, T]$. Let $T^{1,i} := \inf\{t \in I^{i+1}\}$ represent the ith observation time of P^1, and $T^{2,i} := \inf\{t \in J^{i+1}\}$ that of P^2, $i \geq 0$. Let n be an index representing the size of Π^1 and Π^2. Let $r_n := \max_{1\leq i<\infty} |I^i| \vee \max_{1\leq j<\infty} |J^j|$, the largest interval size.

2.1 Consistency

First, we assume that the sampling intervals Π satisfy the following conditions.

Condition (C0): *(i) (I^i) and (J^i) are independent of P^1 and P^2; (ii) As $n \to \infty$, $r_n \to 0$* in probability.

The parameter of interest is the (deterministic) covariation of P^1 and P^2,

$$\langle P^1, P^2\rangle_T = \int_0^T \sigma^1_t \sigma^2_t \rho_t dt =: \theta.$$

[8] have proposed the following estimator for θ constructed from the observations of P^1 and P^2, and the times they were recorded at.

Definition 1 *(Nonsynchronous covariance estimator):*

$$U_n := \sum_{i,j} \left(P^1_{T^{1,i}} - P^1_{T^{1,i-1}}\right)\left(P^2_{T^{2,j}} - P^2_{T^{2,j-1}}\right) 1_{\{I^i \cap J^j \neq \emptyset\}}. \tag{2.2}$$

That is, the product of any pair of increments $(P^1_{T^{1,i}} - P^1_{T^{1,i-1}})$ and $(P^2_{T^{2,j}} - P^2_{T^{2,j-1}})$ will make a contribution to the summation only when the respective observation intervals I^i and J^j are overlapping.

Theorem 2 ([8]) *(Unbiasedness) If $\mu^l_t \equiv 0$, $l = 1, 2$, then U_n is unbiased for θ.*
(Consistency) Suppose (C0) holds.
(1) If $\sup_{0\le t\le T} |\mu^l_t| \in L^4$, $l = 1, 2$, then $U_n \to \theta$ in L^2 as $n \to \infty$.
(2) If $\sup_{0\le t\le T} |\mu^l_t| < \infty$ almost surely, $l = 1, 2$, then $U_n \to \theta$ in probability as $n \to \infty$.

Suppose further that $\rho_t \equiv \rho$ and $\sigma^l_t \equiv \sigma^l$ for some constant, $\rho \in (-1, 1)$ and $\sigma^l > 0$, $l = 1, 2$. We are now interested in estimating the correlation ρ.

Definition 3 *(Nonsynchronous correlation estimators):*

$$R^{(1)}_n := \frac{1}{T}\sum_{i,j} \frac{\left(P^1_{T^{1,i}} - P^1_{T^{1,i-1}}\right)\left(P^2_{T^{2,j}} - P^2_{T^{2,j-1}}\right)}{\sigma^1\sigma^2} 1_{\{I^i \cap J^j \neq \emptyset\}} \quad (\sigma^l \text{ are known}),$$

$$R^{(2)}_n := \frac{\sum_{i,j}\left(P^1_{T^{1,i}} - P^1_{T^{1,i-1}}\right)\left(P^2_{T^{2,j}} - P^2_{T^{2,j-1}}\right) 1_{\{I^i \cap J^j \neq \emptyset\}}}{\left\{\sum_i \left(P^1_{T^{1,i}} - P^1_{T^{1,i-1}}\right)^2\right\}^{1/2}\left\{\sum_j \left(P^2_{T^{2,j}} - P^2_{T^{2,j-1}}\right)^2\right\}^{1/2}} \quad (\sigma^l \text{ are unknown/known}).$$

Corollary 4 ([8]) *Under (C0), $R^{(1)}_n$ and $R^{(2)}_n$ are consistent for ρ as $n \to \infty$.*

Remark: In the financial econometrics literature is recently studied the estimate problem of realized volatility subject to *market microstructure*; see, e.g., [15]. Because nonsynchronicity is a fundamental, salient feature for the multivariate case, we focus on it, without taking microstructure noise into consideration. It is deferred for future research.

2.2 Asymptotic normality

We have also obtained *joint asymptotic normality* of the proposed covariance estimator with the 'raw' realized volatilities (i.e., without synchronization) as the observation interval size shrinks to zero; [6], [7].

We basically maintain the same set-up as stated in the previous section with the following modification regarding U_n and θ: Let $U_n := \left(U^{(0)}_n, U^{(1)}_n, U^{(2)}_n\right)^\top$ where

$$U^{(0)}_n := \sum_{i,j} \left(P^1_{T^{1,i}} - P^1_{T^{1,i-1}}\right)\left(P^2_{T^{2,j}} - P^2_{T^{2,j-1}}\right) 1_{\{I^i \cap J^j \neq \emptyset\}},$$
$$U^{(1)}_n := \sum_i \left(P^1_{T^{1,i}} - P^1_{T^{1,i-1}}\right)^2, \quad U^{(2)}_n := \sum_j \left(P^2_{T^{2,j}} - P^2_{T^{2,j-1}}\right)^2,$$

and $\theta := \left(\theta^{(0)}, \theta^{(1)}, \theta^{(2)}\right)^\top$, where

$$\theta^{(0)} := v^0((0,T]) = \int_0^T \sigma^1_t\sigma^2_t\rho_t dt, \ \theta^{(l)} := v^l((0,T]) := \int_0^T \left(\sigma^l_t\right)^2 dt, \ l = 1, 2.$$

We are interested in asymptotic normality of the three-dimensional vector U_n that consists of the nonsynchronous covariance estimator and the two 'raw' *realized volatility* estimators (without synchronization).

Obviously, (C0) alone is insufficient to establish asymptotic normality of the estimator. We replace (C0) by a stronger set of conditions (C1)–(C4) as follows.

Condition (C1): *(I^i) and (J^i) are independent of P^1 and P^2;*

We define (signed) measures by, for each $I \in \mathcal{B}_{[0,T]}$, where $\mathcal{B}_{[0,T]}$ is the Borel σ-field on $[0,T]$,

$$v(I) := v^0(I) := \int_I \sigma_t^1 \sigma_t^2 \rho_t dt;\ v^l(I) := \int_I \left(\sigma_t^l\right)^2 dt,\ l = 1,2.$$

Now, let $\mathbb{V}_n$ be a (3×3)-*matrix whose elements are*

$$\begin{aligned}
&\mathbb{V}_n^{(0,0)} := b_n^{-1} \left\{ \sum_{i,j} v^1\left(I^i\right) v^2\left(J^j\right) 1_{\{I^i \cap J^j \neq \emptyset\}} + \sum_i v\left(I^i\right)^2 + \sum_j v\left(J^j\right)^2 - \sum_{i,j} v\left(I^i \cap J^j\right)^2 \right\},\\
&\mathbb{V}_n^{(1,1)} := b_n^{-1} \cdot 2 \sum_i v^1\left(I^i\right)^2, \quad \mathbb{V}_n^{(2,2)} := b_n^{-1} \cdot 2 \sum_j v^2\left(J^j\right)^2,\\
&\mathbb{V}_n^{(1,0)} := \mathbb{V}_n^{(0,1)} := b_n^{-1} \cdot 2 \sum_i v^1\left(I^i\right) v\left(I^i\right), \quad \mathbb{V}_n^{(2,0)} := \mathbb{V}_n^{(0,2)} := b_n^{-1} \cdot 2 \sum_j v^2\left(J^j\right) v\left(J^j\right),\\
&\mathbb{V}_n^{(2,1)} := \mathbb{V}_n^{(1,2)} := b_n^{-1} \cdot 2 \sum_{i,j} v\left(I^i \cap J^j\right)^2.
\end{aligned} \tag{2.3}$$

Condition (C2): *There exist a sequence of positive numbers (b_n) and some non-random, nontrivial, symmetric, positive semi-definite, (3×3)-matrix $\mathbb{C}$ such that, as $n \to \infty$, $b_n \to 0$* and

$$\mathbb{V}_n \xrightarrow{P} \mathbb{C}. \tag{2.4}$$

The condition (C2) postulates the (asymptotic) connection between the observation intervals Π and the variance-covariance structure of the given processes, $(v^1(\cdot), v^2(\cdot), v(\cdot))$. When $\mu^l \equiv 0$, (C2) is equivalent to the condition that $b_n^{-1} var^{\Pi}[U_n] \xrightarrow{P} c$ as $n \to \infty$.

Condition (C3): *There exists some $\alpha \in (0, 1/4)$ such that*

$$r_n = o_P\left(b_n^{\frac{3}{4}+\alpha}\right).$$

That is, we allow the random mesh size r_n of Π to tend to zero slowly relative to the (deterministic) b_n, but not too slowly.

For a continuous stochastic process X, we define, for each $\omega \in \Omega$ and $\Delta > 0$, the *modulus of continuity* on $[0,T]$, by

$$\delta(X(\omega); \Delta) := \sup\left\{|X_t(\omega) - X_s(\omega)|; |t-s| \leq \Delta, 0 \leq s,t \leq T\right\}.$$

The following is a condition postulating that the (random) drifts of the underlying processes are sufficiently smooth so that their contribution to U_n in (2.2) would be asymptotically negligible (and that asymptotic normality for the zero drift case would be generalized to the non-zero drift case).

Condition (C4): *For $l = 1, 2$, μ^l is continuous and adapted, such that*

$$\delta(\mu^l; r_n) = O_P\left(r_n^{\frac{1}{2}} b_n^{-(\frac{1}{4}+\alpha)}\right)$$

for α given in (C3).

For instance, an obvious sufficient condition for (C4) is that $t \longmapsto \mu_t^l(\omega)$ is Lipschitz continuous.

Theorem 5 ([6], [7]) *Under the Conditions (C1) - (C4), as $n \to \infty$,*

$$b_n^{-1/2}\left(U_n - \theta\right) \xrightarrow{\mathcal{L}} N(0, \mathbb{C}). \tag{2.5}$$

2.3 Refinement

[7] proposed how to upgrade the condition (C2) so as to make the central limit theory more applicable in practice. Let us define

$$H_n^1(t) := \sum_{i:T^{1,i}\le t} |I^i|^2,\ H_n^2(t) := \sum_{j:T^{2,j}\le t} |J^j|^2,$$
$$H_n^{1\cap 2}(t) := \sum_{i:T^{1,i}\le t}\sum_{j:T^{2,j}\le t} |I^i \cap J^j|^2,\ H_n^{1*2}(t) := \sum_{i:T^{1,i}\le t}\sum_{j:T^{2,j}\le t} |I^i||J^j| 1_{\{I^i\cap J^j\neq\emptyset\}}.$$

The four functions describe the 'distributions' over $[0,T]$ of the sampling times for the bivariate process (P^1, P^2). The functions are piecewise constant, nondecreasing, right-continuous functions starting from 0 at $t = 0$ $(\forall n)$; their "jumps" occur at (subsets of) the sampling times $\left(T^{1,i}, T^{2,j}, i \ge 1, j \ge 1\right)$. They are all finite at $t = T$ $(\forall n)$ and tend to zero as $n \to \infty$ (under (C0)(ii); see [8]). Now we will replaces (C2) with the following condition.

Condition $(\widetilde{C2})$: *There exist a sequence of positive numbers (b_n) with $b_n \to 0$ as $n \to \infty$ and non-random, nondecreasing, right-continuous functions $H^1, H^2, H^{1\cap 2}$,* and *H^{1*2}, respectively mapping from $[0,T]$ into $[0,\infty)$, such that $f(0) = 0$, $0 < f(T) < \infty$, and $b_n^{-1} f_n(t) \xrightarrow{P} f(t)$ as $n \to \infty$ for all continuity points of f, for $f_n = H_n^1, H_n^2, H_n^{1\cap 2}, H_n^{1*2}$* and *$f = H^1, H^2, H^{1\cap 2}, H^{1*2}$*, in turn.

Notice that $(\widetilde{C2})$ is stated in light of the observation times alone, which would make the condition more convenient than **(C2)**. Evidently, $(\widetilde{C2})$ is more stringent than **(C2)** – the former requires (local) convergence of the four functions, regarded as stochastic processes. Its direct implication is that a stronger conclusion than Theorem 5 can be drawn. As its expense we need additionally to impose the continuity condition on the volatility and correlation functions as follows.

Condition (C5): $\sigma^l, l = 1, 2$, *and ρ are continuous in t.*

Theorem 6 *Under the Conditions (C1)$(\widetilde{C2})$(C3)(C4)(C5), as $n \to \infty$,*

$$b_n^{-1/2}\left(U_n - \theta\right) \xrightarrow{\mathcal{L}} N(0, \mathbb{C}),$$

where the 3×3 matrix $\mathbb{C} := \left(\mathbb{C}^{(l,k)}\right)_{0\le l,k\le 2}$ comprises

$$\mathbb{C}^{(0,0)} := \int_0^T \left(\sigma_t^1\sigma_t^2\right)^2 dH^{1*2}(t) + \int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 d(H^1 + H^2 - H^{1\cap 2})(t),\ \mathbb{C}^{(l,l)} := 2\int_0^T \left(\sigma_t^l\right)^4 dH^l(t),\ l = 1, 2,$$
$$\mathbb{C}^{(l,0)} := \mathbb{C}^{(0,l)} := 2\int_0^T \left(\sigma_t^l\right)^2 \left(\sigma_t^1\sigma_t^2\rho_t\right) dH^l(t),\ l = 1, 2,\ \mathbb{C}^{(2,1)} := \mathbb{C}^{(1,2)} := 2\int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 dH^{1\cap 2}(t). \tag{2.6}$$

Therefore, in practice to invoke Theorem 6 the major task is to identify the limiting functions H^1, H^2, $H^{1\cap 2}$, and H^{1*2}.

2.4 Correlation estimation

Suppose further that $\rho_t \equiv \rho$ and $\sigma_t^l \equiv \sigma^l$ for some constant, $\rho \in (-1,1)$ and $\sigma^l > 0$, $l = 1,2$. We are now interested in estimating the correlation ρ of the two Brownian motions W^1 and W^2. Let us recall the correlation estimators we have proposed. $R_n^{(1)} := \frac{1}{T}\sum_{i,j} \frac{\Delta P^1(I^i)\Delta P^2(J^j)}{\sigma^1\sigma^2} K_{ij}$ when σ^l are known, and

$$R_n^{(2)} := \frac{\sum_{i,j} \Delta P^1(I^i)\Delta P^2(J^j)1_{\{I^i\cap J^j\neq\emptyset\}}}{\{\sum_i \Delta P^1(I^i)^2\}^{1/2}\{\sum_j \Delta P^2(J^j)^2\}^{1/2}} \equiv \frac{U_n^{(0)}}{\sqrt{U_n^{(1)}}\sqrt{U_n^{(2)}}}$$

when either σ^l are unknown or known. The asymptotic distribution of $R_n^{(1)}$ is immediately found by standardizing $U_n^{(0)}$ with the (integrated) volatilities $\sigma^1\sqrt{T}$ and $\sigma^2\sqrt{T}$. Regarding that of $R_n^{(2)}$, we can simply apply the standard delta-method (multi-dimensional). That is,

Theorem 7 ([7]) *Under the Conditions (C1), (C2), (C3) and (C4), as $n \to \infty$,*

$$b_n^{-1/2}\left(R_n^{(k)} - \rho\right) \xrightarrow{\mathcal{L}} N\left(0, c_\rho^{(k)}\right),\ k = 1,2,$$

where

$$c_\rho^{(1)} := \frac{\mathbb{C}^{(0,0)}}{(\sigma^1\sigma^2)^2 T^2},\ c_\rho^{(2)} := \frac{1}{T^2}\left\{\mathbb{C}^{(0,0)}\frac{1}{(\sigma^1\sigma^2)^2} + \mathbb{C}^{(1,1)}\frac{\rho^2}{4(\sigma^1)^4} + \mathbb{C}^{(2,2)}\frac{\rho^2}{4(\sigma^2)^4}\right.$$
$$\left. -\mathbb{C}^{(1,0)}\frac{\rho}{(\sigma^1)^3\sigma^2} - \mathbb{C}^{(2,0)}\frac{\rho}{\sigma^1(\sigma^2)^3} + \mathbb{C}^{(2,1)}\frac{\rho^2}{2(\sigma^1\sigma^2)^2}\right\}.$$

Remark: Suppose now that each diffusion process has *feedback* effect in its diffusion coefficient from P^l, $\sigma^l(t) \equiv \sigma^l(t, P_t^l)$ for a known, Borel-measurable function such that

$$P\left[\int_0^T |\mu^l(s)|ds + \int_0^T \sigma^l(s)^2 ds < \infty\right] = 1, \quad l = 1,2.$$

We assume that $\sigma^l(t,x) > 0$, $\forall(t,x) \in [0,T]\times\mathbb{E}$, with an open subset $\mathbb{E}$ of $\mathbb{R}$ on which the process $P_\cdot^l$ takes its values, and that $\sigma^l(t,x)$ is of $C^{1,2}([0,\infty)\times\mathbb{E})$, $l = 1,2$. Since $\Delta P^l(I^i) \simeq \sigma^l(T^{l,i-1})\Delta W^l(I^i)$, by the "pre-whitening" $\frac{\Delta P^l(I^i)}{\sigma^l(T^{l,i-1})}$, one may expect to extract approximately the variation of W^l over I^i, which leads to an estimator of a similar form to $R^{(1)}$. Note that, since $\sigma^l(t,x)$ are known functions, $\sigma^l(T^{l,i})$ are to be observed for every i; hence we can define as a statistic

$$R_n^{(3)} := \frac{1}{T}\sum_{i,j} \frac{\Delta P^1(I^i)\Delta P^2(J^j)}{\sigma^1(T^{1,i-1})\sigma^2(T^{2,j-1})}1_{\{I^i\cap J^j\neq\emptyset\}}. \tag{2.7}$$

Consistency of $R_n^{(3)}$ is shown, for instance, by direct application of Corollary 2.3 of [5]. However, the limiting distribution of this estimator has yet to be found.

3 Special cases

3.1 Perfectly synchronous sampling

Suppose *synchronous* and *equidistant* sampling, $I^i \equiv J^i$, $|I^i| \equiv \frac{T}{n}$.

Corollary 8 ([6], [7]) *Under (C1), (C4) and (C5), U_n is asymptotically normal with mean 0 and variance*

$$\mathbb{C} := T\begin{bmatrix} \int_0^T \left(\sigma_t^1\sigma_t^2\right)^2(1+\rho_t^2)dt & & \\ 2\int_0^T \left(\sigma_t^1\right)^2(\sigma_t^1\sigma_t^2\rho_t)dt & 2\int_0^T \left(\sigma_t^1\right)^4 dt & \\ 2\int_0^T \left(\sigma_t^2\right)^2(\sigma_t^1\sigma_t^2\rho_t)dt & 2\int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 dt & 2\int_0^T \left(\sigma_t^2\right)^4 dt \end{bmatrix}.$$

This result has been indeed known in the literature (e.g., [9], [2]); i.e., $\mathbb{C}$ is nothing more than the asymptotic variance-covariance matrix of the realized volatilities and covariance.

Moreover, regarding the asymptotic distributions of the correlation estimators, under the assumptions that $\sigma_t^l \equiv \sigma^l > 0$ and $\rho_t \equiv \rho$, Theorem 7 implies that:

Corollary 9 ([7]) *Under the Conditions (C1) and (C4), as $n \to \infty$,*

$$\sqrt{n}\left(R_n^{(k)} - \rho\right) \xrightarrow{\mathcal{L}} N\left(0, c_\rho^{(k)}\right),\ k = 1, 2, \tag{3.1}$$

where $c_\rho^{(1)} := (1+\rho^2)$ and $c_\rho^{(2)} := (1-\rho^2)^2$.

3.2 Nonsynchronous alternating sampling at odd/even times

We now consider the following deterministic, regularly spaced sampling scheme. P^1 is sampled at 'odd' times, i.e., $t = \frac{2k-1}{2n}T$, $k = 1, 2, ..., n$, while P^2 is at 'even' times, $t = \frac{2k}{2n}T$ (Note that $\#\left(\Pi^l\right) \simeq n$). Also, we maintain the assumption that the two processes are observed together at $t = 0$ and T just for convenience, which is not essential to the argument. Hence, P^1 and P^2 are sampled in a *nonsynchronous, alternating* way. Note that the sampling scheme consists only of 'incomplete' pairs (except at the end points, $t = 0, T$) in the sense used in the missing data literature.

Corollary 10 ([7]) *Under (C1), (C4) and (C5), U_n is asymptotically normal with mean 0 and variance*

$$\mathbb{C} := T\begin{bmatrix} \int_0^T \left(\sigma_t^1\sigma_t^2\right)^2(2+\frac{3}{2}\rho_t^2)dt & & \\ 2\int_0^T \left(\sigma_t^1\right)^2(\sigma_t^1\sigma_t^2\rho_t)dt & 2\int_0^T \left(\sigma_t^1\right)^4 dt & \\ 2\int_0^T \left(\sigma_t^2\right)^2(\sigma_t^1\sigma_t^2\rho_t)dt & \int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 dt & 2\int_0^T \left(\sigma_t^2\right)^4 dt \end{bmatrix}.$$

Remark: In case the two processes are identical ($\rho_t \equiv 1$), the sub-matrix $\left(\mathbb{C}^{(k,l)}\right)_{1\le k,l\le 2}$ is equivalent to the asymptotic variance-covariance matrix of two realized volatilities $U^{(l)}$, $l = 1, 2$, based on 'sub-samples' of the univariate process, taken at sub-grids $\mathcal{G}^{(1)}$ (consisting of odd times), say, and another $\mathcal{G}^{(2)}$ (even times), respectively. See [11].

Regarding the asymptotic distributions of the correlation estimators, under the assumptions that $\sigma_t^l \equiv \sigma^l > 0$ and $\rho_t \equiv \rho$, Theorem 7 implies that

Corollary 11 ([7]) *Under the Conditions (C1) and (C4), as $n \to \infty$,*

$$\sqrt{n}\left(R_n^{(k)} - \rho\right) \xrightarrow{\mathcal{L}} N\left(0, c_\rho^{(k)}\right),\ k = 1, 2,$$

where $c_\rho^{(1)} := 2(4+3\rho^2)$ and $c_\rho^{(2)} := 2\left\{(1-\rho^2)^2 + (3-\rho^2)\right\}$.

Remark: In both cases, since $c_\rho^{(2)} \le c_\rho^{(1)}$ ($c_\rho^{(2)} = c_\rho^{(1)}$ if and only if $\rho = 0$), $R_n^{(2)}$ is always (asymptotically) more efficient than $R_n^{(1)}$ even in the case when σ^l are known. Its practical implication is that, *even if σ^l were known, it would probably be better to use $R_n^{(2)}$*. It seems reasonable; because the realized volatilities, $U_n^{(1)}$ and $U_n^{(2)}$, are generally correlated with the covariance estimator $U_n^{(0)}$, the division by $\sqrt{U_n^{(1)}U_n^{(2)}}$ can attenuate the variation of $U_n^{(0)}$.

3.3 Poisson sampling

Consider the case of Poisson arrival time sampling with $\lambda^1 := np^1$ and $\lambda^2 := np^2$, for $p^1 \in (0,\infty)$, $p^2 \in (0,\infty)$, $n \in \mathbb{N}$. Let $\Pi^1 := (I^i)_{i=1,2,\ldots}$ and $\Pi^2 := (J^i)_{i=1,2,\ldots}$ be the corresponding inter-arrival intervals, where $I^i := (\widetilde{T}^{1,i-1}, \widetilde{T}^{1,i}] \cap (0,T]$ and $J^i := (\widetilde{T}^{2,i-1}, \widetilde{T}^{2,i}] \cap (0,T]$. Here $\widetilde{T}^{l,i}$ represent the ith arrival times of the lth Poisson process, $l = 1,2$, with $(\widetilde{T}^{1,i})$ and $(\widetilde{T}^{2,i})$ independent. We assume that P^1 and P^2 are observed at $t = 0$ for simplicity. Accordingly, each I^i (resp. J^i) represents the i-th sampling interval of P^1 (resp. P^2).

Corollary 12 ([6], [7]) *Under (C1)(C4)(C5), U_n is asymptotically normal with mean θ and variance $\mathbb{C} := \left(\mathbb{C}^{(l,k)}\right)_{0\le l,k\le 2}$, where*

$$\begin{aligned}
\mathbb{C}^{(0,0)} &:= \left(\frac{2}{p^1} + \frac{2}{p^2}\right)\int_0^T \left(\sigma_t^1\sigma_t^2\right)^2 dt + \left(\frac{2}{p^1} + \frac{2}{p^2} - \frac{2}{p^1+p^2}\right)\int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 dt,\\
\mathbb{C}^{(l,l)} &:= \frac{4}{p^l}\int_0^T \left(\sigma_t^l\right)^4 dt,\ l = 1,2,\\
\mathbb{C}^{(l,0)} &:= \frac{4}{p^l}\int_0^T \left(\sigma_t^l\right)^2\left(\sigma_t^1\sigma_t^2\rho_t\right) dt,\ l = 1,2,\ \mathbb{C}^{(2,1)} := \frac{4}{p^1+p^2}\int_0^T \left(\sigma_t^1\sigma_t^2\rho_t\right)^2 dt.
\end{aligned}$$

For the asymptotic distributions of the correlation estimators, under the assumptions that $\sigma_t^l \equiv \sigma^l > 0$ and $\rho_t \equiv \rho$, we have the following result.

Corollary 13 ([7]) *Under the Conditions (C1)(C4), as $n \to \infty$,*

$$\sqrt{n}\left(R_n^{(k)} - \rho\right) \xrightarrow{\mathcal{L}} N\left(0, c_\rho^{(k)}\right),\ k = 1,2,$$

where

$$c_\rho^{(1)} := \frac{2}{T}\left\{\left(\frac{1}{p^1} + \frac{1}{p^2}\right) + \left(\frac{1}{p^1} + \frac{1}{p^2} - \frac{1}{p^1+p^2}\right)\rho^2\right\}.$$

It can be shown in all the three cases that $c_\rho^{(2)} \le c_\rho^{(1)}$ ($c_\rho^{(2)} = c_\rho^{(1)}$ if and only if $\rho = 0$); therefore, even when σ^l are known, it is more desirable to use $R_n^{(2)}$. It seems reasonable; because the realized volatilities $U_n^{(1)}$ and $U_n^{(2)}$(appearing in the denominator of $R_n^{(1)}$) are generally correlated with the numerator $U_n^{(0)}$, the division by $\sqrt{U_n^{(1)}U_n^{(2)}}$ can attenuate the variation of $U_n^{(0)}$.

4 Concluding remarks

We presented an estimation procedure for the covariance/correlation of two diffusion-type processes when they are observed only at discrete times in a nonsynchronous manner. Consistency and asymptotic normality of the proposed estimators were discussed.

[5] extended [8] to a general case where underlying processes are *continuous semimartingales* and observation times are *stopping times* and showed consistency of the estimators is preserved. It is worth pursuing asymptotic distributions under such a situation. Consideration of jumps will also be a rewarding research project. Since the theory is at the 'hatchling' stage, there are lots to be done.

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