数理解析研究所講究録
1488 巻 2006 年 150-155

THE DAWN OF MODERN THEORY OF GAMES - SUMMARY

Mikio Nakayama
Department of Economics
Keio University
2-15-45 Mita, Tokyo 108-8345 Japan
E-mail: nakayama@econ.keio.ac.jp

Abstract

The modem theory of games initiated by John von Neumann with the minimax theorem in 1928 has now grown to be an indispensable analytical framework for social sciences, and economics in particular. In this paper, we shall review the early history of game theory from von Neumann to John F. Nash, the founder of the noncooperative game theory, including Émile Borel, Hugo Steinhaus and Oskar Morgenstern, thereby pointing out a hint of why game theory has come to be widely applied in economics.

Introduction. One of the leading game theorists, Robert J. Aumann, has won the Nobel prize in economics last year (2005), mainly for his earlier contribution to market equilibrium through the cooperative game approach [1]. Also, in 1994, the Nobel prize has been awarded to three game theorists, John F. Nash, John C. Harsanyi and Reinhard Selten for the foundation and development of the theory of noncooperative games in the perspective of economic theory. In view of these facts, it seems worthwhile to consider what the theory of games, cooperative and noncooperative, is all about. As a first step, we shall review in this paper how a mathematical theory of games was born in the first half of 20th century.

Around the Minimax Theorem. We will first take up John von Neumann's minimax theorem [28] in 1928. We will see this theorem is a first monumental result in his attempt to formalize social behavior of rational agents in the spirit of formalist approach, just like the work *Mathematische Grundlagen der Quantenmechanik* [29] was an attempt to formalize the new physics at that time. The minimax theorem solves the problem in the two-person case completely.

A famous mathematician, Émile Borel, had studied a two-person game [4], earlier than von Neumann's minimax theorem. He considered an extended form of the game *Paper, Stone, Scissors*, introducing the concept of mixed strategies and the *iterated eliminaion of weakly dominated strategies*, but without minimax theorem. Maurice Fréchet therefore asserted later that Borel should be the initiator of game theory [9, 1953]. Hugo Steinhaus had also studied two-person games slightly before the minimax theorem [39]. His two-person game was one that describes a pursuit of an escaping ship with the conflicting objective of duration of pursuit. He is also interested in the game of fair division [40], and noted a solution due to B.Knaster and S.Banach extending the *divide and choose* method.

We will see that these famous mathematicians did not anticipate the equality expressed in the minimax theorem. This is also the main point in the negative response of von Neumann to the assertion of Fréchet [9, 1953].

Expanding Economy Model and Kakutani Fixed Point Theorem. In 1932, von Neumann contributed to economic theory with *A Model of General Equilibrium* [30]. This model describes the dual relation between the expansion of commodity and value in a linear system. Our main concern will be the lemma formulated to proved the existence of the economic equilibrium, which, as the author noted, can be used as a topological, alternative proof of the minimax theorem. This lemma is soon restated by Sizuo Kakutani as a generalized fixed point theorem [11].

Minimax theorem and the model of expanding economy have become a source of new analytical tools in modern economics after the World War II based on the convexity in the structure of economic models.

Cooperative Game Theory. We next turn to the theory of cooperative games initiated by von Neumann and an economist, Oskar Morgenstern. In the book [31], von Neumann and Morgenstern presented an elementary proof of the minimax theorem, and developed the theory of n-person cooperative games based on the minimax theorem. The essential ingredient when considering n-person situations is, to them, the possibility of forming coalitions. Cooperation is described by a coalition S of players which is a subset of a finite set N of players, and any coalition S is assumed to play a two-person zero-sum game against its complementary coalition $N \setminus S$, thereby obtaining the minimax value describing the *worth* of the coalition. Denoting by $v : 2^N \to \Re$ a real valued function that associates any coalition S with its value $v(S)$, the n-person cooperative game is given by the pair (N, v), in which players try to seek compromise or agreements as to the payoffs in the game, that is, how to divide the total proceeds $v(N)$ obtainable through the cooperation of all players.

As a solution concept, they defined a *stable set*, which is a set K of all payoff n-vectors that are not *dominated* by any payoff n-vector in the set K. Here, payoff n-vector x dominates another payoff n-vector y if for some coalition S, x and y are both obtainable and x gives more than y does to each member of coalition S. They interpreted a stable set as a *standard of behavior* or *social convention* that is logically consistent with rational behavior. This is a solution concept quite unfamiliar in its form in the history of mathematics.

Applying the solution to a three-person game, they discovered that two players colluding and discriminating the remaining player in the payoff distribution can be a stable social convention (a *discriminatory solution*). Also, in a trade of an indivisible commodity with one seller and two buyers, two buyers colluding to lower the price can be a stable standard of behavior.

Noncooperative Game Theory. By May of 1950, there appeared a theory of games with n players acting independently; that is, the theory of *noncooperative games* submitted by John F. Nash to Princeton University as a doctoral thesis. In this theory, players do not form coalitions, and the payoff to each player is explicitly dependent of strategies of all the players. The solution of the game is a profile of strategies of all players such that the payoff to any player does not increase by a unilateral change of the strategy. The solution, now called the *Nash equilibrium*, is not only a generalization of the solution to the two-person zero-sum game, but also has become an indispensable tool in modern economic analyses.

We will first review the works *The Bargaining Problem* [23, 1950], *Noncoopertive Games* [25, 1951], and *Two-Person Cooperative Games* [26, 1953]. The first one presents the well known solution "Nash product maximization" to a problem of two-person bargaining via the *axiomatic approach*, just like in the manner that von Neumann and Morgenstern treated utility theory in [31]. This is the first solution to the problem of bilateral monopoly studied unsuccessfully by famous economists such as Hicks and Edgeworth.

The theory of *noncooperative games* was introduced in the doctoral thesis and published as one-page paper in *Proceedings of the National Academy of Sciences* [24, 1950], and later in *Annals of Mathematics* [25, 1951]. Existence of equilibrium points was proved in the former [24, 1950] via Kakutani's fixed point theorem, and by Brouwer's fixed point theorem in the latter which is much more elegant than the former.

The two-person cooperative game is a reformulation of the Bargaining Problem into a strategic form, generating the Nash product maximization as a unique equilibrium point in the game, which also generalizes the minimax theorem to non zero-sum (but strictly competitive) games. At the same time , this game is an excellent illustration of the analysis according to the "Nash Program," which is proposed in the paper as a methodology of game analysis that cooperation should be described as an equilibrium behavior.

Bounded Rationality and Evolutionary Interpretations. Nash gave a "mass-action" interpretation of how to attain an equilibrium point in the PhD thesis, which was deleted in the publication [25, 1951]. The essential idea is what we now know as the *fictitious play* by boundedly rational players. Another bounded-rationality related idea can be seen in the mapping constructed to prove the existence of equilibrium points by the Brouwer fixed point theorem. The mapping that associates a new mixed strategy with a current mixed strategy can be interpreted as a static version of what we now know as the *replicator dynamic* in evolutionary game theory.

Still, Nash gave bounded-rationality related comments on the result of an experiment conducted by Melvin Dresher and Merrill Flood in the early 1950s (due to A.Roth's paper [36]). The game devised for the experiment is what we now know as the *Prisoner's Dilemma*, and 100 times repetition seemed not to support the Nash equilibrium. The main point of Nash's comment is that the repetition makes the game different from the one-shot play, and the *backward induction* would be hard to be operated by ordinary people so that the result would be better approximated by the repeated-game strategy that is today called the *grim trigger strategy*. Moreover, Nash proposed that the experiment be conducted under the mutual ignorance of opponent's actions and players be randomly matched in each round to observe even more interesting behavior. Such a kind of experiment was in fact conducted thirty years or more later by Robert Axelrod [3, 1984] with players being computer programms playing the Prisoner's Dilemma.

Concluding Remarks. We conclude with a remark that while Nash's noncooperative game theory has been well embedded in modern economic theory, von Neumann's 'program' to formalize social sciences appears to have been only partially fulfilled. We may therefore wish to expect the Robert Aumann's winning the Nobel prize to be the beginning of formalization by the full game theory; that is, by cooperative game theory as well as noncooperative game theory.

References

[1] Aumann, R.J.: Markets with a continuum of traders. Econometrica **32**,39-50 (1964)

[2] Aumann, R.J.: Lectures on Game Theory. Westview Press 1989

[3] Axelrod, R.: The Evolution of Cooperation. Basic Books 1984

[4] Borel, É.: La théorie du jeu et les équations intégrales à noyau symétrique. Comptes Rendus Hebdomadaires des Séances de l'Academie des Sciences **173**, 1304–8 (1921). English Translation (Savage, L.J.) : The Theory of Play and Integral Equations with Skew Symmetric Kernels. Econometrica **21**, 97–100 (1953)

[5] Champernowne, D.G.: A note on J.von Neumann's article. Review of Economic Studies **13**, 10–18 (1945-6)

[6] Dantzig, G.B.: Programming of interdependent ictivities II. mathematical model. Econometrica **17**, 200–11 (1951)

[7] Dantzig, G.B.: Constructive proof of the mini-max theorem. Pacific Journal of Mathematics **6**, 25-33 (1956)

[8] Dorfman, R., Samuelson, P.A., Solow, R.M.: Linear Programming and Economic Analysis. McGraw-Hill 1958

[9] Fréchet, M.: Émile Borel, Initiator of the theory of psychological games and its application. Econometrica **21**, 95–99, 118–24 (1953)

[10] Gamow, G. and Stern, M.: Puzzle-Math. The Viking Press, USA, 1958. Japanese Translation (Yura, T.): Kazu ha Majutsu-shi. Hakuyosha, 1958

[11] Kakutani, S.: A Generalization of Brouwer's fixed point theorem. Duke Mathematical Journal **8**, 457–9 (1941)

[12] Kemeny, J.G., Morgenstern, O., Thompson, G.L.: A generalization of von Neumann's model of an expanding economy. Econometrica **24**, 115–35 (1956)

[13] Kuhn, H.W.: On games of fair division. In : Shubik, M. (ed): Essays in Mathematical Economics in Honor of Oskar Morgenstern. Princeton University Press 1967

[14] Kuhn, H.W., Nasar, S.: The Essential John Nash. Princeton University Press 2002

[15] Leonard, R.J.: Creating a context for game theory. In: Weintraub, E.R. (ed): Toward a History of Game Theory. Duke University Press 1992

[16] Loomis, L.H.: On a theorem of von Neumann. Proceedings of the National Academy of Sciences **32**, 213-215 (1946)

[17] Lucas, W.F.: The proof that a game may not have a solution. Transactions of the American Mathematical Society **137**, 219-229 (1969)

[18] Mirowski, P.: What were von Neumann and Morgenstern trying to accomplish?. In: Weintraub, E.R. (ed.): Toward a History of Game Theory . Duke University Press, 1992
[19] Myerson, R.B.: Nash equilibrium and the history of economic theory. Journal of Economic Literature **37**, 1067-1082 (1999)
[20] Morgenstern, O.: Wirtschaftsprognose: Eine Untersuchung ihrer Voraussetzungen und Möglichkeiten. Julius Springer, Wien 1928
[21] Morgenstern, O.: The collaboration between Oskar Morgenstern and John von Neumann on the theory of games. Journal of Economic Literature **14**, 805-816 (1976)
[22] Morgenstern, O., Thompson, G.L.: Mathematical Theory of Expanding and Contracting Economies. Heath-Lexington, Boston 1976
[23] Nash, J.F.: The Bargaining Problem. Econometrica **18**, 155-162 (1950)
[24] Nash, J.F.: Equilibrium points in n-person games. Proceedings of the National Academy of Sciences **36**, 48-9 (1950)
[25] Nash, J.F.: Non-cooperative games. Annals of Mathematics **54(2)**, 286-295 (1951)
[26] Nash, J.F.: Two-Person Cooperative Games. Econometrica **21**, 128-40 (1953)
[27] Nash, J.F.: Esseys on Game Theory (introduced by K. Binmore). Edward Elgar, UK 1996
[28] von Neumann, J.: Zur theorie der gesellschaftsspiele. Mathematische Annalen **100**, 295-320 (1928). English translation: In: Tucker, A.W et al. (eds.): Contributions to the Theory of Games IV . Annals of Mathematics Studies **40**, 1959
[29] von Neumann, J.: Mathematische Grundlagen der Quantenmechanik. Springer-Verlag, Berlin 1932
[30] von Neumann, J.: "Über ein ökonomisches gleichungssystem und ein verallgemeinerung des Brouwerschen fixpunktsatzes. Ergebnisse eines Mathematischen Kolloquiums **8**, 1937. English translation: A model of general equilibrium. Review of Economic Studies **13**, 1–9 (1945)
[31] von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, 1944
[32] von Neumann, J.: Communications on the Borel Notes. Econometrica **21**, 124–25 (1953)
[33] Punzo, L.F.: Von Neumann and Karl Menger's mathematical colloquium. In : Dore, M. et al. (eds.): John von Neumann and Modern Economics. Oxford University Press, 1989
[34] Robinson, J.: An iterative method of solving a game. Annals of Mathematics **54**, 296-301 (1951)
[35] Samuelson, P.A.: A revisionist view of von Neumann's growth model. In : Dore, M. et al. (eds.): John von Neumann and Modern Economics. Oxford University Press, 1989
[36] Roth, A.E.: The early history of experimental economics. Journal of the History of Economic Thought **15**, 184-209 (1993)
[37] Schwalbe, U., P.Walker, P.: Zermelo and the early history of game theory. Games and Economic Behavior **34** 123–137 (2001)
[38] Shubik, M.: Game theory at Princeton, 1949-1955: a personal peminiscence. In : Weintraub, E.R. (ed): Toward a History of Game Theory. Duke University Press, 1992
[39] Steinhaus, H.: Definitions for a theory of games and pursuit. Mysl Akademicka **1**, 13–14 (1925). English translation (H.Kuhn) : In : Naval Research Logistics Quarterly **7.2**, 105–8 (1959)
[40] Steinhaus, H.: Sur la division pragmatique. Econometrica **17** (supplement), 315–319 (1949)
[41] Stdeinhaus, H.: Mathematical Snapshots, 2nd edition. New York, 1960
[42] Suzuki, M.: Introduction to Game Theory (in Japanese). Kyouritsu Shuppan, 1981

[43] Thompson, G.: Jonn von Neumann's contributions to mathematical programming economics. In : Dore, M. et al. (eds.): John von Neumann and Modern Economics. Clarendon Press, 1989

[44] Ville, J.: Sur la théorie générale des jeux où intervient l'habilité des joueurs. In : Borel, É. et al. (eds): Traité du Calcul des Probabilités et ses Applications Volume IV, Paris:Gautier-Villars, 105-113 (1938)

[45] Wald, A.: Statistical Decision Functions, John Wiley & Sons, New York, 1950.

[46] Zermelo, E.: Uber eine anwendung der mengenlehre auf die theorie des schachspiels. Proceedings of the Fifth International Congress of Mathematicians **2**, 501-4 (1913)