数理解析研究所講究録
1488 巻 2006 年 60-76

Representation of Convex Preferences in a Measure Space: Pareto Optimality and Core in Cake Division*

Nobusumi Sagara† (佐柄 信純)
Faculty of Economics, Hosei University
(法政大学経済学部)
e-mail: nsagara@mt.tama.hosei.ac.jp

Milan Vlach (ミラン・ブラッハ)
Kyoto College of Graduate Studies for Informatics
(京都情報大学院大学)
e-mail: m_vlach@kcg.ac.jp

February 15, 2006

1 Introduction

Convexity plays a crucial role in proving the existence of various equilibria in cooperative and noncooperative game theories. While convex analysis on vector spaces has brought a plenty of fruitful results to optimization theory and its application to economics and game theory, it is apparent that standard convex analysis is inadequate to deal with topological spaces which lack a vector space structure. In particular, not enough investigation has been made concerning convexity in σ-fields of measure spaces.

In this paper we propose a convex-like structure in a nonatomic finite measure space. We first introduce convex combinations of measurable sets,

---

*This is a condensed version of the paper with the same title. The full paper is available upon request. This research is a part of the "International Research Project on Aging (Japan, China and Korea)" at Hosei Institute on Aging, Hosei University, supported by Special Assistance of the Ministry of Education, Culture, Sports, Science and Technology.

†Corresponding author.

and quasi-concave and concave functions on a Borel σ-field and prove Jensen's inequalities, which conform with the standard definitions results in convex analysis. We then introduce the convexity of preference relations on the Borel σ-field and show that a utility function representing the convex preference relation is quasi-concave on the Borel σ-field. While our attention is focused on a nonatomic finite measure space with the Borel σ-field of a topological space, the proposed structure and its basic properties can easily be extended to an arbitrary nonatomic finite measure space.

Having concepts and basic results analogous to those of standard convex analysis, we apply them, together with our previous results from Sagara and Vlach (2006) on topologizing a Borel σ-field and the representation of preference relations on the Borel σ-field by a continuous utility function, to the problems of cake division among a finite number of individuals. In particular, we are concerned with the existence of Pareto optimal partitions, and the existence of core partitions with non-transferable utility (NTU) and transferable utility (TU) games arising in a pure exchange economy in which each individual is endowed with an initial "piece" of the cake. We also provide conditions guaranteeing that every weakly Pareto optimal partition is a solution to the problem of maximizing a weighted sum of individual utilities. Especially, in contrast to Berliant (1985) and Berliant and Dunz (2004), we present a direct proof of the existence of core partitions for the NTU case without introducing any price system.

When preference relations of each individual are represented by non-atomic probability measures, it is relatively simple to show the existence of Pareto optimal partitions and the existence of core partitions with TU by a direct application of Lyapunov's convexity theorem which ensures that the utility possibility set is convex and compact (see Barbanel and Zwicker 1997, Dubins and Spanier 1961, Legut 1986 and Sagara 2006). However, representing a preference relation by a probability measure means that the corresponding utility function is countably additive on the σ-field, and consequently assumes a constant marginal utility. This is obviously a severe restriction on the preference relation that is difficult to justify from an economics viewpoint.

The main purpose of this paper is to obtain the existence result without imposing any additivity requirements on preference relations. Instead, the continuity and convexity of preference relations of each individual play a significant role in guaranteeing the convexity and compactness of the utility possibility set.

2 Convexity in a Measure Space

In this section we propose a new concept of the convexity in a nonatomic finite measure space. We introduce convex combinations of measurable sets, concave and quasi-concave functions on a Borel σ-field in conformity with the standard convex analysis. Although we restrict our attention to a nonatomic finite measure space with the Borel σ-field, all results in this section are valid for any nonatomic finite measure space.

2.1 Convex Combination of Measurable Sets

Let $(\Omega, \mathscr{B}_\Omega, \mu)$ be a finite measure space with Ω a topological space and $\mathscr{B}_\Omega$ the Borel σ-field of Ω. An element $A \in \mathscr{B}_\Omega$ is an *atom* of a measure μ if $\mu(A) > 0$ and for any measurable subset B of A, either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. If μ has no atoms, then μ is called *nonatomic.*

Let μ be a nonatomic measure on $\mathscr{B}_\Omega$. By Lyapunov's convexity theorem, μ has the convex range in $\mathbb{R}$. Therefore, for any $t \in [0, \mu(\Omega)]$ there exists some $A \in \mathscr{B}_\Omega$ satisfying $\mu(A) = t$. Especially, for any $A \in \mathscr{B}_\Omega$ and $t \in [0, \mu(A)]$ there exists a measurable subset E of A satisfying $\mu(E) = t$.

Let $A \in \mathscr{B}_\Omega$ and $t \in [0,1]$ be given arbitrarily. We define the family $\langle tA \rangle$ of subsets of A by

$$\langle tA \rangle = \{E \in \mathscr{B}_\Omega \mid \mu(E) = t\mu(A),\ E \subset A\}.$$

In view of the nonatomicity of μ, it follows that $\langle tA \rangle$ is nonempty for any $A \in \mathscr{B}_\Omega$ and $t \in [0,1]$. Note that $E \in \langle tA \rangle$ if and only if $A \setminus E \in \langle (1-t)A \rangle$, and $\mu(A) = 0$ if and only if $\langle tA \rangle$ contains the empty set for any $t \in [0,1]$.

Theorem 2.1. *For every element A and B in $\mathscr{B}_\Omega$ and any $t \in [0,1]$ there exist disjoint elements $E \in \langle tA \rangle$ and $F \in \langle (1-t)B \rangle$.*

Theorem 2.1 guarantees that for every element A and B in $\mathscr{B}_\Omega$ and any $t \in [0,1]$ there exists some $C \in \mathscr{B}_\Omega$ such that C is a union of disjoint sets E and F satisfying $E \in \langle tA \rangle$ and $F \in \langle (1-t)B \rangle$. The family of all such elements C is denoted by $\mathscr{D}_t(A, B)$.

Let Δ^{n-1} denote the $(n-1)$-dimensional unit simplex in $\mathbb{R}^n$; that is,

$$\Delta^{n-1} = \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i \geq 0,\ i = 1, \dots, n \right\}.$$

The following result is an obvious extension of Theorem 2.1.

Theorem 2.2. *For every finite collection of elements $A_1, \ldots, A_n$ in $\mathscr{B}_\Omega$ and any $(t_1, \ldots, t_n) \in \Delta^{n-1}$, there exist disjoint elements $E_1 \in \langle t_1 A_1 \rangle, \ldots, E_n \in \langle t_n A_n \rangle$.*

Theorem 2.2 guarantees that for every finite collection of elements $A_1, \ldots, A_n$ in $\mathscr{B}_\Omega$ and any $(t_1, \ldots, t_n) \in \Delta^{n-1}$ there exists some E in $\mathscr{B}_\Omega$ such that E is a union of disjoint sets $E_1, \ldots, E_n$ satisfying $E_i \in \langle t_i A_i \rangle$ for each $i = 1, \ldots, n$. The family of all such elements E is denoted by $\mathscr{D}_{t_1,\ldots,t_n}(A_1, \ldots, A_n)$. When $n = 2$, we adhere to using $\mathscr{D}_t(A, B)$ instead of $\mathscr{D}_{t,1-t}(A, B)$.

By a *partition* we always mean an ordered finite collection of disjoint elements in $\mathscr{B}_\Omega$ whose union is Ω. A partition is called an *n-partition* if the number of its members is n.

Theorem 2.3. *Let $(X_1, \ldots, X_m)$ be an m-partition. For every finite collection of n-partitions $(A_1^1, \ldots, A_n^1), \ldots, (A_1^l, \ldots, A_n^l)$ and any $(t_1, \ldots, t_l) \in \Delta^{l-1}$ there exists some $A_{ij} \in \mathscr{D}_{t_1,\ldots,t_l}(A_i^1 \cap X_j, \ldots, A_i^l \cap X_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ such that $(\bigcup_{j=1}^m A_{1j}, \ldots, \bigcup_{j=1}^m A_{nj})$ is an n-partition satisfying $\bigcup_{j=1}^m A_{ij} \in \mathscr{D}_{t_1,\ldots,t_l}(A_i^1, \ldots, A_i^l)$ for each $i = 1, \ldots, n$.*

Corollary 2.1. *Let $(X_1, \ldots, X_m)$ be an m-partition. For every pair of n-partitions $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ and any $t \in [0, 1]$ there exists some $C_{ij} \in \mathscr{D}_t(A_i \cap X_j, B_i \cap X_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ such that $(\bigcup_{j=1}^m C_{1j}, \ldots, \bigcup_{j=1}^m C_{nj})$ is an n-partition satisfying $\bigcup_{j=1}^m C_{ij} \in \mathscr{D}_t(A_i, B_i)$ for each $i = 1, \ldots, n$.*

2.2 Concave Functions on a Borel σ-Field

Let $A \triangle B = (A \cup B) \setminus (A \cap B)$ be the symmetric difference of A and B.

The following definitions of the (strict) μ-quasi-concavity and (strict) μ-concavity of functions on $\mathscr{B}_\Omega$ are analogues of the standard definitions in convex analysis.

Definition 2.1. A function f on $\mathscr{B}_\Omega$ is:

(i) *μ-quasi-concave* if $A, B \in \mathscr{B}_\Omega$ and $t \in (0, 1)$ imply

$$\min\{f(A), f(B)\} \leq f(C) \quad \text{for any } C \in \mathscr{D}_t(A, B).$$

(ii) *Strictly μ-quasi-concave* if $\mu(A \triangle B) > 0$ and $t \in (0, 1)$ imply

$$\min\{f(A), f(B)\} < f(C) \quad \text{for any } C \in \mathscr{D}_t(A, B).$$

(iii) μ-*concave* if $A, B \in \mathscr{B}_\Omega$ and $t \in (0,1)$ imply

$$tf(A) + (1-t)f(B) \leq f(C) \quad \text{for any } C \in \mathscr{D}_t(A, B).$$

(iv) *Strictly* μ-*concave* if $\mu(A \triangle B) > 0$ and $t \in (0,1)$ imply

$$tf(A) + (1-t)f(B) < f(C) \quad \text{for any } C \in \mathscr{D}_t(A, B).$$

A function f on $\mathscr{B}_\Omega$ is called to be (*strictly*) μ-*quasi-convex* if $-f$ is (strictly) μ-quasi-concave and f is called to be (*strictly*) μ-*convex* if $-f$ is (strictly) μ-concave.

Example 2.1. A trivial example of a μ-concave and also μ-convex function on $\mathscr{B}_\Omega$ is μ itself. It is immediate that μ is neither strictly μ-quasi-concave, strictly μ-quasi-convex, strictly μ-concave, nor strictly μ-convex by its additivity.

Example 2.2. Let φ be a real function on $[0, \mu(\Omega)]$ and define the function f_φ on $\mathscr{B}_\Omega$ by $f_\varphi(A) = \varphi(\mu(A))$. Then f_φ is (strictly) μ-quasi-concave on $\mathscr{B}_\Omega$ if and only if φ is (strictly) quasi-concave on $[0, \mu(\Omega)]$.

A partition $(X_1, \dots, X_n)$ is μ-*positive* if $\mu(X_i) > 0$ for each $i = 1, \dots, n$.

Definition 2.2. Let $(X_1, \dots, X_n)$ be a μ-positive partition. A function f on $\mathscr{B}_\Omega$ is:

(i) μ-*quasi-concave at* $(X_1, \dots, X_n)$ if $A, B \in \mathscr{B}_\Omega$, $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply

$$\min\{f(A), f(B)\} \leq f\left(\bigcup_{i=1}^n C_i\right).$$

(ii) *Strictly* μ-*quasi-concave at* $(X_1, \dots, X_n)$ if $\mu(A \triangle B) > 0$, $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply

$$\min\{f(A), f(B)\} < f\left(\bigcup_{i=1}^n C_i\right).$$

(iii) μ-*concave at* $(X_1, \dots, X_n)$ if $\mu(A \triangle B) > 0$ if $A, B \in \mathscr{B}_\Omega$, $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply

$$tf(A) + (1-t)f(B) \leq f\left(\bigcup_{i=1}^n C_i\right).$$

(iv) *Strictly μ-concave at* $(X_1, \dots, X_n)$ if $\mu(A \triangle B) > 0$, $t \in (0,1)$ and $C_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply

$$tf(A) + (1-t)f(B) < f\left(\bigcup_{i=1}^{n} C_i\right).$$

It can be shown that for every μ-positive n-partition $(X_1, \dots, X_n)$ it follows that $\bigcup_{i=1}^{n} \mathscr{D}_t(A \cap X_i, B \cap X_i) \subset \mathscr{D}_t(A, B)$ for any $t \in (0,1)$ and $A, B \in \mathscr{B}_\Omega$. Therefore, (strict) μ-quasi-concavity [resp. (strict) μ-concavity] implies (strict) μ-quasi-concavity [resp. (strict) μ-concavity] at $(X_1, \dots, X_n)$. However, for arbitrary $n \geq 2$ and for any $A, B \in \mathscr{B}_\Omega$ and $t \in (0,1)$ we can easily find an n-partition $(X_1, \dots, X_n)$ such that $\bigcup_{i=1}^{n} \mathscr{D}_t(A \cap X_i, B \cap X_i) \not\subset \mathscr{D}_t(A, B)$. Thus, (strict) μ-quasi-concavity [resp. (strict) μ-concavity] at some μ-positive partition does not imply (strict) μ-quasi-concavity [resp. (strict) μ-concavity]; The former is a "local" property while the latter is "global". When $n = 1$, Definition 2.2 is equivalent to Definition 2.1.

Theorem 2.4. *A function on $\mathscr{B}_\Omega$ is μ-quasi-concave if and only if it is μ-quasi-concave at any μ-positive n-partition.*

Example 2.3. Let $(X_1, \dots, X_n)$ be a μ-positive partition and let φ be a real function on $[0, \mu(X_1)] \times \cdots \times [0, \mu(X_n)]$. Define the function f_φ on $\mathscr{B}_\Omega$ by

$$f_\varphi(A) = \varphi(\mu(A \cap X_1), \dots, \mu(A \cap X_n)).$$

When $n = 1$, this case reduces to Example 2.2. Define the set S by

$$S = \{(\mu(A \cap X_1), \dots, \mu(A \cap X_n)) \in \mathbb{R}^n \mid A \in \mathscr{B}_\Omega\}.$$

Since the measure μ_i defined by $\mu_i(A) = \mu(A \cap X_i)$ is nonatomic and S is the range of the vector measure $(\mu_1, \dots, \mu_n)$, by Lyapunov's convexity theorem, it follows that S is convex and compact in $\mathbb{R}^n$. It can be shown that f_φ is μ-quasi-concave on $\mathscr{B}_\Omega$ at $(X_1, \dots, X_n)$ if and only if φ is quasi-concave on S. Similarly, f_φ is strictly μ-quasi-concave [resp. (strictly) μ-concave] at $(X_1, \dots, X_n)$ if and only if φ is strictly quasi-concave [resp. (strictly) concave] on S.

Recall that if a function on a real vector space is both concave and convex, then it is an additive function. Similar property holds for a function on $\mathscr{B}_\Omega$ which is both μ-concave and μ-convex at some μ-positive n-partition.

Theorem 2.5. *If f is both μ-concave and μ-convex at some μ-positive partition and $f(\emptyset) = 0$, then f is finitely additive on $\mathscr{B}_\Omega$.*

Denote the interior of Δ^{n-1} by

$$\operatorname{int}\Delta^{n-1} = \{(\alpha_1, \ldots, \alpha_n) \in \Delta^{n-1} \mid \alpha_i > 0,\ i = 1, \ldots, n\}.$$

The following result, a variant of Jensen's inequality, also justifies the introduction of the μ-quasi-concavity and μ-concavity of functions on $\mathscr{B}_\Omega$.

Theorem 2.6 (Jensen's inequality). *Let $(X_1, \ldots, X_m)$ be a μ-positive m-partition. A function f on $\mathscr{B}_\Omega$ is:*

(i) *μ-concave if and only if for every finite collection of elements $A_1, \ldots, A_n$ in $\mathscr{B}_\Omega$ and any $(t_1, \ldots, t_n) \in \operatorname{int}\Delta^{n-1}$,*

$$\sum_{i=1}^{n} t_i f(A_i) \le f(Y) \quad \text{for any } Y \in \mathscr{D}_{t_1,\ldots,t_n}(A_1, \ldots, A_n).$$

(ii) *μ-quasi-concave if and only if for every finite collection of elements $A_1, \ldots, A_n$ in $\mathscr{B}_\Omega$ and any $(t_1, \ldots, t_n) \in \operatorname{int}\Delta^{n-1}$,*

$$\min_{1\le i\le n}\{f(A_i)\} \le f(Y) \quad \text{for any } Y \in \mathscr{D}_{t_1,\ldots,t_n}(A_1, \ldots, A_n).$$

(iii) *μ-concave at $(X_1, \ldots, X_m)$ if and only if for every finite collection of elements $A_1, \ldots, A_n$ in $\mathscr{B}_\Omega$ and any $(t_1, \ldots, t_n) \in \operatorname{int}\Delta^{n-1}$, $Y_j \in \mathscr{D}_{t_1,\ldots,t_n}(A_1 \cap X_j, \ldots, A_n \cap X_j)$ for each $j = 1, \ldots, m$ implies*

$$\sum_{i=1}^{n} t_i f(A_i) \le f\left(\bigcup_{j=1}^{m} Y_j\right).$$

(iv) *μ-quasi-concave at $(X_1, \ldots, X_m)$ if and only if for every finite collection of elements $A_1, \ldots, A_n$ in $\mathscr{B}_\Omega$ and any $(t_1, \ldots, t_n) \in \operatorname{int}\Delta^{n-1}$, $Y_j \in \mathscr{D}_{t_1,\ldots,t_n}(A_1 \cap X_j, \ldots, A_n \cap X_j)$ for each $j = 1, \ldots, m$ implies*

$$\min_{1\le i\le n}\{f(A_i)\} \le f\left(\bigcup_{j=1}^{m} Y_j\right).$$

It is obvious from the above proof that Jensen's inequality is also valid for strictly μ-quasi-concave and strictly μ-concave functions by replacing the inequalities in Theorem 2.6 with strict inequalities and adding the condition that $\mu(A_i \,\triangle\, A_j) > 0$ for some $i \ne j$.

3 Preference Relations on a Borel σ-Field

In this section we first define the convexity of preference relations on $\mathscr{B}_\Omega$. Convex preferences are in conformity with the representation by a μ-quasi-concave function discussed in Subsection 2.2. We then show that maximal elements in $\mathscr{B}_\Omega$ are essentially unique with respect to the μ-strictly convex preferences. We next introduce a metric on $\mathscr{B}_\Omega$ which is identified with the L^1-norm metric of characteristic functions. We then define the continuity of preference relations on $\mathscr{B}_\Omega$ under which the existence of a continuous utility function representing the continuous preferences is guaranteed when Ω is a compact subset of a locally compact topological group with a regular Haar measure. The topological argument in this section is based on Sagara and Vlach (2006).

3.1 Convexity of Preference Relations

A *preference relation* $\succsim$ on $\mathscr{B}_\Omega$ is a complete transitive binary relation on $\mathscr{B}_\Omega$. The strict preference $A \succ B$ means that $A \succsim B$ and $B \not\succsim A$. The indifference $A \sim B$ means that $A \succsim B$ and $B \succsim A$. A real-valued set function f on $\mathscr{B}_\Omega$ *represents* $\succsim$ if $f(A) \geq f(B)$ holds if and only if $A \succsim B$ does, and such f is called a *utility function* representing $\succsim$.

The following definition of the (strictly) μ-convexity of preference relations are analogues of the (strict) convexity of preference relations on a standard commodity space.

Definition 3.1. A preference relation $\succsim$ on $\mathscr{B}_\Omega$ is:

(i) *μ-convex* if $A \succsim C$, $B \succsim C$, and $t \in (0,1)$ imply $D \succsim C$ for any $D \in \mathscr{D}_t(A,B)$.

(ii) *Strictly μ-convex* if $A \succsim C$, $B \succsim C$, $\mu(A \triangle B) > 0$, and $t \in (0,1)$ imply $D \succ C$ for any $D \in \mathscr{D}_t(A,B)$.

Definition 3.2. Let $(X_1, \dots, X_n)$ be a μ-positive partition. A preference relation $\succsim$ on $\mathscr{B}_\Omega$ is:

(i) *μ-convex at $(X_1, \dots, X_n)$* if $A \succsim C$, $B \succsim C$, $t \in (0,1)$, and $D_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply $\bigcup_{i=1}^n D \succsim C$.

(ii) *Strictly μ-convex at $(X_1, \dots, X_n)$* if $A \succsim C$, $B \succsim C$, $\mu(A \triangle B) > 0$, $t \in (0,1)$, and $D_i \in \mathscr{D}_t(A \cap X_i, B \cap X_i)$ for each $i = 1, \dots, n$ imply $\bigcup_{i=1}^n D \succ C$.

Theorem 3.1. *A preference relation is (strictly) μ-quasi-convex if and only if it is (strictly) μ-convex at any μ-positive n-partition.*

The following result characterizes (strictly) μ-quasi-concave and (strictly) μ-concave utility functions.

Theorem 3.2. *Let $(X_1, \dots, X_n)$ be a μ-positive partition. A utility function representing a preference relation $\succsim$ is:*

(i) *(Strictly) μ-quasi-concave if and only if $\succsim$ is (strictly) μ-convex.*

(ii) *(Strictly) μ-concave at $(X_1, \dots, X_n)$ if and only if $\succsim$ is (strictly) μ-convex at $(X_1, \dots, X_n)$.*

An element $A \in \mathscr{B}_\Omega$ is *maximal* with respect to $\succsim$ if there exists no element $B \in \mathscr{B}_\Omega$ such that $B \succ A$. Since $\succsim$ is complete, this is equivalent to saying that $A \succsim B$ for every $B \in \mathscr{B}_\Omega$.

Two measurable sets A and B in $\mathscr{B}_\Omega$ are *μ-equivalent* if $\mu(A \triangle B) = 0$. The μ-equivalence defines an equivalence relation on $\mathscr{B}_\Omega$.

Theorem 3.3. *If a preference relation on $\mathscr{B}_\Omega$ is strictly μ-convex at some μ-positive partition, then its maximal element is unique up to μ-equivalence.*

Remark 3.1. In this paper we have not pursued the representability of μ-convex preferences by a μ-concave utility function. The situation here is similar to the possibility in which convex preferences may not have the representation by a concave utility function on a commodity space. For a finite dimensional commodity space, Kannai (1977) characterized the representability of convex preferences by a concave utility function. At present we do not know whether the approach of Kannai is applicable to the convex preferences on measure spaces in our framework.

3.2 Continuity of Preference Relations

Let $(X, \mathscr{B}_X, \mu)$ be a measure space, where X is a topological space, $\mathscr{B}_X$ is the Borel σ-field of X, and μ is a Borel measure on $\mathscr{B}_X$. Let Ω be a compact subset of X. When Ω is endowed with the relative topology from X, the Borel σ-field $\mathscr{B}_\Omega$ of Ω is given by $\mathscr{B}_\Omega = \{E \cap \Omega \mid E \in \mathscr{B}_X\}$ and the restriction μ, which we denote again μ, to the Borel measurable space $(\Omega, \mathscr{B}_\Omega)$ makes $(\Omega, \mathscr{B}_\Omega, \mu)$ a finite Borel measure space. Each element f in $L^1(\Omega, \mathscr{B}_\Omega, \mu)$ is identified with an element $\tilde{f}$ in $L^1(X, \mathscr{B}_X, \mu)$ by the embedding $f \mapsto \tilde{f}$ satisfying $\tilde{f} = f$ on Ω and $\tilde{f} = 0$ on $X \setminus \Omega$. This embedding yields an isometry on $L^1(\Omega, \mathscr{B}_\Omega, \mu)$ into $L^1(X, \mathscr{B}_X, \mu)$ and under this identification $L^1(\Omega, \mathscr{B}_\Omega, \mu)$ is a closed vector subspace of $L^1(X, \mathscr{B}_X, \mu)$.

We denote the μ-equivalence class of $A \in \mathscr{B}_\Omega$ by $[A]$ and the set of μ-equivalence classes in $\mathscr{B}_\Omega$ by $\mathscr{B}_\Omega[\mu]$. If, for any two μ-equivalence classes $\mathbf{A}$ and $\mathbf{B}$, we define the metric d by $d(\mathbf{A}, \mathbf{B}) = \mu(A \triangle B)$ where A and B are arbitrarily selected elements of $\mathbf{A}$ and $\mathbf{B}$, then $\mathscr{B}_\Omega[\mu]$ becomes a complete metric space. Since $\mu(A \triangle B) = \int |\chi_A - \chi_B| d\mu$ where χ_A and χ_B are characteristic functions of A and B respectively, we know that two measurable sets A and B are μ-equivalent if, and only if, their characteristic functions differ by a μ-null function. Therefore, the mapping $\mathbf{A} \mapsto \chi_A$ where A is an arbitrarily selected element of $\mathbf{A}$ is an isometry on $\mathscr{B}_\Omega[\mu]$ into $L^1(\Omega, \mathscr{B}_\Omega, \mu)$.

Definition 3.3. A preference relation $\succsim$ on $\mathscr{B}_\Omega$ is *μ-indifferent* if $\mu(A \triangle B) = 0$ implies $A \sim B$.

A μ-indifferent preference relation $\succsim$ *induces* a preference relation $\succsim_\mu$ on $\mathscr{B}_\Omega[\mu]$ defined by $\mathbf{A} \succsim_\mu \mathbf{B}$ if and only if there exist $A \in \mathbf{A}$ and $B \in \mathbf{B}$ such that $A \succsim B$. This is equivalent to saying that $\mathbf{A} \succsim_\mu \mathbf{B}$ if and only if $A \succsim B$ for any $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Thus, any utility function f representing $\succsim$ on $\mathscr{B}_\Omega$ induces a utility function f_μ representing $\succsim_\mu$ on $\mathscr{B}_\Omega[\mu]$ by $f_\mu(\mathbf{A}) = f(A)$ where A is an arbitrary element in $\mathbf{A}$.

Definition 3.4. A preference relation $\succsim$ on $\mathscr{B}_\Omega$ is *μ-continuous* if it is μ-indifferent and for any $\mathbf{A} \in \mathscr{B}_\Omega[\mu]$ both the upper contour set $\{\mathbf{B} \in \mathscr{B}_\Omega[\mu] \mid \mathbf{B} \succsim_\mu \mathbf{A}\}$ and the lower contour set $\{\mathbf{B} \in \mathscr{B}_\Omega[\mu] \mid \mathbf{A} \succsim_\mu \mathbf{B}\}$ are closed in $\mathscr{B}_\Omega[\mu]$.

The μ-continuity of $\succsim$ implies that the preference relation $\succsim_\mu$ induced by $\succsim$ satisfies the standard continuity axiom for preference relations.

Definition 3.5. A function f on $\mathscr{B}_\Omega$ is:

(i) *μ-indifferent* if $\mu(A \triangle B) = 0$ implies $f(A) = f(B)$.

(ii) *μ-continuous* if it is μ-indifferent and induces a continuous function f_μ on $\mathscr{B}_\Omega[\mu]$.

The following result from Sagara and Vlach (2006) guarantees the existence of a μ-continuous utility function representing μ-continuous preferences.

Proposition 3.1. *Let $(X, \mathscr{B}_X, \mu)$ be a Borel measure space with X a locally compact topological group and μ a regular Haar measure. Moreover, let Ω be a compact subset of X and $(\Omega, \mathscr{B}_\Omega, \mu)$ be the finite measure space induced by the restriction of $(X, \mathscr{B}_X, \mu)$. Then, for any μ-continuous preference relation $\succsim$ on $\mathscr{B}_\Omega$, there exists a μ-continuous utility function representing $\succsim$.*

Example 3.1. Let $\mu_1, \dots, \mu_n$ be finite measures of a measurable space $(\Omega, \mathscr{B}_\Omega)$. Define $\mu = \frac{1}{n}\sum_{i=1}^{n} \mu_i$. Let f be a continuous function on $[0, \mu_1(\Omega)] \times \cdots \times [0, \mu_n(\Omega)]$. A preference relation on $\mathscr{B}_\Omega$ defined by

$$A \succsim B \overset{\text{def}}{\iff} f(\mu_1(A), \dots, \mu_n(A)) \geq f(\mu_1(B), \dots, \mu_n(B))$$

is μ-continuous.

Example 3.2. Let $\mu_1, \dots, \mu_n$ and μ be defined as in Example 3.1 and let $(X_1, \dots, X_n)$ be a partition. Let f be a continuous function on $[0, \mu_1(X_1)] \times \cdots \times [0, \mu_n(X_n)]$. Consider a preference relation on $\mathscr{B}_\Omega$ defined by

$$A \succsim B \overset{\text{def}}{\iff} f(\mu_1(A \cap X_1), \dots, \mu_n(A \cap X_n)) \geq f(\mu_1(B \cap X_1), \dots, \mu_n(B \cap X_n)).$$

This is a numerical representation of preference relations studied by Sprumont (2004). As in Example 3.1, it can be shown that $\succsim$ is μ-continuous. See for details Sagara and Vlach (2006).

The (strict) μ-monotonicity of preference relations on $\mathscr{B}_\Omega$ in the following definition are analogues of the (strict) monotonicity of preference relations on a standard commodity space.

Definition 3.6. A preference relation $\succsim$ on $\mathscr{B}_\Omega$ is:

(i) *μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $A \succsim B$.

(ii) *Strictly μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $A \succ B$.

Similar to Definition 3.6, the (strict) μ-monotonicity of functions on $\mathscr{B}_\Omega$ are defined as follows.

Definition 3.7. A function f on $\mathscr{B}_\Omega$ is:

(i) *μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $f(A) \geq f(B)$.

(ii) *Strictly μ-monotone* if $A \supset B$ and $\mu(A) > \mu(B)$ implies $f(A) > f(B)$.

Example 3.3. Let f_φ be a set function on $\mathscr{B}_\Omega$ introduced in Example 2.3. Then f_φ is (strictly) μ-monotone on $\mathscr{B}_\Omega$ if and only if φ is (strictly) increasing on S.

Note that preference relations on a standard commodity space are strictly monotone if they are continuous, monotone and strictly convex. As the following result shows, the similar property holds for preference relations on $\mathscr{B}_\Omega$.

Theorem 3.4. *If a preference relation is μ-continuous, μ-monotone, and strictly μ-convex at some μ-positive partition, then it is strictly μ-monotone.*

4 Pareto Optimal Partitions

This section is concerned with the existence and characterization of a Pareto optimal partition. The existence of a weakly Pareto optimal partition follows from the μ-continuity of the utility function of each individual and the compactness of the set of partitions in L^1. It is shown that if each individual has a μ-continuous and strictly μ-monotone utility function, then weak Pareto optimality is equivalent to Pareto optimality. We also show that if each individual has a μ-concave utility function, then the utility possibility set is a convex set, and consequently every weakly Pareto optimal partition is a solution to the maximization problem of a weighted utility sum of each individual by the supporting hyperplane theorem.

Note that a preference relation is represented by a (strictly) μ-monotone utility function if and only if the preference relation is (strictly) μ-monotone. By Proposition 3.1, a preference relation is represented by a μ-continuous utility function if and only if the preference relation is μ-continuous, and by Theorem 3.2, a preference relation is represented by a (strictly) μ-quasi-concave utility function if and only if the preference relation is (strictly) μ-convex. Therefore, it is legitimate in the sequel to employ utility functions of individuals instead of their preference relations.

4.1 Characterization of Pareto Optimality

Let $(X, \mathscr{B}_X, \mu)$ be a Borel measure space with X a locally compact topological group and μ a nonatomic regular Haar measure. Let Ω be a compact subset of X and $(\Omega, \mathscr{B}_\Omega, \mu)$ be the nonatomic finite measure space induced from $(X, \mathscr{B}_X, \mu)$ as in Subsection 3.2. A typical example of this structure is the Lebesgue measure space of $\mathbb{R}^n$ with any compact subset of $\mathbb{R}^n$ in which $\mathbb{R}^n$ is locally compact topological Abelian group under the vector addition and the Lebesgue measure is a nonatomic regular Haar measure. Denote the finite set of individuals by $I = \{1, \ldots, n\}$. A utility function of individual $i \in I$ on $\mathscr{B}_\Omega$ is denoted by u_i and the set of n-partitions of Ω by $\mathscr{P}_n$.

Definition 4.1. A partition $(A_1, \ldots, A_n)$ is:

(i) *Weakly Pareto optimal* if there exists no partition $(B_1, \ldots, B_n)$ such that $u_i(A_i) < u_i(B_i)$ for each $i \in I$.

(ii) *Pareto optimal* if no partition exists $(B_1, \ldots, B_n)$ such that $u_i(A_i) \leq u_i(B_i)$ for each $i \in I$ and $u_i(A_i) < u_i(B_i)$ for some $i \in I$.

We denote the n-times Cartesian product of $\mathscr{B}_\Omega[\mu]$ by $\mathscr{B}_\Omega^n[\mu]$ and define the set $\mathscr{P}_n[\mu]$ of μ-equivalence classes of partitions by

$$\mathscr{P}_n[\mu] = \{(\mathbf{A}_1, \dots, \mathbf{A}_n) \in \mathscr{B}_\Omega^n[\mu] \mid \exists (A_1, \dots, A_n) \in \mathscr{P}_n : A_i \in \mathbf{A}_i \ \forall i \in I\}.$$

The following result from Sagara and Vlach (2006) plays a crucial role in the analysis in the sequel.

Proposition 4.1. *Let $(X, \mathscr{B}_X, \mu)$ be a Borel measure space with X a locally compact topological group and μ a regular Haar measure. If Ω is a compact subset of X and $(\Omega, \mathscr{B}_\Omega, \mu)$ is the finite measure space induced by the restriction of $(X, \mathscr{B}_X, \mu)$, then $\mathscr{P}_n[\mu]$ is a compact metric space.*

Define the *utility possibility set* U by

$$U = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \exists (A_1, \dots, A_n) \in \mathscr{P}_n : x_i \le u_i(A_i) \ \forall i \in I\}.$$

Note that if u_i is a nonatomic finite measure for each $i \in I$, then the convexity of U trivially follows from Lyapunov's convexity theorem without imposing any concavity on u_i. Thus, the next theorem is regarded as a variant of this result for the case that u_i is not necessarily additive for each $i \in I$.

Theorem 4.1. *If u_i is μ-continuous and μ-concave at some μ-positive partition for each $i \in I$, then U is a closed convex subset of $\mathbb{R}^n$.*

The main results of this section are the following.

Theorem 4.2. (i) *If u_i is μ-continuous for each $i \in I$, then there exists a weakly Pareto optimal partition.*

(ii) *If u_i is μ-continuous and strictly μ-monotone for each $i \in I$, then a partition is Pareto optimal if and only if it is weakly Pareto optimal.*

(iii) *If u_i is μ-concave at some μ-positive partition for each $i \in I$, then a partition is weakly Pareto optimal if and only if it solves the problem*

$$\max \left\{ \sum_{i \in I} \alpha_i u_i(A_i) \mid (A_1, \dots, A_n) \in \mathscr{P}_n \right\} \tag{P_α}$$

for some $\alpha \in \Delta^{n-1}$.

Example 4.1. Let $(\Omega, \mathscr{B}_\Omega, \mu)$ be a Lebesgue measure space with Ω a compact subset of $\mathbb{R}^l$ and $\mathscr{B}_\Omega$ the σ-field of Borel subsets of Ω. Suppose that Ω is decomposed into disjoint sets $X_1, \dots, X_m$ with $\mu(X_1), \dots, \mu(X_m) > 0$. Let utility functions of each individual be given by

$$u_i(A) = f_i(\mu(A \cap X_1), \dots, \mu(A \cap X_m)),$$

where f_i is real-valued functions defined on $[0, \mu(X_1)] \times \cdots \times [0, \mu(X_m)]$ for each $i \in I$. This representation of preferences is a special case of Example 3.2. Note that this economy is analogous to a pure exchange economy with n individuals, m commodities and total endowment Ω. If f_i is continuous, then u_i is μ-continuous (Example 3.1). Define the set by

$$S = \{(\mu(A \cap X_1), \dots, \mu(A \cap X_m)) \in \mathbb{R}^m \mid A \in \mathscr{B}_\Omega\}.$$

Then S is convex and compact, and f_i is concave and strictly increasing on S if and only if u_i is strictly μ-concave at $(X_1, \dots, X_n)$ and strictly μ-monotone (Examples 2.3 and 3.3). Therefore, Theorem 4.2 is true for this economy.

Remark 4.1. The existence of a weakly Pareto optimal partition was established first by Dubins and Spanier (1961) for the case of additive preferences represented by a nonatomic finite measure. The equivalence between Pareto optimality and weak Pareto optimality is guaranteed for the case of additive preferences if a nonatomic finite measure of each individual is mutually absolutely continuous (see Sagara 2006). A characterization of weak Pareto optimality in terms of the maximization problem of a weighted utility sum using the supporting hyperplane theorem was provided by Barbanel and Zwicker (1997) for the case of additive preferences. Without imposing any topological structure on a σ-field, Sagara (2006) extended these results for the case of nonadditive preferences with a concave transformation of a nonatomic finite measure by employing Lyapunov's convexity theorem.

5 Core Partitions in a Cooperative Game

This section introduces cooperative games with NTU and with TU in a pure exchange economy in which the initial individual endowments form a partition. We show the existence of a core partition with NTU under the assumption of μ-continuity and μ-quasi-concavity of utility functions of each individual and the existence of a core partition with TU under the assumption of μ-continuity and μ-concavity of utility functions of each individual.

5.1 NTU Game

A nonempty subset of I is called a *coalition*. We denote the collection of coalitions by $\mathscr{N}$. Let $(\Omega_1, \dots, \Omega_n) \in \mathscr{P}_n$ be an *initial partition* in which individual $i \in I$ is endowed with a measurable subset Ω_i of Ω. A partition $(A_1, \dots, A_n)$ is an *S-partition* if $\bigcup_{i \in S} A_i = \bigcup_{i \in S} \Omega_i$ for coalition S.

Definition 5.1. A coalition S *improves upon* a partition $(A_1, \dots, A_n)$ with NTU if there exists some S-partition $(B_1, \dots, B_n)$ such that $u_i(A_i) < u_i(B_i)$ for each $i \in S$. A partition with NTU that cannot be improved upon by any coalition is a *core partition* with NTU.

It is obvious from the definitions that a core partition with NTU is weakly Pareto optimal. Note that if u_i is μ-continuous and strictly μ-monotone for each $i \in I$, then a core partition with NTU is also Pareto optimal by Theorem 4.2(ii).

Theorem 5.1. *If u_i is μ-continuous and μ-quasi-concave at some μ-positive partition for each $i \in I$, then there exists a core partition with NTU.*

Remark 5.1. Berliant (1985) identified a measurable set with a characteristic function in L^∞ and introduced a price system in L^1 as a weak* continuous linear functional on a commodity space in L^∞ to show the existence of an equilibrium for the case of additive preferences by the standard argument of Bewley (1972). The existence of an equilibrium implies the nonemptiness of a core partition with NTU. Berliant and Dunz (2004) embedded characteristic functions in L^1 with a price system in L^∞ as the norm dual of a commodity space in L^1 to show the existence of an equilibrium for the case of nonadditive preferences by the fixed point argument under the continuity assumption of preferences and the strong convexity assumption that the upper contour set is separated by hyperplanes in L^∞. Dunz (1991) proved balancedness of the NTU game for the case of nonadditive preferences with a specific integral form and Sagara (2006) also gave a proof of the balancedness for the case of nonadditive preferences with a concave transformation of a nonatomic finite measure.

5.2 TU Game

TU game developed here is a variant of a market game introduced by Shapley and Shubik (1969), who showed the balancedness of the market game with a finite dimensional commodity space.

Definition 5.2. A coalition S *improves upon* a partition $(A_1, \dots, A_n)$ with TU if there exists some S-partition $(B_1, \dots, B_n)$ such that $\sum_{i \in S} u_i(A_i) < \sum_{i \in S} u_i(B_i)$. A partition with TU that cannot be improved upon by any coalition is a *core partition* with TU.

It is obvious from the definitions that a core partition with TU is weakly Pareto optimal and that a core partition with TU is a core partition with NTU. Note that if u_i is μ-continuous and strictly μ-monotone for each $i \in I$, then a core partition with TU is also Pareto optimal by Theorem 4.2(ii).

Theorem 5.2. *If u_i μ-continuous and μ-concave at some μ-positive partition for each $i \in I$, then there exists a core partition with TU.*

Remark 5.2. Legut (1990) characterized payoff vectors in the core of the TU game for the case of additive preferences with a nonatomic finite measure. Legut (1985) proved the balancedness of the TU game with countably infinite individuals for the case of additive preferences with a nonatomic finite measure, and Legut (1986) and Sagara (2006) showed the balancedness of the TU game with finitely many individuals for the case of nonadditive preferences with a concave transformation of nonatomic finite measures.

References

Barbanel, J. B. and W. S. Zwicker, (1997). "Two applications of a theorem of Dvoretsky, Wald, and Wolfovitz to cake division", *Theory and Decision*, vol. 43, pp. 203–207.

Berliant, M., (1985). "An equilibrium existence result for an economy with land", *Journal of Mathematical Economics*, vol. 14, pp. 53–56.

Berliant, M. and K. Dunz, (2004). "A foundation of location theory: existence of equilibrium, the welfare theorems, and core", *Journal of Mathematical Economics*, vol. 40, pp. 593–618.

Bewley, T. F., (1972). "Existence of equilibria in economies with infinitely many commodity spaces", *Journal of Economic Theory*, vol. 4, pp. 514–540.

Bondareva, O. N., (1963). "Some applications of linear programming methods to the theory of cooperative games", *Problemy Kibernitiki*, vol. 10, pp. 119–139 [in Russian].

Dubins, L. E. and E. H. Spanier, (1961). "How to cut a cake fairly", *American Mathematical Monthly*, vol. 68, pp. 1–17.

Dunz, K., (1991). "On the land trading game", *Regional Science and Urban Economics*, vol. 21, pp. 73–88.

Kannai, Y., (1977). "Concavifiability and constructions of concave utility functions", *Journal of Mathematical Economics*, vol. 4, pp. 1–56.

Legut, J., (1985). "The problem of fair division for countably many participants", *Journal of Mathematical Analysis and Applications*, vol. 109, pp. 83–89.

Legut, J., (1986). "Market games with a continuum of indivisible commodities", *International Journal of Game Theory*, vol. 15, pp. 1–7.

Legut, J., (1990). "On totally balanced games arising from cooperation in fair division", *Games and Economic Behavior*, vol. 2, pp. 47–60.

Sagara, N., (2006). "An existence result on partitioning of a measurable space: Pareto optimality and core", accepted for publication in *Kybernetika*.

Sagara, N. and M. Vlach, (2006). "Equity and efficiency in a measure space with nonadditive preferences: the problem of cake division", Faculty of Economics, Hosei University, mimeo.

Scarf, H. E., (1967). "The core of an N person game", *Econometrica*, vol. 35, pp. 50–69.

Shapley, L., (1967). "On balanced sets and cores", *Naval Research Logistics Quarterly*, vol. 14, pp. 453–460.

Shapley, L. and M. Shubik, (1969). "On market games", *Journal of Economic Theory*, vol. 1, pp. 9–25.

Sprumont, Y. (2004). "What is a commodity? Two axiomatic answers", *Economic Theory*, vol. 23, pp. 429–437.