

## HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGY COBORDISMS OF SURFACES

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### 1. INTRODUCTION

The Alexander polynomial is one of the most fundamental invariants for finitely presentable groups. It can be easily computed from any finite presentation of a group. By considering the fundamental group of a manifold, we can regard it as a polynomial invariant of manifolds. Moreover, especially in the cases of low dimensional manifolds, it gives some kinds of geometrical information.

One method for computing the Alexander polynomial of a finitely presentable group  $G$  goes as follows. Take a finite presentation  $\langle x_1, \dots, x_l \mid r_1, \dots, r_m \rangle$  of  $G$ . We compute the Jacobi matrix  $(\frac{\partial r_i}{\partial x_j})_{i,j}$  at  $\mathbb{Z}G$  of the presentation by using free differentials. Applying the natural map  $\alpha : G \rightarrow H := H_1(G)/(\text{torsion})$  to each entry of the matrix, we obtain so called the Alexander matrix of the presentation. Then the Alexander polynomial of  $G$  is the greatest common divisor of all  $(l-1)$ -minors of the Alexander matrix. It is defined uniquely up to units of  $\mathbb{Z}H$  and does not depend on the finite presentation of  $G$ .

In the above process of a computation, the map  $\alpha : G \rightarrow H$  makes the situation much easier—From non-commutative algebra to commutative one. It enables us to use the determinant of matrices and take the greatest common divisor of a set of elements of  $\mathbb{Z}H$ . On the other hand, it is reasonable to ask what informations on  $G$   $\alpha$  loses. For that, some generalizations of the Alexander polynomial have been defined by several people. One of the most famous ones is the twisted Alexander polynomial. However, in this paper, we concern the theory of higher-order Alexander invariants defined by using localizations of some kinds of non-commutative rings located between  $\mathbb{Z}G$  and  $\mathbb{Z}H$ .

Higher-order Alexander invariants were first defined by Cochran in [1] for knot groups, and then generalized for arbitrary finitely presentable groups by Harvey in [7, 8]. They are numerical invariants interpreted as degrees of “non-commutative Alexander polynomials”, which have some unclear ambiguity except their degrees in difficulties of non-commutative rings. Using them, Harvey obtained various sharper results than those given by the ordinary Alexander invariants — lower bounds on the Thurston norm, necessary conditions for realizing a given group as the fundamental group of some 3-manifold, and so on. Leidy-Maxim [12] studied these invariants for plane algebraic curves.

In this paper, we give an application of higher-order Alexander invariants to homology cobordisms of surfaces<sup>1</sup>. The set of homology cobordisms of a fixed surface has a natural

<sup>1</sup>The word “surface” means a real 2-dimensional manifold.

monoid structure, and moreover, by considering them up to homology cobordism, we can construct a group (see Section 3 for details). The aim of this paper is to obtain some informations on their structures by defining and studying variants of higher-order Alexander invariants associated to homology cobordisms of surfaces.

## 2. HIGHER-ORDER ALEXANDER INVARIANTS AND TORSION-DEGREE FUNCTIONS

We begin by reviewing the theory of higher-order Alexander invariants along the lines of Harvey's papers [7, 8]. Then we generalize them to functions of matrices called *torsion-degree functions*. A key ingredient of this generalization is the Dieudonné determinant of skew fields, which enables us to proceed our argument by using non-commutative linear algebra.

Before starting our discussion, we summarize our notation. For a matrix  $A$  with coefficients in a ring  $R$ , and a homomorphism  $\varphi : R \rightarrow R'$ , we denote by  ${}^\varphi A$  the matrix obtained from  $A$  by applying  $\varphi$  to each entry.  $A^T$  denotes the transpose of  $A$ . When  $R = \mathbb{Z}G$  for a group  $G$  or its right field of fractions (if exists), we denote by  $\bar{A}$  the matrix obtained from  $A$  by applying the involution induced from  $(x \mapsto x^{-1}, x \in G)$  to each entry.

For a module  $M$ ,  $M^n$  (resp.  $M_n$ ) denotes the module of column (resp. row) vectors with  $n$  entries.

For a finite CW-complex  $X$  and its regular covering  $X_\Gamma$  with respect to a homomorphism  $\pi_1 X \rightarrow \Gamma$ ,  $\Gamma$  acts on  $X_\Gamma$  from the right through its deck transformation group. Therefore we regard the  $\mathbb{Z}\Gamma$ -cellular chain complex  $C_*(X_\Gamma)$  of  $X_\Gamma$  as a collection of free right  $\mathbb{Z}\Gamma$ -modules consisting of column vectors together with differentials given by left multiplications of matrices. For each  $\mathbb{Z}\Gamma$ -bimodule  $A$ , the twisted chain complex  $C_*(X; A)$  is given by the tensor product of the right  $\mathbb{Z}\Gamma$ -module  $C_*(X_\Gamma)$  and the left  $\mathbb{Z}\Gamma$ -module  $A$ , so that  $C_*(X; A)$  and  $H_*(X; A)$  are right  $\mathbb{Z}\Gamma$ -modules.

**2.1. Review of Harvey's higher-order Alexander invariants.** Here we review Harvey's setting of higher-order Alexander invariants in [7, 8]. A group  $\Gamma$  is *poly-torsion-free-abelian* (PTFA, for short) if  $\Gamma$  has a normal series of finite length whose successive quotients are all torsion-free abelian. Any subgroup of a PTFA group is also PTFA.

We recall some properties of the group ring  $\mathbb{Z}\Gamma$  of a PTFA group  $\Gamma$  from the theory of non-commutative rings, for which we refer to [2], [3], [14], [17].

A multiplicatively closed set  $S$  of a ring  $R$  is called a *right divisor set* of  $R$  if it satisfies

- (1)  $0 \notin S, 1 \in S$ ,
- (2) For any  $r \in R, s \in S$ , the set  $sR \cap rS$  is not empty.

For each right divisor set  $S$  of  $R$ , we can construct its *right quotient ring*  $RS^{-1}$ . An integral domain  $R$  is called a *right Ore domain* if  $R - \{0\}$  is a right divisor set.

For each PTFA group  $\Gamma$ ,  $\mathbb{Z}\Gamma$  is known to be an Ore domain, so that it can be embedded in the *right field of fractions*  $\mathcal{K}_\Gamma := \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1}$ , which is a skew field.

We will also use the following localizations of  $\mathbb{Z}\Gamma$  placed between  $\mathbb{Z}\Gamma$  and  $\mathcal{K}_\Gamma$ . Let  $\psi \in H^1(\Gamma)$  be a primitive element. This means the corresponding homomorphism, which is denoted by  $\psi$  again, under  $H^1(\Gamma) \cong \text{Hom}(\Gamma, \mathbb{Z})$  is onto. We write  $\Gamma^\psi := \text{Ker } \psi$ . Then we have an exact sequence

$$1 \longrightarrow \Gamma^\psi \longrightarrow \Gamma \xrightarrow{\psi} \mathbb{Z} \longrightarrow 1.$$

We take a splitting  $\xi : \mathbb{Z} \rightarrow \Gamma$  of this sequence and put  $t := \xi(1) \in \Gamma$ . Since  $\Gamma^\psi$  is again a PTFA group,  $\mathbb{Z}\Gamma^\psi - \{0\}$  is a right divisor set of  $\mathbb{Z}\Gamma^\psi$ . Hence  $\mathbb{Z}\Gamma^\psi$  can be embedded in its right field of fractions  $\mathcal{K}_{\Gamma^\psi} = \mathbb{Z}\Gamma^\psi(\mathbb{Z}\Gamma^\psi - \{0\})^{-1}$ . Moreover  $\mathbb{Z}\Gamma^\psi - \{0\}$  is also a right divisor set of  $\mathbb{Z}\Gamma$ , so that we can construct a right quotient ring  $\mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1}$ . Then the splitting  $\xi$  gives an isomorphism between  $\mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1}$  and the skew Laurent polynomial ring  $\mathcal{K}_{\Gamma^\psi}[t^\pm]$ , in which  $at = t(t^{-1}at)$  holds for each  $a \in \Gamma$ .  $\mathcal{K}_{\Gamma^\psi}[t^\pm]$  is known to be a non-commutative right and left principal ideal domain. By definition, we have inclusions

$$\mathbb{Z}\Gamma \hookrightarrow \mathcal{K}_{\Gamma^\psi}[t^\pm] \hookrightarrow \mathcal{K}_\Gamma.$$

$\mathcal{K}_{\Gamma^\psi}[t^\pm]$  and  $\mathcal{K}_\Gamma$  are known to be flat  $\mathbb{Z}\Gamma$ -modules.

On  $\mathcal{K}_{\Gamma^\psi}[t^\pm]$ , we have a map  $\text{deg}^\psi : \mathcal{K}_{\Gamma^\psi}[t^\pm] \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$  assigning to each polynomial its degree. We put  $\text{deg}^\psi(0) := \infty$ . Note that the composite  $\mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1} \xrightarrow{\cong} \mathcal{K}_{\Gamma^\psi}[t^\pm] \xrightarrow{\text{deg}^\psi} \mathbb{Z}_{\geq 0} \cup \{\infty\}$  does not depend on the choice of the splitting  $\xi$ .

Harvey's higher-order Alexander invariants [8] are defined as follows. Let  $G$  be a finitely presentable group, and let  $\varphi : G \rightarrow \mathbb{Z}$  be an epimorphism. For a PTFA group  $\Gamma$  and an epimorphism  $\varphi_\Gamma : G \rightarrow \Gamma$ ,  $(\varphi_\Gamma, \varphi)$  is called an *admissible pair* for  $G$  if there exists an epimorphism  $\psi : \Gamma \rightarrow \mathbb{Z}$  satisfying  $\varphi = \psi \circ \varphi_\Gamma$ . For each admissible pair  $(\varphi_\Gamma, \varphi)$  for  $G$ , we regard  $\mathcal{K}_{\Gamma^\psi}[t^\pm] = \mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1}$  as a  $\mathbb{Z}G$ -module, and we define higher-order Alexander invariants for  $(\varphi_\Gamma, \varphi)$  by

$$\begin{aligned} \bar{\delta}_\Gamma^\psi(G) &= \dim_{\mathcal{K}_{\Gamma^\psi}}(H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm])) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}, \\ \delta_\Gamma^\psi(G) &= \dim_{\mathcal{K}_{\Gamma^\psi}}(T_{\mathcal{K}_{\Gamma^\psi}[t^\pm]}H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm])) \in \mathbb{Z}_{\geq 0}, \end{aligned}$$

where  $T_{\mathcal{K}_{\Gamma^\psi}[t^\pm]}M$  denotes the  $\mathcal{K}_{\Gamma^\psi}[t^\pm]$ -torsion part for each  $\mathcal{K}_{\Gamma^\psi}[t^\pm]$ -module  $M$ . We call  $\bar{\delta}_\Gamma^\psi(G)$  the  $\Gamma$ -degree<sup>2</sup>, and call  $\delta_\Gamma^\psi(G)$  the *refined*  $\Gamma$ -degree. Note that the right  $\mathcal{K}_{\Gamma^\psi}[t^\pm]$ -module  $H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm])$  are decomposed into

$$H_1(G; \mathcal{K}_{\Gamma^\psi}[t^\pm]) = (\mathcal{K}_{\Gamma^\psi}[t^\pm])^r \oplus \left( \bigoplus_{i=1}^l \frac{\mathcal{K}_{\Gamma^\psi}[t^\pm]}{p_i(t)\mathcal{K}_{\Gamma^\psi}[t^\pm]} \right)$$

<sup>2</sup>Our definition is slightly different from that in [8].

for some  $r \in \mathbb{Z}_{\geq 0}$  and  $p_i(t) \in \mathcal{K}_{\Gamma\psi}[t^{\pm}]$ , and then

$$\bar{\delta}_{\Gamma}^{\psi}(G) = \begin{cases} \sum_{i=1}^l \deg^{\psi}(p_i(t)) & (r = 0), \\ \infty & (r > 0) \end{cases},$$

$$\delta_{\Gamma}^{\psi}(G) = \sum_{i=1}^l \deg^{\psi}(p_i(t)).$$

For a space  $X$  and an admissible pair for  $\pi_1 X$ , we define  $\bar{\delta}_{\Gamma}^{\psi}(X) := \bar{\delta}_{\Gamma}^{\psi}(\pi_1 X)$  and  $\delta_{\Gamma}^{\psi}(X) := \delta_{\Gamma}^{\psi}(\pi_1 X)$ .

**2.2. Torsion-degree functions.** We fix a finitely presentable group  $G$  and an admissible pair  $(\varphi_{\Gamma}, \varphi)$  for  $G$ . The (refined)  $\Gamma$ -degree can be computed from a presentation matrix of the right  $\mathcal{K}_{\Gamma\psi}[t^{\pm}]$ -module  $H_1(G; \mathcal{K}_{\Gamma\psi}[t^{\pm}])$ . Therefore we can consider it to be a function on the set  $M(\mathcal{K}_{\Gamma\psi}[t^{\pm}])$  of all matrices with entries in  $\mathcal{K}_{\Gamma\psi}[t^{\pm}]$ . In this subsection, we generalize it to a function on  $M(\mathcal{K}_{\Gamma})$ .

First, we extend  $\deg^{\psi} : \mathcal{K}_{\Gamma\psi}[t^{\pm}] \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$  to  $\deg^{\psi} : \mathcal{K}_{\Gamma} \rightarrow \mathbb{Z} \cup \{\infty\}$  by setting  $\deg^{\psi}(fg^{-1}) = \deg^{\psi}(f) - \deg^{\psi}(g)$  for  $f \in \mathbb{Z}\Gamma, g \in \mathbb{Z}\Gamma - \{0\}$  (see Proposition 9.1.1 in [3], for example). It induces a group homomorphism  $\deg^{\psi} : (\mathcal{K}_{\Gamma}^{\times})_{\text{ab}} \rightarrow \mathbb{Z}$ , where  $(\mathcal{K}_{\Gamma}^{\times})_{\text{ab}}$  is the abelianization of the multiplicative group  $\mathcal{K}_{\Gamma}^{\times} = \mathcal{K}_{\Gamma} - \{0\}$ .

For the skew field  $\mathcal{K}_{\Gamma}$ , we have the Dieudonné determinant

$$\det : GL(\mathcal{K}_{\Gamma}) \longrightarrow (\mathcal{K}_{\Gamma}^{\times})_{\text{ab}},$$

which is a homomorphism. This map is characterized by the following three properties:

- (a)  $\det I = 1$ ,
- (b) If  $A'$  is obtained by multiplying a row of a matrix  $A \in GL(\mathcal{K}_{\Gamma})$  by  $a \in \mathcal{K}_{\Gamma}^{\times}$  from the left, then  $\det A' = a \cdot \det A$ .
- (c) If  $A'$  is obtained by adding to a row of a matrix  $A$  a left  $\mathcal{K}_{\Gamma}$ -linear combination of other rows, then  $\det A' = \det A$ .

It is well known that this determinant induces an isomorphism  $K_1(\mathcal{K}_{\Gamma}) \xrightarrow{\cong} (\mathcal{K}_{\Gamma}^{\times})_{\text{ab}}$ .

The following lemma will be used in our generalization of Harvey's invariants. We take  $A \in M(m, n, \mathcal{K}_{\Gamma})$ , where  $M(m, n, \mathcal{K}_{\Gamma})$  is the set of all  $m \times n$  matrices with entries in  $\mathcal{K}_{\Gamma}$ . Assume that  $\text{rank}_{\mathcal{K}_{\Gamma}} A = k$ .

**Lemma 2.1** ([15, Lemma 10.1]). *Let  $U \in M(m - k, m, \mathcal{K}_{\Gamma})$ ,  $V \in M(n, n - k, \mathcal{K}_{\Gamma})$  be matrices satisfying*

$$\begin{cases} UA = 0, & \text{rank}_{\mathcal{K}_{\Gamma}} U = m - k, \\ AV = 0, & \text{rank}_{\mathcal{K}_{\Gamma}} V = n - k. \end{cases}$$

For each  $I \subset \{1, 2, \dots, m\}$ ,  $J \subset \{1, 2, \dots, n\}$  with  $\#I = m - k$ ,  $\#J = n - k$ , let  $U_I$  denote the square matrix defined by taking  $i$ -th columns from  $U$  for all  $i \in I$ , and  $V_J$  denote the one defined by taking  $j$ -th rows from  $V$  for all  $j \in J$ . We also denote by  $A_{I^c J^c}$  the one defined by taking  $i$ -th rows from  $A$  for all  $i \in I^c := \{1, 2, \dots, m\} - I$  and then taking  $j$ -th columns for all  $j \in J^c := \{1, 2, \dots, n\} - J$ .

- (1) If  $U_I$  or  $V_J$  is not invertible, then  $A_{I^c J^c}$  is not invertible.

(2) Otherwise,

$$\Delta(A; U, V) := \operatorname{sgn}(II^c) \operatorname{sgn}(JJ^c) \frac{\det A_{I^c J^c}}{\det U_I \det V_J} \in (\mathcal{K}_\Gamma^\times)_{\text{ab}}$$

is independent of the choice of  $I$  and  $J$  such that  $U_I, V_J$  are invertible, where  $\operatorname{sgn}(II^c) \in \{\pm 1\}$  (resp.  $\operatorname{sgn}(JJ^c)$ ) is the signature of the juxtaposition of  $I$  and  $I^c$  (resp.  $J$  and  $J^c$ ), and we put  $\det \emptyset := 1$ .

(3) For  $P_1 \in GL(m, \mathcal{K}_\Gamma)$ ,  $P_2 \in GL(n, \mathcal{K}_\Gamma)$ ,  $Q_1 \in GL(m-k, \mathcal{K}_\Gamma)$  and  $Q_2 \in GL(n-k, \mathcal{K}_\Gamma)$ , we have

$$\Delta(P_1^{-1}AP_2^{-1}; Q_1UP_1, P_2VQ_2) = \frac{\Delta(A; U, V)}{\det P_1 \det P_2 \det Q_1 \det Q_2}.$$

As we see in Lemma 2.1 (3),  $\Delta(A; U, V)$  does depend on  $U$  and  $V$ . The following definition and lemma give particular choices of  $U$  and  $V$  for our purpose. Recall that  $\mathcal{K}_{\Gamma\psi}[t^\pm] \subset \mathcal{K}_\Gamma$ .

**Definition 2.2.**  $(U, V)$  is said to be  $\psi$ -primitive for  $A$  if

- (1)  $U, V$  have entries in  $\mathcal{K}_{\Gamma\psi}[t^\pm]$ .
- (2) The row vectors  $u_1, \dots, u_{m-k} \in (\mathcal{K}_{\Gamma\psi}[t^\pm])_m$  of  $U$  generate  $\operatorname{Ker}(\cdot A) \cap (\mathcal{K}_{\Gamma\psi}[t^\pm])_m$  in  $(\mathcal{K}_\Gamma)_m$  as a left  $\mathcal{K}_{\Gamma\psi}[t^\pm]$ -module.
- (3)  $V$  has a property similar to (2) with respect to the column vectors.

**Lemma 2.3** ([15, Lemma 10.3]). (1) There exists a pair  $(U, V)$  which is  $\psi$ -primitive for  $A$ .

(2) If  $(U', V')$  is also  $\psi$ -primitive for  $A$ , then there exist  $P_1 \in GL(m, \mathcal{K}_{\Gamma\psi}[t^\pm])$ ,  $P_2 \in GL(n, \mathcal{K}_{\Gamma\psi}[t^\pm])$ ,  $Q_1 \in GL(m-k, \mathcal{K}_{\Gamma\psi}[t^\pm])$  and  $Q_2 \in GL(n-k, \mathcal{K}_{\Gamma\psi}[t^\pm])$  such that

$$UP_1 = U', \quad P_2V = V', \quad Q_1U = U', \quad VQ_2 = V'.$$

**Definition 2.4.** Let  $G$  be a PTFA group, and let  $\psi : \Gamma \rightarrow \mathbb{Z}$  is an epimorphism.

(1) The torsion-degree function  $d_\Gamma^\psi : M(\mathcal{K}_\Gamma) \rightarrow \mathbb{Z}$  is defined by

$$d_\Gamma^\psi(A) := \deg^\psi(\Delta(A; U, V))$$

for a pair  $(U, V)$  which is  $\psi$ -primitive for  $A$ .

(2) The truncated torsion-degree function  $\bar{d}_\Gamma^\psi : M(\mathcal{K}_\Gamma) \rightarrow \mathbb{Z} \cup \{\infty\}$  is defined by

$$\bar{d}_\Gamma^\psi(A) := \begin{cases} d_\Gamma^\psi(A) & \text{if } \operatorname{rank} A \geq m-1, \\ \infty & \text{otherwise} \end{cases}$$

for  $A \in M(m, n, \mathcal{K}_\Gamma)$ .

Lemma 2.3 together with the fact that  $\deg^\psi(\det P) = 0$  for any  $P \in GL(\mathcal{K}_{\Gamma\psi}[t^\pm])$  shows that these functions are well-defined.

**Example 2.5.** (1) For  $A \in GL(\mathcal{K}_\Gamma)$ , we have  $d_\Gamma^\psi(A) = \bar{d}_\Gamma^\psi(A) = \deg^\psi(\det A)$ .

(2) Let  $M$  be a finitely generated right  $\mathcal{K}_{\Gamma\psi}[t^\pm]$ -module, and let  $A$  be a presentation matrix of  $M$ . Then we have  $d_\Gamma^\psi(A) = \dim_{\mathcal{K}_{\Gamma\psi}[t^\pm]}(T_{\mathcal{K}_{\Gamma\psi}[t^\pm]}M)$ . As for  $\bar{d}_\Gamma^\psi(A)$ , we can see that  $\bar{d}_\Gamma^\psi(A) \in \mathbb{Z}$  if and only if the rank of the  $\mathcal{K}_{\Gamma\psi}[t^\pm]$ -free part of  $M$  is less than 2.

(3) Let  $G$  be a finitely presentable group and we take a presentation  $\langle x_1, \dots, x_l \mid r_1, \dots, r_m \rangle$

of  $G$ . Let  $(\varphi_\Gamma, \varphi)$  be an admissible pair for  $G$ . The Jacobi matrix  $A := \left( \frac{\partial r_j}{\partial x_i} \right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}}^{\varphi_\Gamma}$  at  $\mathcal{K}_{\Gamma\psi}[t^\pm]$  gives a presentation matrix of  $H_1(G, \{1\}; \mathcal{K}_{\Gamma\psi}[t^\pm])$ . Then Harvey's invariants are given by

$$\begin{aligned} \delta_\Gamma^\psi(G) &= \dim_{\mathcal{K}_{\Gamma\psi}}(T_{\mathcal{K}_{\Gamma\psi}[t^\pm]} H_1(G; \mathcal{K}_{\Gamma\psi}[t^\pm])) = d_\Gamma^\psi(A), \\ \bar{\delta}_\Gamma^\psi(G) &= \dim_{\mathcal{K}_{\Gamma\psi}}(H_1(G; \mathcal{K}_{\Gamma\psi}[t^\pm])) = \bar{d}_\Gamma^\psi(A), \end{aligned}$$

where the second equality of each case follows from the direct sum decomposition

$$H_1(G, \{1\}; \mathcal{K}_{\Gamma\psi}[t^\pm]) \cong H_1(G; \mathcal{K}_{\Gamma\psi}[t^\pm]) \oplus \mathcal{K}_{\Gamma\psi}[t^\pm]$$

shown by Harvey in [7].

**Remark 2.6.** Friedl [4] gave an interpretation of Harvey's invariants by Reidemeister torsions. The definition of our truncated torsion-degree functions has some overlaps with his description.

### 3. HOMOLOGY COBORDISMS OF SURFACES

We proceed all our discussion in PL or smooth category.

Let  $\Sigma_{g,1}$  ( $g \geq 0$ ) be a compact connected oriented surface of genus  $g$  with one boundary component. We take a base point  $p$  on the boundary of  $\Sigma_{g,1}$ , and take  $2g$  loops  $\gamma_1, \dots, \gamma_{2g}$  of  $\Sigma_{g,1}$  as shown in Figure 1. We consider them to be an embedded bouquet  $R_{2g}$  of  $2g$ -circles tied at the base point  $p \in \partial\Sigma_{g,1}$ . Then  $R_{2g}$  and the boundary loop  $\zeta$  of  $\Sigma_{g,1}$  together with one 2-cell make up a standard CW-decomposition of  $\Sigma_{g,1}$ . It is well-known that the fundamental group  $\pi_1\Sigma_{g,1}$  of  $\Sigma_{g,1}$  is isomorphic to the free group  $F_{2g}$  of rank  $2g$  generated by  $\gamma_1, \dots, \gamma_{2g}$ , in which  $\zeta = \prod_{i=1}^g [\gamma_i, \gamma_{g+i}]$ .

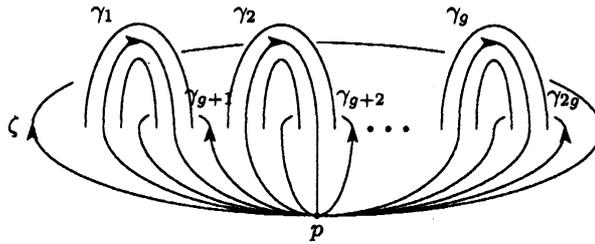


Figure 1

A *homology cylinder*  $(M, i_+, i_-)$  over  $\Sigma_{g,1}$ , whose origin is in Habiro [6], Garoufalidis-Levine [5] and Levine [11], is a homology cobordism  $M$  from  $\Sigma_{g,1}$  to itself together with two markings of boundaries, namely it consists of a compact oriented 3-manifold  $M$  and two embeddings  $i_+, i_- : \Sigma_{g,1} \rightarrow \partial M$  satisfying that

- (1)  $i_+$  is orientation-preserving and  $i_-$  is orientation-reversing,
- (2)  $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$  and  $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$ ,
- (3)  $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$ ,

(4)  $i_+, i_- : H_*(\Sigma_{g,1}) \rightarrow H_*(M)$  are isomorphisms.

We denote  $i_+(p) = i_-(p)$  by  $p \in \partial M$  again and consider it to be the base point of  $M$ . We write a homology cylinder by  $(M, i_+, i_-)$  or simply by  $M$ .

Two homology cylinders are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism between the underlying 3-manifolds which is compatible with the markings. We denote the set of isomorphism classes of homology cylinders by  $\mathcal{C}_{g,1}$ . Given two homology cylinders  $M = (M, i_+, i_-)$  and  $N = (N, j_+, j_-)$ , we can define a new homology cylinder  $M \cdot N$  by

$$M \cdot N = (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$

Then  $\mathcal{C}_{g,1}$  becomes a monoid with the identity element  $1_{\mathcal{C}_{g,1}} := (\Sigma_{g,1} \times I, \text{id} \times 1, \text{id} \times 0)$ .

From the monoid  $\mathcal{C}_{g,1}$ , we can construct the *homology cobordism group*  $\mathcal{H}_{g,1}$  of homology cylinders as in the following way. Two homology cylinders  $M = (M, i_+, i_-)$  and  $N = (N, j_+, j_-)$  are *homology cobordant* if there exists a compact oriented 4-manifold  $W$  such that

- (1)  $\partial W = M \cup (-N) / (i_+(x) = j_+(x), i_-(x) = j_-(x)) \quad x \in \Sigma_{g,1}$ ,
- (2) the inclusions  $M \hookrightarrow W, N \hookrightarrow W$  induce isomorphisms on the homology,

where  $-N$  is  $N$  with opposite orientation. We denote by  $\mathcal{H}_{g,1}$  the quotient set of  $\mathcal{C}_{g,1}$  with respect to the equivalence relation of homology cobordism. The monoid structure of  $\mathcal{C}_{g,1}$  induces a group structure of  $\mathcal{H}_{g,1}$ . In the group  $\mathcal{H}_{g,1}$ , the inverse of  $(M, i_+, i_-)$  is given by  $(-M, i_-, i_+)$ .

**Example 3.1.** For each element  $\varphi$  of the mapping class group  $\mathcal{M}_{g,1}$  of  $\Sigma_{g,1}$ , we can construct a homology cylinder  $M_\varphi \in \mathcal{C}_{g,1}$  defined by

$$M_\varphi := (\Sigma_{g,1} \times I, \text{id} \times 1, \varphi \times 0),$$

where collars of  $i_+(\Sigma_{g,1})$  and  $i_-(\Sigma_{g,1})$  are stretched half-way along  $\partial \Sigma_{g,1} \times I$ . This gives an injective monoid homomorphism  $\mathcal{M}_{g,1} \hookrightarrow \mathcal{C}_{g,1}$  and also  $\mathcal{M}_{g,1} \hookrightarrow \mathcal{H}_{g,1}$ . We consider  $\mathcal{C}_{g,1}$  and  $\mathcal{H}_{g,1}$  to be enlargements of  $\mathcal{M}_{g,1}$ .

Let  $N_k(G) := G/(\Gamma^k G)$  be the  $k$ -th nilpotent quotient of a group  $G$ , where we define  $\Gamma^1 G = G$  and  $\Gamma^{i+1} G = [\Gamma^i G, G]$  for  $i \geq 1$ . For simplicity, we write  $N_k(X)$  for  $N_k(\pi_1 X)$  where  $X$  is a CW-complex, and write  $N_k$  for  $N_k(F_{2g}) = N_k(\Sigma_{g,1})$ . It is known that  $N_k$  is a torsion-free nilpotent group for each  $k \geq 2$ .

Let  $(M, i_+, i_-)$  be a homology cylinder. By definition,  $i_+, i_- : \pi_1 \Sigma_{g,1} \rightarrow \pi_1 M$  are 2-connected, namely they induce isomorphisms on  $H_1$  and epimorphisms on  $H_2$ . Then, by Stallings' theorem [16],  $i_+, i_- : N_k \xrightarrow{\cong} N_k(M)$  are isomorphisms for each  $k \geq 2$ . Using them, we obtain a monoid homomorphism

$$\sigma_k : \mathcal{C}_{g,1} \longrightarrow \text{Aut} N_k \quad ((M, i_+, i_-) \mapsto (i_+)^{-1} \circ i_-).$$

It is easily checked that  $\sigma_k$  induces a group homomorphism  $\sigma_k : \mathcal{H}_{g,1} \rightarrow \text{Aut}N_k$ . We define filtrations of  $\mathcal{C}_{g,1}$  and  $\mathcal{H}_{g,1}$  by

$$\begin{aligned} \mathcal{C}_{g,1}[1] &:= \mathcal{C}_{g,1}, & \mathcal{C}_{g,1}[k] &:= \text{Ker} \left( \mathcal{C}_{g,1} \xrightarrow{\sigma_k} \text{Aut}N_k \right) \text{ for } k \geq 2, \\ \mathcal{H}_{g,1}[1] &:= \mathcal{H}_{g,1}, & \mathcal{H}_{g,1}[k] &:= \text{Ker} \left( \mathcal{H}_{g,1} \xrightarrow{\sigma_k} \text{Aut}N_k \right) \text{ for } k \geq 2. \end{aligned}$$

#### 4. APPLICATIONS OF TORSION-DEGREE FUNCTIONS TO HOMOLOGY CYLINDERS

In this section, we define and study some invariants of homology cylinders arising from the Magnus representation, twisted homology groups of related manifolds and (truncated) torsion-degree functions associated to nilpotent quotients  $N_k$  of  $\pi_1 \Sigma_{g,1}$ . For each  $k \geq 2$ ,  $N_k$  is known to be a finitely generated torsion-free nilpotent group. In particular, it is PTFA. Since  $H_1(N_k) = H_1(N_2) = H_1(\Sigma_{g,1})$  and  $H^1(N_k) = H^1(N_2) = H^1(\Sigma_{g,1})$ , taking an epimorphism  $N_k \rightarrow \mathbb{Z}$ , which is needed in the definition of a torsion-degree function, is done by choosing a primitive element of  $H^1(\Sigma_{g,1})$ .

Let  $(M, i_+, i_-) \in \mathcal{C}_{g,1}$  be a homology cylinder. By Stallings' theorem,  $N_k$  and  $N_k(M)$  are isomorphic. We consider the right quotient field  $\mathcal{K}_{N_k}$  (resp.  $\mathcal{K}_{N_k(M)}$ ) of  $\mathbb{Z}N_k$  (resp.  $\mathbb{Z}N_k(M)$ ) to be a local coefficient system on  $\Sigma_{g,1}$  (resp.  $M$ ). By a simple argument using covering spaces, we have the following.

**Lemma 4.1.**  $i_{\pm} : H_*(\Sigma_{g,1}, p; i_{\pm}^* \mathcal{K}_{N_k(M)}) \rightarrow H_*(M, p; \mathcal{K}_{N_k(M)})$  are isomorphisms as right  $\mathcal{K}_{N_k(M)}$ -vector spaces.

This lemma yields various applications of torsion-degree functions to homology cylinders.

**4.1. Magnus representations and torsion-degree functions.** As a first application of Lemma 4.1, we define a matrix-valued invariant of  $\mathcal{C}_{g,1}$  and  $\mathcal{H}_{g,1}$ . The following construction is based on Kirk-Livingston-Wang's paper [9].

We fix an integer  $k \geq 2$ . Since  $R_{2g} \subset \Sigma_{g,1}$  is a deformation retract, we have

$$H_1(\Sigma_{g,1}, p; i_{\pm}^* \mathcal{K}_{N_k(M)}) \cong H_1(R_{2g}, p; i_{\pm}^* \mathcal{K}_{N_k(M)}) = C_1(\widetilde{R_{2g}}) \otimes_{\pi_1 R_{2g}} i_{\pm}^* \mathcal{K}_{N_k(M)} \cong \mathcal{K}_{N_k(M)}^{2g}$$

with a basis

$$\{\overline{\gamma_1} \otimes 1, \dots, \overline{\gamma_{2g}} \otimes 1\} \subset C_1(\widetilde{R_{2g}}) \otimes_{\pi_1 R_{2g}} i_{\pm}^* \mathcal{K}_{N_k(M)}$$

as a right free  $\mathcal{K}_{N_k(M)}$ -module, where  $\overline{\gamma_i}$  is a lift of  $\gamma_i$  on the universal covering  $\widetilde{R_{2g}}$ .

**Definition 4.2.** (1) For each  $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ , we denote by  $r'_k(M) \in GL(2g, \mathcal{K}_{N_k(M)})$  the representation matrix of the right  $\mathcal{K}_{N_k(M)}$ -isomorphism

$$\mathcal{K}_{N_k(M)}^{2g} \cong H_1(\Sigma_{g,1}, p; i_-^* \mathcal{K}_{N_k(M)}) \xrightarrow[i_+^{-1} \circ i_-]{\cong} H_1(\Sigma_{g,1}, p; i_+^* \mathcal{K}_{N_k(M)}) \cong \mathcal{K}_{N_k(M)}^{2g}$$

(2) The *Magnus representation* for  $\mathcal{C}_{g,1}$  is the map  $r_k : \mathcal{C}_{g,1} \rightarrow GL(2g, \mathcal{K}_{N_k})$  which assigns to  $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$  the matrix  $i_+^{-1} r'_k(M)$ .

While we call  $r_k(M)$  the Magnus "representation", it is actually a crossed homomorphism.

**Theorem 4.3** ([15, Theorem 7.12]). For  $M_1 = (M_1, i_+, i_-)$ ,  $M_2 = (M_2, j_+, j_-) \in \mathcal{C}_{g,1}$ , we have

$$r_k(M_1 \cdot M_2) = r_k(M_1) \cdot {}^{\sigma_k(M_1)}r_k(M_2).$$

Moreover, we can show the following.

**Theorem 4.4** ([15, Theorem 7.13]).  $r_k : \mathcal{C}_{g,1} \rightarrow GL(2g, \mathcal{K}_{N_k})$  factors through  $\mathcal{H}_{g,1}$ .

Consequently, we obtain the Magnus representation  $r_k : \mathcal{H}_{g,1} \rightarrow GL(2g, \mathcal{K}_{N_k})$ , which is a crossed homomorphism. Note that if we restrict  $r_k$  to  $\mathcal{C}_{g,1}[k]$  (and  $\mathcal{H}_{g,1}[k]$ ), it becomes a homomorphism. In what follows, we use  $\tilde{r}_k := \overline{r_k(\cdot)}^T$  instead of  $r_k$  by a technical reason.  $\tilde{r}_k$  is a crossed-anti-homomorphism.

We now define some numerical invariants by using  $I_{2g} - \tilde{r}_k(M)$  for  $(M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$ . Recall that for every homology cylinder  $(M, i_+, i_-)$  belonging to  $\mathcal{C}_{g,1}[k]$ , two inclusions  $i_+$  and  $i_-$  induce the same isomorphism  $i_+ = i_- : N_k \xrightarrow{\cong} N_k(M)$ , so that we can naturally identify them. Under this identification, we have the following.

**Lemma 4.5** ([15, Theorem 11.1]). Let  $M$  be a homology cylinder belonging to  $\mathcal{C}_{g,1}[k]$ .

- (1)  $(I_{2g} - \tilde{r}_k(M))(1 - \gamma_1, \dots, 1 - \gamma_{2g})^T = 0$ ,
- (2)  $\left( \frac{\partial \zeta}{\partial \gamma_1}, \dots, \frac{\partial \zeta}{\partial \gamma_{2g}} \right) (I_{2g} - \tilde{r}_k(M)) = 0$ ,

where  $\partial/\partial \gamma_i$  is the ordinary free differential (and we send it to  $\mathbb{Z}N_k$ ).

We consider  $\bar{d}_{N_k}^\psi(I_{2g} - \tilde{r}_k(M))$  to be an invariant of  $M$ . By Lemma 4.5, the rank of  $I_{2g} - \tilde{r}_k(M)$  is at most  $2g - 1$ . As  $I_{2g} - \tilde{r}_k(1_{\mathcal{C}_{g,1}}) = 0_{2g}$  indicates, however, the rank is not necessarily equal to  $2g - 1$ . That is,  $\bar{d}_{N_k}^\psi(I_{2g} - \tilde{r}_k(M))$  has a possibility of being  $\infty$ . Such a situation corresponds to the vanishing of the Alexander polynomial of the closing of a homology cylinder as we will see in Remark 4.9.

Note that  $\bar{d}_{N_k}^\psi(I_{2g} - \tilde{r}_k(M))$  is a homology cobordism invariant since  $\tilde{r}_k(M)$  is. We can show that it does not depend on the choice of a generating system of  $\pi_1 \Sigma_{g,1}$ .

**4.2.  $N_k$ -torsions and torsion-degree functions.** In this subsection, we identify  $N_k$  with  $N_k(M)$  by using  $i_+$  for each homology cylinder  $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ .

Since the relative complex  $C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k})$  obtained from any smooth triangulation of  $(M, i_+(\Sigma_{g,1}))$  is acyclic by Lemma 4.1, we can consider its Reidemeister torsion

$$\tau_{N_k}(M) := \tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k})) \in K_1(\mathcal{K}_{N_k})/(\pm N_k).$$

We now call this the  $N_k$ -torsion of  $M$ . Recall that Reidemeister torsions are invariant under subdivision of the CW-complex  $(M, i_+(\Sigma_{g,1}))$  and simple homotopy equivalence. We refer to [13] and [18] for generalities of Reidemeister torsions.

By a topological consideration, we can show that

$$\bar{d}_{N_k}^\psi(\tau_{N_k}(M)) = \bar{d}_{N_k}^\psi(\tau_{N_k}(M)) = \text{rank}_{\mathcal{K}_{N_k}^\psi} H_1(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k}^\psi[t^\pm]),$$

which can be computed from a presentation of  $\pi_1 M$ .

**Proposition 4.6** ([15, Proposition 11.2]). *Let  $M_1, M_2 \in \mathcal{C}_{g,1}$ . Then*

$$d_{N_k}^\psi(\tau_{N_k}(M_1 \cdot M_2)) = d_{N_k}^\psi(\tau_{N_k}(M_1)) + d_{N_k}^{\psi \cdot \sigma_2(M_1)}(\tau_{N_k}(M_2))$$

*holds for every primitive element  $\psi \in H^1(\Sigma_{g,1})$ .*

Note that if we restrict  $d_{N_k}^\psi(\tau_{N_k}(\cdot))$  to  $\mathcal{C}_{g,1}[2]$ , we obtain a monoid homomorphism from  $\mathcal{C}_{g,1}[2]$  to  $\mathbb{Z}_{\geq 0}$ .

**Remark 4.7.** Proposition 4.6 can be seen as a generalization of [10, Proposition 1.11].

### 4.3. Factorization formula of $N_k$ -degrees for the closing of a homology cylinder.

For each homology cylinder  $(M, i_+, i_-)$ , we can construct a closed 3-manifold

$$C_M := M / (i_+(x) = i_-(x)), \quad x \in \Sigma_{g,1}$$

called the *closing* of  $M$ . Note that if  $M \in \mathcal{C}_{g,1}[k]$ , we have a natural isomorphism

$$N_k = N_k(\Sigma_{g,1}) \cong N_k(M) \cong N_k(C_M).$$

In particular, we have  $H_1(\Sigma_{g,1}) = H_1(M) = H_1(C_M)$ .

**Theorem 4.8** ([15, Proposition 11.4]). *Let  $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$ . For each primitive element  $\psi \in H^1(N_k) = H^1(C_M)$ , we have*

$$\bar{\delta}_{N_k}^\psi(C_M) = d_{N_k}^\psi(\tau_{N_k}(M)) + \bar{d}_{N_k}^\psi(I_{2g} - \tilde{r}_k(M)) \in \mathbb{Z} \cup \{\infty\}.$$

**Remark 4.9** (The case of  $k = 2$ ). Since  $\mathbb{Z}N_2 = \mathbb{Z}N_2(\Sigma_g)$  and  $\mathcal{K}_{N_2} = \mathcal{K}_{N_2(\Sigma_g)}$  are commutative, we can use the ordinary determinant to calculate the invariants seen above. The following is a direct generalization of the formula for string links given in [9, Theorem 6.2]. For  $M \in \mathcal{C}_{g,1}[2]$ , we put

$$\Delta_{N_2}(M) := -\frac{\det((I_{2g} - \tilde{r}_2(M))_{(1,1)})}{(1 - \gamma_1)(1 - \gamma_{g+1})} \in \mathcal{K}_{N_2},$$

where  $A_{(i,j)}$  denotes the matrix obtained from a matrix  $A$  by removing its  $i$ -th row and  $j$ -th column. We call  $\Delta_{N_2}(M)$  the *Alexander rational function* of  $M$ . Then the Alexander polynomial  $\Delta_{C_M}$  of  $C_M$  decomposes as

$$\Delta_{C_M} \doteq \overline{\tau_{N_2}(M)} \cdot \Delta_{N_2}(M),$$

where  $\doteq$  means that these equalities hold in  $\mathcal{K}_{N_2}$  up to  $\pm N_2$ .

**4.4. Examples.** The formula in Theorem 4.8 holds as elements of  $\mathbb{Z} \cup \{\infty\}$ , so that the additivity loses its meaning when the value is  $\infty$ . Note that  $\bar{\delta}_{N_k}^\psi(C_M) = \infty$  if and only if  $\bar{d}_{N_k}^\psi(I_{2g} - \tilde{r}_k(M)) = \infty$ , and this occurs when  $H_1(C_M; \mathcal{K}_{N_k}^\psi[t^\pm])$  has a non-trivial free part.

The following are some examples of homology cylinders which have non-trivial Alexander rational functions. By using Theorem 4.12 in the next subsection, we obtain many situations where the formula sufficiently works. The computations for the cases of  $k \geq 3$  are generally quite difficult.

**Example 4.10.** Assume that  $g = 1$ . We denote by  $\tau_\zeta \in \mathcal{M}_{1,1}$  the Dehn twist along  $\zeta$ , which belongs to  $\mathcal{C}_{1,1}[3]$ . Then, we have

$$\tilde{r}_2(\tau_\zeta) = \begin{pmatrix} \gamma_1 + \gamma_2 - \gamma_1\gamma_2 & 1 - 2\gamma_1 + \gamma_1^2 \\ -1 + 2\gamma_2 - \gamma_2^2 & 2 - \gamma_1 - \gamma_2 + \gamma_1\gamma_2 \end{pmatrix}.$$

Then  $\Delta_{N_2}(\tau_\zeta) = -1 \in \mathbb{Z}N_2$ , which is non-trivial.

**Example 4.11.** Assume that  $g \geq 2$ . Let  $\tau_1, \tau_2$  and  $\tau_3$  be Dehn twists along simple closed curves  $c_1, c_2$  and  $c_3$  as in Figure 2, respectively.

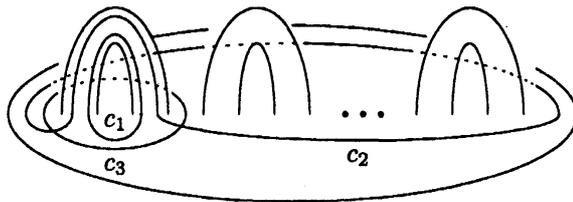


Figure 2

Then  $\tau_1\tau_2^{-1}, \tau_3 \in \mathcal{C}_{g,1}[2]$ . By a direct computation, we can check that

$$\Delta_{N_2}(\tau_1\tau_2^{-1} \cdot \tau_3) \doteq -(\gamma_1^{-1} - 1)^{2g-2},$$

while  $\Delta_{N_2}(\tau_1\tau_2^{-1}) = \Delta_{N_2}(\tau_3) = 0$ .

**4.5.  $N_k$ -torsion and Harvey's Realization Theorem.** As seen in Theorem 4.6, the degree of the  $N_k$ -torsion gives a monoid homomorphism

$$d_{N_k}^\psi(\tau_{N_k}(\cdot)) : \mathcal{C}_{g,1}[2] \longrightarrow \mathbb{Z}_{\geq 0}$$

for each primitive element  $\psi \in H^1(\Sigma_{g,1})$  and an integer  $k \geq 2$ . To see some properties of these homomorphisms, we use a variant of Harvey's Realization Theorem [7, Theorem 11.2], which gives a method for performing surgery on a compact orientable 3-manifold to obtain a homology cobordant one having distinct higher-order Alexander invariants. By Theorem 4.8, we can expect that a similar result holds for the degrees of  $N_k$ -torsions, and this is indeed the case.

**Theorem 4.12.** *Let  $M \in \mathcal{C}_{g,1}$  be a homology cylinder. For each primitive element  $x \in H_1(\Sigma_{g,1})$  and any integers  $n \geq 2$  and  $k \geq 1$ , there exists a homology cylinder  $M(n, k; x)$  such that*

- (1)  $M(n, k; x)$  is homology cobordant to  $M$ ,
- (2)  $d_{N_k}^\psi(\tau_{N_k}(M(n, k; x))) = d_{N_k}^\psi(\tau_{N_k}(M))$  for  $2 \leq i \leq n-1$ ,
- (3)  $d_{N_k}^\psi(\tau_{N_n}(M(n, k; x))) \geq d_{N_k}^\psi(\tau_{N_n}(M)) + k|p|$

for any primitive element  $\psi \in H^1(\Sigma_{g,1})$  satisfying  $\psi(x) = p$ .

**Corollary 4.13.** *The maps  $\{d_{N_k}^\psi(\tau_{N_k}(\cdot)) : \mathcal{C}_{g,1}[2] \rightarrow \mathbb{Z}_{\geq 0}\}_{k \geq 2}$  are all non-trivial homomorphisms, and independent of each other for any primitive element  $\psi \in H^1(\Sigma_{g,1})$ .*

In fact, we can show it by constructing homology cylinders that are homology cobordant to the unit  $1_{\mathcal{C}_{g,1}}$ . From this we see that  $\mathcal{C}_{g,1}[2], \mathcal{C}_{g,1}[3], \dots, \text{Ker}(\mathcal{C}_{g,1} \rightarrow \mathcal{H}_{g,1})$  are not finitely

generated as monoids. Note that  $d_{N_k}^\psi(\tau_{N_k}(M)) = 0$  if  $M \in \mathcal{M}_{g,1}$ , since  $\Sigma_{g,1} \times I$  is simple homotopy equivalent to  $\Sigma_{g,1}$ .

## 5. PROBLEMS

Finally, we raise the following problems.

**Problem 5.1.** Generalize the factorization formula (Theorem 4.8) to  $\delta_{N_k}^\psi(C_M)$ . Can we write it in terms of the Magnus representation and  $N_k$ -torsion?

Some partial answers to this problem are already obtained. For example, it is easily checked that  $\delta_{N_k}^\psi(C_{M_\varphi}) = d_{N_k}^\psi(I_{2g} - \tilde{r}_k(M_\varphi))$  for  $\varphi \in \mathcal{M}_{g,1}$ .

**Problem 5.2.** Compute higher-order Alexander invariants explicitly.

General cases seem to be quite difficult. In our setting, we need to consider only the cases of free nilpotent quotients  $N_k$ , whose group rings  $\mathbb{Z}N_k$  have somewhat easier structures. Difficulties are concentrated on Ore properties of  $\mathbb{Z}N_k$ .

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