BOUNDARY ESTIMATES OF $p$-HARMONIC FUNCTIONS IN A METRIC MEASURE SPACE

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1. INTRODUCTION

The purpose of this note is two-fold. First we discuss Carleson type estimates, which provide control of the bound of positive harmonic functions vanishing on a portion of the boundary. Such an estimate is well-known for harmonic functions in certain Euclidean domains. We shall prove a Carleson type estimate for $p$-harmonic functions on bounded John domains in a complete metric space equipped with an Ahlfors $Q$-regular measure supporting a $(1, p)$-Poincaré inequality for some $1 < p \leq Q$. This part is based on [4].

Secondly, we discuss the Hölder continuity of $p$-harmonic functions up to the boundary. It is classical that a domain is regular, then the Dirichlet solution of a continuous boundary function is continuous up to the boundary. It may be natural to think that the better continuity of a boundary function ensures the better continuity of the Dirichlet solution. We shall investigate conditions on a domain for every Hölder continuous boundary function to have Hölder continuous solution with the same Hölder exponent. Our results are new even in the Euclidean setting when $p \neq 2$. This part is based on [5].

2. CARLESON ESTIMATES FOR HARMONIC FUNCTIONS

Let us begin with the classical result due to Carleson.

**Theorem A** (Carleson [11]). Let $D$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists a constant $A > 1$ with the following property: for $\xi \in \partial D$ and $R > 0$ small, take a point $y_R \in D$ such that $|y_R - \xi| = R$ and $\text{dist}(y_R, \partial D) \geq R/A$. Then

$$u \leq Au(y_R) \quad \text{on} \ D \cap B(\xi, R).$$

2000 Mathematics Subject Classification. 31B05, 31B25, 31C35.

Keywords and phrases. Carleson estimate, $p$-harmonic function, metric measure space.

This work was supported in part by Grant-in-Aid for Scientific Research (B) (No. 15340046) Japan Society for the Promotion of Science.
whenever $u$ is a positive harmonic function in $D \cap B(\xi, AR)$ with $u = 0$ on $\partial D \cap B(\xi, AR)$.

Ever since the Carleson's work there have been a large number of studies on this subjects. Most of them generalize the domain $D$ and exploited harmonic analysis on non-smooth domains. There are several ways to prove the Carleson estimates in non-smooth domains:

(i) Carleson [11] and Jerison–Kenig [18] employed the uniform barrier. This argument requires the Capacity Density Condition for the complement of the domain.

(ii) In [1], the author prove the Carleson estimate by showing the Boundary Harnack principle first. The boundary Harnack principle was deduced from the estimates of the Green functions and representation of harmonic functions as the Green potential. This approach is not applicable to non-linear equations.

(iii) In the study of the Martin boundary of Denjoy domains, Benedicks [6] observed the Domar method [15] is useful. See Chevallier [13]. The Domar method is a very robust argument based on the sub-mean value property of subharmonic functions. In the sequel, we shall observe that the Domar method is applicable even to solutions of non-linear equations in metric measure spaces.

3. Metric measure space

Let $(X, d, \mu)$ be a proper metric measure space with doubling Borel measure $\mu$. Here we say that $X$ is proper if closed and bounded subsets of $X$ are compact; and that $\mu$ is doubling if there is a constant $A \geq 1$ such that

$$\mu(B(x, 2r)) \leq A\mu(B(x, r)),$$

where $B(x, r) = \{ y \in X : d(x, y) < r \}$ is the open ball with center $x$ and radius $r$. For simplicity, we assume that $X$ is Ahlfors $Q$-regular, i.e.,

$$A^{-1}r^Q \leq \mu(B(x, r)) \leq Ar^Q \quad \text{for every ball } B(x, r).$$

Throughout the note we fix $1 < p \leq Q$. We shall define the notion of $p$-harmonicity.

For a moment let $f$ be a smooth function on $\mathbb{R}^n$ and let $\overline{\gamma}$ be a rectifiable curve. Then

$$|f(x) - f(y)| = \left| \int_{\overline{\gamma}} \nabla f \cdot dx \right| \leq \int_{\overline{\gamma}} |\nabla f| ds.$$

In view of this observation, Heinonen-Koskela [17] defined an upper gradient of a function $f$ on a metric measure space $X$ to be $g \geq 0$ such that for
every rectifiable curve $\overline{xy} \subset X$

(3.1) \[ |f(x) - f(y)| \leq \int_{\overline{xy}} g ds. \]

The above requirement is somewhat too strong for the limiting operation. We say that $g$ is a \textit{weak upper gradient} of $f$ if $g$ satisfies (3.1) for all curves $\overline{xy}$ except for $p$-module zero. By $g_f$ we denote the \textit{minimal $p$-weak upper gradient} of $f$, i.e.,

\[ g_f(x) := \inf_{g} \left( \limsup_{r \to 0^+} \int_{B(x,r)} gd\mu \right). \]

The minimal $p$-weak upper gradient $g_f$ satisfies (3.1) for all curves $\overline{xy}$ except for $p$-module zero. See [23] for these accounts. We assume the following $(1, p)$-Poincaré inequality.

**Definition 1** $(1, p)$-Poincaré inequality. There exist constants $\kappa \geq 1$ (scaling constant) and $A_p \geq 1$ such that

\[ \int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq A_p r \theta_{B(x,\kappa r)}^{1/p} (\int_{B(x,\kappa r)} g_{u}^{p} d\mu)^{1/p} \]

whenever $B(x, r) \subset X$.

By the Hölder inequality $(1, q)$-Poincaré inequality with $q < p$ implies the $(1, p)$-Poincaré inequality. Conversely, Keith-Zhong [19] showed that if $X$ supports a $(1, p)$-Poincaré inequality, then there is $q < p$ such that $X$ supports a $(1, q)$-Poincaré inequality. Define the Sobolev space on $X$ as follows.

**Definition 2** (Sobolev or Newtonian space [23]). Define

\[ \|u\|_{N^{1,p}} := \left( \int_{X} |u|^{p} d\mu \right)^{1/p} + \left( \int_{X} g_{u}^{p} d\mu \right)^{1/p}. \]

If $\|u - v\|_{N^{1,p}} = 0$, then we write $u \sim v$. The \textit{Newtonian space} of $X$ is the quotient

\[ N^{1,p}(X) = \{ u : \|u\|_{N^{1,p}} < \infty \}/\sim \]

The space $N^{1,p}(X)$ equipped with the norm $\| \cdot \|_{N^{1,p}}$ is a Banach space and a lattice. Cheeger [12] gave an alternative definition of Sobolev space, which coincides with the above Newtonian space for $1 < p < \infty$. Moreover, the modulus of the Cheeger derivative and the minimum upper gradient are comparable:

\[ A^{-1}|df(x)| \leq g_f(x) \leq A|df(x)| \]

([24, Corollary 3.7]). If $f = A$ on $E$, then $g_f = |df| = 0 \mu$-a.e. on $E$ ([12, Proposition 2.2]).
**Definition 3.** Define the \( p \)-capacity of \( E \subset X \) by

\[
\text{Cap}_p(E) := \inf_u \left( \int_X |u|^p \, d\mu + \int_X |du|^p \, d\mu \right)
\]

Here \( \inf \) is taken over all \( u \in N^{1,p}(X) \) such that \( u = 1 \) on \( E \). We say that a property holds \( p \)-q.e. if it holds except for \( E \) with \( \text{Cap}_p(E) = 0 \).

Hereafter let \( \Omega \subset X \) be a bounded domain in \( X \) with \( \text{Cap}_p(X \setminus \Omega) > 0 \). The null-Sobolev space for \( \Omega \) is defined by

\[
N_{0}^{1,p}(\Omega) = \{ u \in N^{1,p}(X) : u = 0 \text{ p-q.e. on } X \setminus \Omega \}.
\]

**Definition 4.** We say that \( u \) is \( p \)-harmonic in \( \Omega \) if \( u \in N_{1_{\text{loc}}}^{1,p}(\Omega) \) and

\[
\int_U g_u^p \, d\mu \leq \int_U g_{u+\varphi}^p \, d\mu
\]

for all relatively compact subsets \( U \) of \( \Omega \) and for every function \( \varphi \in N_{0}^{1,p}(U) \). We say that \( u \) is Cheeger \( p \)-harmonic in \( \Omega \) if \( u \in N_{1_{\text{loc}}}^{1,p}(\Omega) \) and

\[
\int_U |du|^p \, d\mu \leq \int_U |d(u + \varphi)|^p \, d\mu
\]

for all relatively compact subsets \( U \) of \( \Omega \) and for every function \( \varphi \in N_{0}^{1,p}(U) \). This is equivalent to the Euler equation:

\[
\int_U |du|^{p-2} du \cdot d\varphi \, d\mu = 0.
\]

**Remark 1.** If \( p = 2 \), then the above Euler equation is linear and hence Cheeger 2-harmonicity is a linear property. On the other hand, the \( p \)-harmonicity based on the upper gradient has no Euler equation and hence it is non-linear even if \( p = 2 \).

**Definition 5.** We say that \( u \) is a \( p \)-subsolution if

\[
\int_U g_u^p \, d\mu \leq \int_U g_{u+\varphi}^p \, d\mu
\]

for all relatively compact subsets \( U \) of \( \Omega \) and for every function \( \varphi \in N_{0}^{1,p}(U) \). We say that \( u \) is a \( p \)-quasiminimizer if there exists \( A_{qm} \geq 1 \) such that

\[
\int_U g_u^p \, d\mu \leq A_{qm} \int_U g_{u+\varphi}^p \, d\mu
\]

for all relatively compact subsets \( U \) of \( \Omega \) and for every nonpositive function \( \varphi \in N_{0}^{1,p}(U) \). If the inequality holds for every nonpositive function \( \varphi \in N_{0}^{1,p}(U) \), then \( u \) is called \( p \)-quasisubminimizer.
It is easy to see that a Cheeger p-(sub)harmonic function is a p-quasi(sub)minimizer. Basic properties will be given for p-quasi(sub)minimizers, and hence p-(sub)harmonic functions and Cheeger p-(sub)harmonic functions can be treated simultaneously.

**Definition 6.** By $H_p^U f$ we denote the solution to the $p$-Dirichlet problem on the open set $U$ with boundary data $f \in N^{1,p}(U)$, i.e., $H_p^U f$ is $p$-harmonic in $U$ and $H_p^U f - f \in N_0^{1,p}(U)$. An upper semicontinuous function $u$ is said to be $p$-subharmonic in $\Omega$ if the comparison principle holds, i.e., if $f \in N^{1,p}(U)$ is continuous up to $\partial U$ and $u \leq f$ on $\partial U$, then $u \leq H_p^U f$ on $U$ for all relatively compact subsets $U$ of $\Omega$.

**Remark 2.** We summarize functions:

(i) A (Cheeger) $p$-harmonic function is a $p$-quasiminimizer.

(ii) A (Cheeger) $p$-subsolution is a $p$-quasisubminimizer.

(iii) A bounded (Cheeger) $p$-subharmonic function is a $p$-quasisubminimizer.

4. **DOMAR ARGUMENT**

Let $u \geq 0$ be a locally bounded $p$-quasisubminimizer. Then $u$ is in the De Giorgi class, $DG_p(\Omega)$, i.e., if $B(x,R) \subset \Omega$, then

$$\int_{\{v \in B(x,R) : u(y) > k\}} g_{u}^{p} \, d\mu \leq \frac{A}{(r-\rho)^p} \int_{\{v \in B(x,R) : u(y) > k\}} (u - k)^{p} \, d\mu$$

for every $k \in \mathbb{R}$ and $0 < \rho < r < R/\kappa$. Here $g_{u}$ is the minimal $p$-weak upper gradient of $u$ and $\kappa \geq 1$ is the scaling constant for the Poincaré inequality ([22, 20, 21]).

The above inequality is very strong; its repeated application, together with the De Giorgi method [14] yields the following estimate ([22]):

If $u \in DG_p(\Omega)$, $0 < R < \text{diam}(X)/3$, $B(x,R) \subset \Omega$, then for every $k_0 \in \mathbb{R}$

$$\sup_{B(x,R/2)} u \leq k_0 + A \left( \int_{B(x,R)} (u - k_0)^{p} \, d\mu \right)^{1/p}.$$  

Let $k_0 = 0$ and $u \geq 0$. We obtain the weak submean value inequality:

$$(\text{wsmv}) \quad u(x) \leq A_s \left( \int_{B(x,R)} u^{p} \, d\mu \right)^{1/p}.$$  

Here $A_s \geq 1$ is independent of $x$, $R$ and $u$. This inequality may be regarded as a sort of the mean value inequality for $p$-subharmonic functions. Although it is weak ($A_s > 1$), it is sufficient to employ the Domar method and to give the Carleson estimate.
Lemma 1 ([15]). Let $\Omega$ be a bounded open set and let $\delta_\Omega(x) = \text{dist}(x, X\setminus \Omega)$. Suppose $u \geq 0$ locally bounded on $\Omega$ satisfies (wsmv). If there exists a positive constant $\epsilon$ such that

$$I := \int_\Omega (\log^+ u)^{Q-1/\epsilon} d\mu < \infty,$$

then

$$u(x) \leq A \exp(\frac{AI^{1/\epsilon} \delta_\Omega(x)^{-Q/\epsilon}}{\log^+ u(x)}) \text{ for all } x \in \Omega.$$

Let us prepare the following estimate.

Lemma 2. Suppose $u \geq 0$ satisfies (wsmv) and locally bounded on $B(x, R)$. Let $a > 2A_s$ and $0 < t \leq u(x)$. If

$$\mu((y \in B(x, R) : \frac{t}{a} < u(y) \leq at)) \leq \frac{\mu(B(x, R))}{a^{2p}},$$

then there exists a point $x' \in B(x, R)$ with $u(x') > at$.

Proof. Suppose $u \leq at$ on $B(x, R)$. Then (wsmv) gives

$$t \leq \frac{A_s}{\mu(B(x, R))} \left( \int_{B(x, R) \cap \{u \leq a^{-1}t\}} u(y)^{p} dy + \int_{B(x, R) \cap \{u > a^{-1}t\}} u(y)^{p} dy \right)^{1/p}$$

$$\leq A_s \left( \frac{t^p}{a} + \frac{(at)^p}{a^{2p}} \right)^{1/p} \leq \frac{2^{1/p}A_s}{a} t < 2^{1/p-1}t.$$

This is a contradiction. $\square$

Proof of Lemma 1. Observe $\mu(B(y, r)) \geq \frac{r^Q}{A_1}$ for $0 < r < 2 \text{ diam}(\Omega)$. Let

$$R_j = (A_1a^{2p}\mu((y \in \Omega : a^{j-2}u(x) < u(y) \leq a^j u(x))))^{1/Q},$$

which means

$$\mu((y \in \Omega : a^{j-2}u(x) < u(y) \leq a^j u(x))) \leq \frac{R_j^Q}{A_1a^{2p}} \leq \frac{\mu(B(x, R_j))}{a^{2p}}.$$

Then the lemma is proved by the following procedure:

- $\delta_\Omega(x) \leq 2 \sum_{j=1}^\infty R_j.$
- $\sum_{j=1}^\infty R_j \leq AI^{1/Q}(\log^+ u(x))^{-\epsilon/Q}.$
- $u(x) \leq \exp(\frac{AI^{1/\epsilon} \delta_\Omega(x)^{-Q/\epsilon}}{\log^+ u(x)}).$
Let us illustrate the most crucial step (i): Let \( x_1 = x, \ t = u(x_1) \). If \( \delta_\Omega(x_1) < R_1 \), then STOP. Otherwise \( B(x_1, R_1) \subset \Omega \), so
\[
\mu(\{y \in B(x_1, R_1) : a^{-1}u(x) < u(y) \leq au(x)\}) \leq \mu(B(x_1, R_1)) \leq \frac{\mu(B(x_1, R_1))}{a^{2p}}.
\]
By Lemma 2 we find \( x_2 \in B(x_1, R_1) \) with \( u(x_2) > au(x_1) \). If \( \delta_\Omega(x_2) < R_2 \), then STOP. Otherwise \( B(x_2, R_2) \subset \Omega \), and we find \( x_3 \in B(x_2, R_2) \) with \( u(x_3) > au(x_2) > a^2u(x_1) \). Repeat the procedure. Since \( u \) is locally bounded above, \( \{x_j\} \) is finite or \( x_j \to \partial \Omega \). This gives \( \delta_\Omega(x) \leq 2 \sum_{j=1}^\infty R_j \). □

5. CARLESON ESTIMATE FOR \( p \)-HARMONIC FUNCTIONS

A bounded domain \( D \) is called a uniform domain if for every couple of points \( x, y \in D \) there exists a curve \( \gamma \subset D \) connecting \( x \) and \( y \) such that
\[
\ell(y) \leq Ad(x, y), \\
\min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \leq A\delta_\Omega(z) \quad (z \in \gamma).
\]
A Lipschitz domain and an NTA domain are uniform domains. Roughly speaking, a uniform domain is a domain satisfying the interior conditions for an NTA domain.

A bounded domain \( D \) is called a John domain with John center \( x_0 \) if the above condition holds with one fixed point \( y = x_0 \) and varying \( x \in D \). Define the quasi hyperbolic metric by
\[
k_D(x, y) = \inf_{\bar{xy}} \int_{\bar{xy}} \frac{ds}{\delta_D(z)},
\]
where \( \inf \) is taken over all curves \( \bar{xy} \) connecting \( x \) and \( y \) in \( D \). A John domain \( D \) satisfies the quasihyperbolic boundary condition
\[
k_D(x, x_0) \leq A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A.
\]
This condition can be localized as follows.

**Definition 7** (Local reference points [3]). A boundary point \( \xi \in \partial D \) is said to have a system of local reference points of order \( N \) if there exist \( R_\xi > 0, \lambda_\xi > 1 \) and \( A_\xi > 1 \) with the following property: if \( 0 < R < R_\xi \), then we find
$N$ points $y_1, \ldots, y_N \in D \cap S(\xi, R)$ such that $\delta_D(y_j) \geq R/A_{\xi}$ and such that for every $x \in D \cap B(\xi, R/2)$ there is $i \in \{1, \ldots, N\}$ such that

$$k_D(x, y_i) = k_{D \cap B(\xi, A_{\xi}R)}(x, y_i) \leq A_{\xi} \left[ \log \left( \frac{R}{\delta_D(x)} \right) + 1 \right].$$

**Remark 3.** If $D$ is a uniform domain, then every boundary point $\xi \in \partial D$ has a system of local reference points of order 1; the constants $R_{\xi}, A_{\xi}, A_{\xi}$ can be taken independently on $\xi$.

If $D$ is a John domain, then there exists a finite number $N$ such that each $\xi \in \partial D$ has a system of local reference points of order $N$; the constants $R_{\xi}, A_{\xi}, A_{\xi}$ can be taken independently on $\xi$. In general $N \geq 2$. If $D$ is a Denjoy domain, then $N = 2$.

**Theorem 1** (Carleson estimate for a John domain). Let $D$ be a John domain with $\xi \in \partial D$. For small $R > 0$ take local reference points $y_1, \ldots, y_N \in D \cap S(\xi, R)$. Suppose $h > 0$ is a bounded $p$-harmonic function on $D \cap B(\xi, 16R)$ with $h = 0$ on $\partial D \cap B(\xi, 16R)$.

Then $h(x) \leq A \sum_{i=1}^{N} h(y_i)$ for $x \in D \cap B(\xi, R/4)$.

**Corollary 1** (Carleson estimate for a uniform domain). Let $D$ be a uniform domain with $\xi \in \partial D$. For small $R > 0$ take a nontangential point $y_R \in D \cap S(\xi, R)$, i.e., $\delta_D(y_R) \geq R/A$. Suppose $h > 0$ is a bounded $p$-harmonic function on $D \cap B(\xi, A_{\xi}R)$ with $h = 0$ on $\partial D \cap B(\xi, A_{\xi}R)$. Then $h(x) \leq Ah(y_R)$ for $x \in D \cap B(\xi, R)$. Here $A > 1$ depends only on $D$.

**Proof.** Let us give a sketch of the proof. In view of the geometry of a uniform domain, we have

$$k_D(x, y_R) \leq A \log \frac{R}{\delta_D(x)} + A \quad \text{for} \quad x \in D \cap B(\xi, A_{\xi}R).$$
Then the Harnack inequality gives
\[ u(x) = \frac{h(x)}{h(y_R)} \leq A \left( \frac{R}{\delta_D(x)} \right)^\lambda. \]

Extend \( u \) by \( u = 0 \) on \( B(\xi, AR) \setminus D \). Then the extended function is a \( p \)-subsolution \( h \) on \( \Omega = B(\xi, AR) \) with (wsnv).

An elementary geometrical observation gives
\[ I = \int_{\Omega} \left( \log^+ \left( \frac{h(x)}{h(y_R)} \right) \right)^{Q-1+\epsilon} d\mu \leq A \int_{D \cap B(\xi, AR)} \left( \log^+ \left( \frac{R}{\delta_D(x)} \right)^\lambda \right)^{Q-1+\epsilon} d\mu \leq AR^{Q}. \]

Hence the Domar theorem yields
\[ \frac{h(x)}{h(y_R)} = u(x) \leq A \exp(IA^{1/\epsilon} \delta_D(x)^{-Q/\epsilon}) \leq A \exp(AR^{Q/\epsilon} R^{-Q/\epsilon}) = A \]
for \( x \in D \cap B(\xi, R) \). See [4] for details.

\[ \square \]

6. Hölder estimates of \( p \)-harmonic extension operators

Let \( D \subset \mathbb{R}^n \) be a bounded open set and let \( f \) be a function on \( \partial D \). Let \( P_D f \) be the (Perron-Wiener-Brelot) Dirichlet solution of \( f \) over \( D \). A boundary point \( \xi \in \partial D \) is said to be regular if
\[ \lim_{x \to \xi} P_D f(x) = f(\xi) \]
for every \( f \in C(\partial D) \). We say that \( D \) is a regular domain if every boundary point \( \xi \in \partial D \) is regular. If \( D \) is regular, then \( P_D \) maps \( C(\partial D) \) to \( \mathcal{H}(D) \cap C(\overline{D}) \). It is natural to raise the following question: Does the better continuity of a boundary function \( f \) guarantee the better continuity of \( P_D f \)?

An answer to this question was given in [2] for classical harmonic functions on Euclidean domains with Hölder continuity. In this note we investigate the same problem \( p \)-harmonic functions in metric measure space.

As was observed in the first part, the notions of \( p \)-harmonicity, \( p \)-Dirichlet problem, \( p \)-Perron solution, \( p \)-regularity, \( p \)-capacity, \( p \)-Wiener criterion are available (A. Björn, J. Björn, P. MacManus, and N. Shanmugalingam [10], [8], [9] and [7]).

Let \( 0 < \beta \leq \alpha \leq 1 \). Consider the family \( \Lambda_\alpha(E) \) of all bounded \( \alpha \)-Hölder continuous functions \( u \) on \( E \) with norm
\[ \|u\|_{\Lambda_\alpha(E)} := \sup_{x \in E} |u(x)| + \sup_{x,y \in E \atop x \neq y} \frac{|u(x) - u(y)|}{d(x,y)^\alpha} < \infty. \]

We shall study the operator norm:
\[ \|P_D\|_{\alpha \to \beta} := \sup_{f \in \Lambda_\alpha(\partial D) \atop \|f\|_{\Lambda_\alpha(\partial D)} \neq 0} \frac{\|P_D f\|_{\Lambda_\beta(D)}}{\|f\|_{\Lambda_\alpha(\partial D)}}. \]
Heinonen, Kilpeläinen and Martio [16, Theorem 6.44] studied the condition for $\|P_D\|_{\alpha \rightarrow \beta} < \infty$ for $\beta < \alpha$ in Euclidean setting. The case most interesting case $\alpha = \beta$ has remained open.

7. **Trivial boundary points**

Is it true $\|P_D\|_{\alpha \rightarrow \beta} < \infty \implies D$ is $p$-regular?

This is not the case ([2]). A punctured ball $D$ is $p$-irregular and yet $\|P_D\|_{\alpha \rightarrow \beta} < \infty$. To avoid such a pathological example we rule out $p$-trivial boundary points. We say that $a \in \partial D$ is a $p$-trivial boundary point if there is $r > 0$ such that $\text{Cap}_p(\partial D \cap B(a, r)) = 0$.

**Proposition 1.** Suppose $\|P_D\|_{\alpha \rightarrow \beta} < \infty$ for some $0 < \beta \leq \alpha$. Then $D$ is a $p$-regular domain if and only if $\partial D$ has no $p$-trivial points.

Hereafter let $D$ be $p$-regular. Let $\alpha = \beta$. We shall study several conditions for $\|P_D\|_{\alpha \rightarrow \alpha} < \infty$. We have the local or interior Hölder continuity of $p$-harmonic functions ([22, Theorem 5.2]): There exists $\alpha_0 > 0$ such that every $p$-harmonic function in $D$ is locally $\alpha_0$-Hölder continuous in $D$. This constant $\alpha_0$ depends only on $p$ and the constants associated with the doubling property of $\mu$ and the Poincaré inequality, but not on $D$. In general, $\alpha_0 < 1$. In order to have $\|P_D\|_{\alpha \rightarrow \alpha} < \infty$, we restrict ourselves to $\alpha \leq \alpha_0$.

8. **Relationships among several conditions**

The conditions for $\|P_D\|_{\alpha \rightarrow \alpha} < \infty$ involve the $p$-harmonic measure.

**Definition 8.** By the $p$-harmonic measure $\omega_p(E; U)$ we mean the upper Perron solution $\overline{P}_U \chi_E$ of the boundary function $\chi_E$ in $U$ ([9]).

**Remark 4.** The $p$-harmonic measure $\omega_p(E; U)$ need not be a measure unless $p = 2$ and the Cheeger harmonicity is adopted because of the non-linear nature of $p$-harmonicity.

**Definition 9.** Global Harmonic Measure Decay Property: $\text{GHMD}(\alpha)$

We say that $D$ satisfies the global harmonic measure decay property with exponent $\alpha$ if there exist $A_2 \geq 1$ and $r_0 > 0$ such that if $a \in \partial D$ and $0 < r < r_0$, then

$$\omega_p(x; \partial D \setminus B(a, r), D) \leq A_2 \left(\frac{d(x, a)}{r}\right)^\alpha$$

for all $x \in D \cap B(a, r)$.

**Definition 10.** Local Harmonic Measure Decay Property: $\text{LHMD}(\alpha)$
We say that \( D \) satisfies the local harmonic measure decay property with exponent \( \alpha \) if there exist \( A_3 \geq 1 \) and \( r_0 > 0 \) such that if \( a \in \partial D \) and \( 0 < r < r_0 \), then
\[
\omega_\alpha(x; D \cap S(a,r), D \cap B(a,r)) \leq A_3 \left( \frac{d(x,a)}{r} \right)^\alpha
\]
for all \( x \in D \cap B(a,r) \).

We shall use \( \varphi_{a,\alpha}(x) = \min\{d(x,a)^\alpha, 1\} \) for \( a \in \partial D \) as a test boundary function with respect to \( \alpha \)-Hölder continuity.

**Theorem 2.** Consider the following four conditions.

(i) \( \|P_D\|_{\alpha \rightarrow \infty} < \infty \).
(ii) There exists \( A_4 \) such that \( P_D \varphi_{a,\alpha}(x) \leq A_4 d(x,a)^\alpha \) for all \( x \in D \).
(iii) Global Harmonic Measure Decay of order \( \alpha \).
(iv) Local Harmonic Measure Decay of order \( \alpha \).

Then we have
\[
(i) \iff (ii) \iff (iii) \iff (iv).
\]
If (iv) holds for some \( \alpha' > \alpha \), then (i) and (ii) hold.

As an immediate corollary, we observe that the larger \( \alpha \) is the stronger the property \( \|P_D\|_{\alpha \rightarrow \infty} < \infty \) is.

**Corollary 2.** If \( 0 < \beta < \alpha \leq \alpha_0 \) and \( \|P_D\|_{\alpha \rightarrow \infty} < \infty \), then \( \|P_D\|_{\beta \rightarrow \infty} < \infty \).

**Remark 5.** It is not true that LHMD(\( \alpha \)) \( \Rightarrow \|P_D\|_{\alpha \rightarrow \infty} < \infty \). There is a domain \( D \) for which the LHMD(\( \alpha \)) holds and yet \( \|P_D\|_{\alpha \rightarrow \infty} = \infty \).

In fact let \( D = \{z \in \mathbb{C} : |z| < 1, |\arg z| < \pi/(2\alpha)\} \) for \( 0 < \alpha \leq 1 \). Then the LHMD(\( \alpha \)) with respect to the classical harmonic measure holds. Nevertheless \( \|P_D\|_{\alpha \rightarrow \alpha} = \infty \); if \( \varphi(z) = |z|^\alpha \) for \( \partial D \). Then \( \|\varphi\|_{\alpha_0(\partial D)} < \infty \) and yet \( P_D \varphi(x) \approx x^\alpha \log(1/x) \) as \( x \downarrow 0 \) on the positive real axis, so \( \|P_D \varphi\|_{\alpha_0(D)} = \infty \).

Let us consider some exterior conditions of the domain \( D \) in terms of the relative capacity:
\[
\text{Cap}_p(E, U) := \inf \left\{ \int_U g_u^p \, d\mu : u \in N_0^{1,p}(U) \text{ and } u \geq 1 \text{ on } E \right\}.
\]

**Definition 11.** We say that \( E \) is uniformly \( p \)-fat or satisfies the \( p \)-capacity density condition if there exist \( A_5 > 0 \) and \( r_0 > 0 \) such that
\[
\frac{\text{Cap}_p(E \cap B(a,r), B(a,2r))}{\text{Cap}_p(B(a,r), B(a,2r))} \geq A_5
\]
whenever \( a \in E \) and \( 0 < r < r_0 \).
Theorem 3. The following five conditions are equivalent:

(i) $\|P_D\|_{\alpha \to \alpha} < \infty$ for some $\alpha > 0$.
(ii) $P_D \varphi_{a, \alpha}(x) \leq A_4 d(x, a)^a$ holds for some $\alpha > 0$.
(iii) GHMD$(\alpha)$ holds for some $\alpha > 0$.
(iv) LHMD$(\alpha)$ holds for some $\alpha > 0$.
(v) $X \setminus D$ satisfies the capacity density condition.

Corollary 3. If $X \setminus D$ satisfies the volume density condition:

$$\frac{\mu(B(a, r) \setminus D)}{\mu(B(a, r))} \geq A,$$
for every $a \in \partial D$ and $r < r_0$,
then $\|P_D\|_{\alpha \to \alpha} < \infty$ for some $\alpha > 0$.

Remark 6. Our arguments are based mostly on the comparison principle for $p$-harmonic functions and the variational properties of the De Giorgi class, which includes $p$-harmonic functions. The crucial part is GHMD $\Rightarrow$ LHMD for which we need the refinement of the submean value property for the De Giorgi class.

Remark 7. The comparison principle implies LHMD $\Rightarrow$ GHMD. The converse implication GHMD $\Rightarrow$ LHMD is crucial. Let us illustrate its proof:

Let $u = \omega_p(\partial D \cap B(a, r); D)$. Suppose $\zeta \in \partial D \cap S(a, Ar)$ (UP). Then $u \leq \frac{1}{2}$ on $B(\zeta, cr)$, so $u \leq 1 - \varepsilon$ on a small ball intersecting $B(\zeta, cr)$ by some argument based on the De Giorgi class. Repeating the same argument, we obtain $u \leq 1 - \varepsilon$ on $S(a, Ar)$. Hence $\omega_p(\partial D \cap B(a, r); D) \geq \varepsilon$ on $D \cap S(a, Ar)$; in other words

$$\omega_p(D \cap S(a, Ar); D \cap B(a, Ar)) \leq \varepsilon^{-1} \omega_p(\partial D \cap B(a, r); D)$$
on $D \cap B(a, Ar)$.

Hence GHMD $\Rightarrow$ LHMD.

References


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