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Results on relative expandability and relative pseudocompactness

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1. Introduction

This report is a summary of [17], [18] and [19], and a continuation of [21].

Throughout this note all spaces are assumed to be T_1 and the symbol γ denotes an infinite cardinal. Moreover, the symbols $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{I}$ denote the set of real numbers, the set of natural numbers and the closed unit interval, respectively. Let $\mathcal{T}_2$ (respectively, $\mathcal{T}_3$, $\mathcal{T}_{3\frac{1}{2}}$) be the class of all Hausdorff (respectively, regular, Tychonoff) spaces.

A subspace Y is said to be 1- (respectively, 2-) *paracompact in* X if for every open cover $\mathcal{U}$ of X, there exists a collection $\mathcal{V}$ of open subsets of X with $X = \bigcup \mathcal{V}$ (respectively, $Y \subset \bigcup \mathcal{V}$) such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y in X. Here, $\mathcal{V}$ is said to be a *partial refinement* of $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ containing V, and $\mathcal{V}$ of subsets of X is *locally finite* (respectively, *discrete*) *at* y *in* X if there exists a neighborhood U_y of y in X which intersects at most finitely many members (respectively, at most one member) of $\mathcal{V}$ ([3]). Y is said to be 3-*paracompact in* X if for every open cover $\mathcal{U}$ of X, there exists a locally finite (in Y) open cover $\mathcal{V}$ of Y such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ ([3]).

Yasui [35], [36] introduced 1- or 2-countable paracompactness of a subspace in a space. Aull [6] defined α-paracompactness and α-countably paracompactness of a subspace in a space. 1- and α-paracompactness need not imply each other, but for a closed subspace Y of a regular space X, these are mutually equivalent ([25, Theorem 1.3], see also [21]). Meanwhile, 1- and α-countable paracompactness do not imply each other even if Y is a closed subspace of a regular space X. Characterizations of absolute embeddings of 1- and α-countable paracompactness were given in [27] and [17], respectively (see Theorems 2.1 and 2.2 below).

In [17], notions of relative expandability and relative discrete expandability were introduced. In particular, the notions of 1- (respectively, α-) expandability lies between 1- (respectively, α-) paracompactness and 1- (respectively, α-) countable paracompactness ([17]). In Section 3, their absolute embeddings are considered. Moreover, 2- and strong expandability of Y in X were defined in [18] and results on relative discrete expandability are also given.

In Section 4, we discuss potential pseudocompactness and relative pseudocompactness. Arhangel'skiĭ and Genedi [4] introduced the notions of strong pseudocompactness of a subspace in a space and potential pseudocompactness of a space. They proved that under CH the discrete space of cardinality ω_1 is potentially pseudocompact (Corollary 3.2) and posed a problem whether the assumption CH can be omitted or not. Answering this problem, Grarcía-Ferreira and Just [10] proved that for any uncountable cardinal κ the discrete space of cardinality κ is potentially pseudocompact (Theorem 3.3). But their proof of this theorem in [10] uses a set-theoretic technique (such as the Fichtenholz-Kantorovich-Hausdorff theorem). In Section 4, an alternative simple proof of this theorem is given. Moreover, we consider the relative versions of well-known Scott-Watson theorem: every pseudocompact metacompact Tychonoff space is compact ([30], [31]).

Recall that a Tychonoff space X is *almost compact* if $|\beta X \setminus X| \leq 1$, where βX is the Stone-Čech compactification of X.

For a subset Y of a space X, $\overline{Y}^X$ denotes the closure of Y in X. Other undefined notations and terminology are used as in [9] and [21].

2. Relative countable paracompactness and relative (discrete) expandability

Yasui [35], [36] defined that a subspace Y of a space X is 1- (respectively, 2-) *countably paracompact in* X if for every countable open cover $\mathcal{U}$ of X, there exists a collection $\mathcal{V}$ of open subsets of X with $X = \bigcup \mathcal{V}$ (respectively, $Y \subset \bigcup \mathcal{V}$) such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y. It is clear that if Y is 1- (respectively, 2-) paracompact in X, then Y is countably 1- (respectively, 2-) paracompact in X.

Aull [6] defined that a subspace Y of a space X is *α-countably paracompact in* X if for every countable collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection $\mathcal{V}$ of open subsets of X such that $Y \subset \bigcup \mathcal{V}$, $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite in X. It is obvious that if Y is α-paracompact in X, then Y is α-countably paracompact in X.

Recall that 1- and α-paracompactness do not imply each other in general, but for a closed subspace Y of a regular space X, Y is 1-paracompact in X if and only if Y is α-paracompact in X ([25, Theorem 1.3], see also [21]). The following results should be compared with [21, Corollary 3.7].

Theorem 2.1 (Matveev [27]). *A Tychonoff (respectively, regular) space Y is*

1-countably paracompact in every larger Tychonoff (respectively, regular) space if and only if Y is Lindelöf.

Theorem 2.2 ([17]). *A Tychonoff (respectively, regular) space Y is α-countably paracompact in every larger Tychonoff (respectively, regular) space if and only if Y is countably compact.*

Krajewski [23] defined that a space X is *γ-expandable* if for every locally finite collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X, there exists a locally finite collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for every $\alpha < \gamma$. A space X is *expandable* if X is γ-expandable for every γ. It is known that every paracompact or every countably compact space is expandable. Moreover, it is also known that a space X is countably paracompact if and only if X is ω-expandable ([23]).

As relative notions of expandability, Y is said to be *1-γ-expandable in* X if for each locally finite collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite at each point of Y in X. If Y is 1-γ-expandable in X for every γ, Y is said to be *1-expandable in* X. A subspace Y of a space X is said to be *α-γ-expandable in* X if for each collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X which is locally finite at every point of Y in X, there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite in X. If Y is α-γ-expandable in X for every γ, Y is said to be *α-expandable in* X ([17]). Notice that if a subspace Y of a space X is α-paracompact in X, then for every collection $\{F_\alpha \mid \alpha \in \Omega\}$ of closed subsets of X which is locally finite at every $y \in Y$, $\{F_\alpha \cap Y \mid \alpha \in \Omega\}$ is locally finite in X. Note that 1-countable paracompactness and α-countable paracompactness need not imply each other even if Y is a closed subspace of a regular space X ([17]).

Theorem 2.3 ([17]). *A Tychonoff (respectively, regular) space Y is 1-expandable in every larger Tychonoff (respectively, regular) space if and only if Y is compact.*

Theorem 2.4 ([17]). *A Tychonoff (respectively, regular) space Y is α-expandable in every larger Tychonoff (respectively, regular) space if and only if Y is countably compact.*

Remark 2.5. Similarly to the proof of [20, Proposition 3.19], we have that a Hausdorff space Y is 1-expandable (or equivalently, 1-countably paracompact) in every larger Hausdorff space if and only if $Y = \emptyset$.

Remark 2.6. The proof of Theorems 2.2 and 2.4 works to show that for a Hausdorff space Y, the following statements are equivalent:

(a) Y is α-expandable in every larger Hausdorff space.
(b) Y is α-countably paracompact in every larger Hausdorff space.
(c) Y is countably compact.

Smith and Krajewski [29] defined that a space X is *discretely γ-expandable* if for every discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X, there exists a locally finite collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for every $\alpha < \gamma$. A space X is *discretely expandable* if X is discretely γ-expandable for every γ. It is easy to see that every expandable or every collectionwise normal space is discretely expandable ([29]).

As relative version of these notions, we define that a subspace Y a space X is 1-*discretely γ-expandable* if for each discrete collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite at each point of Y in X. Moreover, Y is said to be *α-discretely γ-expandable in X* if for each collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X which is discrete at every point of Y in X, there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite in X. Moreover, 1- and *α-discretely expandability* of a subspace in a space are now easy to be understood. It is easy to see that if Y is 1- (respectively, α-) γ-expandable in X, then Y is 1- (respectively, α-) discretely γ-expandable in X ([17]). Notice that 1-discrete expandability and α-discrete expandability of Y in X do not imply each other.

The proofs of Theorems 2.3 and 2.4 essentially show the following.

Theorem 2.7 **([17]).** *A Tychonoff (respectively, regular) space Y is 1-discretely expandable in every larger Tychonoff (respectively, regular) space if and only if Y is compact.*

Theorem 2.8 **([17]).** *A Tychonoff (respectively, regular) space Y is α-discretely expandable in every larger Tychonoff (respectively, regular) space if and only if Y is countably compact.*

Remark 2.9. As in Remark 2.5, we have that a Hausdorff space Y is 1-discretely expandable in every larger Hausdorff space if and only if $Y = \emptyset$.

Remark 2.10. In Theorems 2.1, 2.2, 2.3, 2.4, 2.7 and 2.8, and Remarks 2.5, 2.6 and 2.9, "in every larger Tychonoff (respectively, regular, Hausdorff) space" can be replaced by "in every larger Tychonoff (respectively, regular, Hausdorff) space containing Y as a closed subspace".

Remark 2.11. In [15], E. Grabner et. al. asked the following question; suppose that Y is a closed subspace of a regular space X. If Y is 1-discretely expandable in X and metacompact in itself, is Y 1-paracompact in X? In [17], a negative answer to this question was given.

We define that a subspace Y is *2-γ-expandable* (respectively, *2-discretely γ-expandable*) *in* X if for each locally finite (respectively, discrete) collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of X there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \cap Y \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite at each point of Y in X. If Y is 2-γ-expandable (respectively, 2-discretely γ-expandable) in X for every γ, Y is said to be *2-expandable* (respectively, *2-discretely expandable*) *in* X ([19], see also [15]*).

Moreover, Y is said to be *strongly γ-expandable* (respectively, *strongly discretely γ-expandable*) in X if for each locally finite (respectively, discrete) collection $\{F_\alpha \mid \alpha < \gamma\}$ of closed subsets of Y there exists a collection $\{G_\alpha \mid \alpha < \gamma\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for each $\alpha < \gamma$ and $\{G_\alpha \mid \alpha < \gamma\}$ is locally finite at each point of Y in X. If Y is strongly (respectively, strongly discretely) γ-expandable in X for every γ, we say that Y is *strongly* (respectively, *strongly discretely*) *expandable in* X.

We also define that Y is *countably Aull-paracompact in* X if for every countable collection $\mathcal{U}$ of open subsets of X with $Y \subset \bigcup \mathcal{U}$, there exists a collection $\mathcal{V}$ of open subsets of X with $Y \subset \bigcup \mathcal{V}$ such that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally finite at each point of Y. It is clear that if Y is countably Aull-paracompact in X, then Y is 2-countably paracompact in X ([19]).

If Y is 2-paracompact in X, then Y is 2-expandable in X ([19] and see also [15] assuming that all spaces are Hausdorff). Moreover, it is easy to see that Y is 2-countably paracompact (respectively, countably Aull-paracompact) in X if and only Y is 2-ω-expandable (respectively, strongly ω-expandable) in X. For other basic properties of these notions, see [19].

Let X_Y denote the space obtained from the space X, with the topology generated by a subbase $\{U \mid U$ is open in X or $U \subset X \setminus Y\}$. Hence, points in $X \setminus Y$ are isolated and Y is closed in X_Y. Moreover, X and X_Y generate the same topology on Y ([9]). As is seen in [1] and [20], the space X_Y is often useful in discussing several relative topological properties. The following results should be compared with [21, Lemmas 2.1, 2.2 and 2.3].

Lemma 2.12 **([19]).** *For a subspace Y of a space X, the following statements are equivalent.*

(*a*) *Y is strongly (respectively, strongly discretely) γ-expandable in X.*

(*b*) *Y is 2- (respectively, 2-discretely) γ-expandable in G for every open subset G of X with $Y \subset G$.*

(*c*) *X_Y is (respectively, discretely) γ-expandable.*

(*d*) *Y is 2- (respectively, 2-discretely) γ-expandable in X_Y.*

*Note that E. Grabner, G. Grabner, K. Miyazaki and J. Tartir [15] called 2-discretely expandability of Y in X "discrete expandability of Y in X".

(*e*) *Y is strongly (respectively, strongly discretely) γ-expandable in X_Y.*

Corollary 2.13 ([19]). *For a subspace Y of a space X, the following statements are equivalent.*

(*a*) *Y is countably Aull-paracompact in X.*

(*b*) *Y is 2-counatbly paracompact in G for every open subset G of X with $Y \subset G$.*

(*c*) *X_Y is countably paracompact.*

(*d*) *Y is 2-countably paracompact in X_Y.*

(*e*) *Y is countably Aull-paracompact in X_Y.*

These results and definitions above admit the implications in Diagram 1 (see the next page) for a subspace Y of a space X; for brevity "d-expandable", "st- (d-) expandable" and "c- (Aull-) paracompact" means "discrete expandable", "strongly (discretely) expandable" and "countably (Aull-) paracompact", respectively.

Here, we characterize absolute embeddings of 2-, strong (discrete) expandability and 2-, strong countable paracompactness for Hausdorff case as follows.

Proposition 2.14 ([19]). *For a Hausdorff space Y, the following statements are equivalent.*

(*a*) *Y is strongly expandable in every larger Hausdorff space.*

(*b*) *Y is 2-expandable in every larger Hausdorff space.*

(*c*) *Y is strongly discretely expandable in every larger Hausdorff space.*

(*d*) *Y is 2-discretely expandable in every larger Hausdorff space.*

(*e*) *Y is countably compact.*

Proposition 2.15 ([19]). *For a Hausdorff space Y, the following statements are equivalent.*

(*a*) *Y is countably Aull-paracompact in every larger Hausdorff space.*

(*b*) *Y is 2-countably paracompact in every larger Hausdorff space.*

(*c*) *Y is countably compact.*

Remark 2.16. In Propositions 2.14 and 2.15, "in every larger Hausdorff space" can be replaced by "in every larger Hausdorff space containing Y as a closed subspace".

For the case of Tychonoff or regular spaces, see [19].

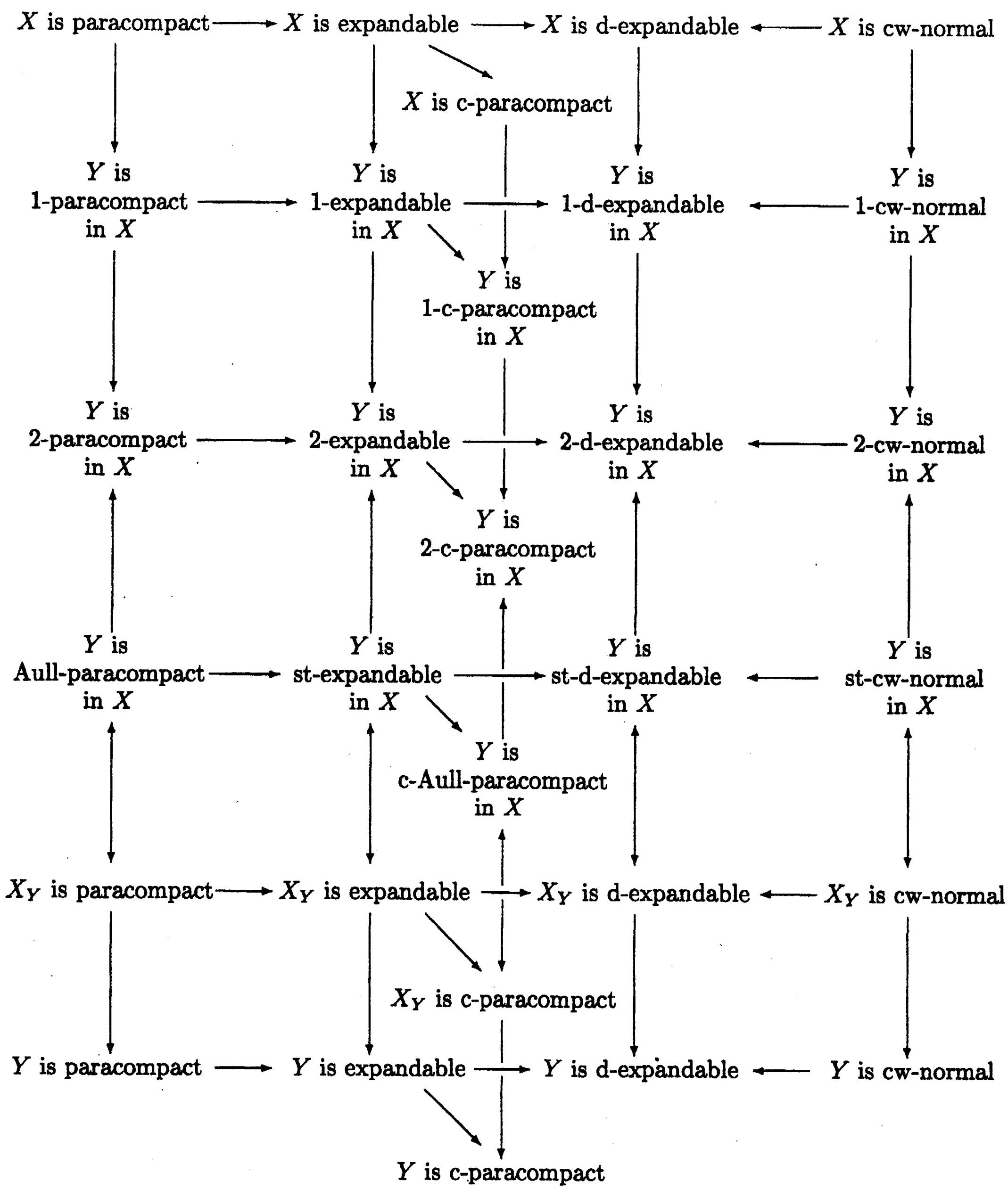

DIAGRAM 1

Here, we list results on absolute embeddings discussed above as follows. All results in the following Tables 1 and 2 can be referred to [21], and the results mentioned in Section 2 are listed in Table 3.

In the following tables, for each relative topological property $\mathcal{P}$ and the class $\mathcal{T}_i$ $(i = 2, 3, 3\frac{1}{2})$, the corresponding property indicates characterizations of absolute embedding of $\mathcal{P}$ in the class $\mathcal{T}_i$. In Table 1, for example, the property "normal and almost compact" is the characterization of absolute 1-normality in the class $\mathcal{T}_{3\frac{1}{2}}$ or $\mathcal{T}_3$. That is, it means the following statement; *a Tychonoff (respectively, regular) space Y is 1-normal in every larger Tychonoff (respectively, regular) space if and only if Y is normal and almost compact.*

Moreover, since absolute embeddings of 3-paracompactness and 2- or 3-metacompactness are trivial, these properties are omitted in the tables.

TABLE 1. Relative (collectionwise) normality

Property	$\mathcal{T}_{3\frac{1}{2}}$ or $\mathcal{T}_3$	$\mathcal{T}_2$
1-normality	normal and almost compact ([20])	$\lvert Y\rvert \leq 1$ ([20])
2-normality	either Lindelöf or normal and almost compact ([20])	compact ([20])
(strong) normality	either Lindelöf or normal and almost compact ([7], [28])	compact ([32], see also [20])
1-collectionwise normality	normal and almost compact ([20])	$\lvert Y\rvert \leq 1$ ([20])
2-collectionwise normality	either Lindelöf or normal and almost compact ([20])	compact ([20])
(strong) collectionwise normality	either Lindelöf or normal and almost compact ([16])	compact ([32], see also [20])

TABLE 2. Relative paracompactness

Property	$\mathcal{T}_{3\frac{1}{2}}$ or $\mathcal{T}_3$	$\mathcal{T}_2$
1-paracompactness	compact ([24], [25])	$Y = \emptyset$ ([20])
α-paracompactness	compact ([24])	
2-paracompactness	Lindelöf ([3], [13])	compact ([32], see also [20])
Aull-paracompactness	Lindelöf ([33])	compact ([33])

TABLE 3. Relative countable paracompactness and relative (discrete) expandability

Property	$\mathcal{T}_{3\frac{1}{2}}$ or $\mathcal{T}_3$	$\mathcal{T}_2$
1-countable paracompactness	Lindelöf	$Y=\emptyset$
α-countable paracompactness	countably compact	
2-countale paracompactness	unknown	countably compact
countable Aull-paracompactness	unknown	countably compact
1-(discrete) expandability	compact	$Y=\emptyset$
α-(discrete) expandability	countably compact	
2-(discrete) expandability	unknown	countably compact
strong (discrete) expandability	unknown	countably compact

TABLE 4. Other relative topological properties

Property	$\mathcal{T}_{3\frac{1}{2}}$ or $\mathcal{T}_3$	$\mathcal{T}_2$
regularity	all spaces	compact ([5])
1-metacompactness	compact ([20])	

3. Relative pseudocompactness

A space X is said to be *pseudocompact* if every continuous real-valued function on X is bounded. For a Tychonoff space X, pseudocompactness of X is equivalent that every locally finite collection of non-empty open subsets of X is finite ([9], [26]); the latter condition is often called *feeble compactness* of X.

Arhangel'skiĭ and Genedi [4] defined that a subspace Y of a space X is *strongly pseudocompact in* X if every collection $\mathcal{U}$ of open subsets of X which is locally finite at every $y \in Y$ in X and such that $U \cap Y \neq \emptyset$ for all $U \in \mathcal{U}$ is finite. Y is said to be *pseudocompact in* X if every locally finite collection of open subsets of X which satisfies $U \cap Y \neq \emptyset$ for all $U \in \mathcal{U}$ is finite. In [26], pseudocompactness of Y in X is called *feeble compactness of* Y *in* X. Strong pseudocompactness of Y in X clearly implies its pseudocompactness in X.

Recall that a subspace Y of a space X is *compact in* X if every open cover of X has a finite subcollection which covers Y ([3]). Y is said to be *countably compact in* X if every infinite subset of Y has an accumulation point in X. It is well-known that Y is countably compact in X if and only if every countable open cover of X has a finite subcollection which covers Y. It is also known that Y is compact in X if and only if every infinite subset of Y has a complete accumulation point in X ([22]).

Let $\mathcal{P}$ be some class of spaces. A space Y is said to be *potentially pseudocompact in the class* $\mathcal{P}$ if there exists a space $X \in \mathcal{P}$ containing Y such that Y is strongly pseudocompact in X. In particular, if Y is potentially pseudocompact in the class $\mathcal{T}_3$, Y is said to be *potentially pseudocompact* ([4]). Arhangel'skiĭ and Genedi [4] proved that the discrete space of cardinality ω is not potentially pseudocompact. They also proved the following.

Theorem 3.1 (Arhangel'skiĭ and Genedi [4]). *The discrete space of cardinality $\mathfrak{c}$ is potentially pseudocompact.*

Corollary 3.2 (Arhangel'skiĭ and Genedi [4]). *Assuming CH, the discrete space of cardinality of ω_1 is potentially pseudocompact.*

In [4], a problem was posed whether it is possible to drop the assumption CH. García-Ferreira and Just [10] gave an affirmative answer to this problem in ZFC as follows.

Theorem 3.3 (García-Ferreira and Just [10]). *Let κ be an uncountable cardinal. Then the discrete space of cardinality κ is potentially pseudocompact.*

Although the proof in [10] of Theorem 3.3 needs an involved construction making a sort of Ψ-spaces and uses a set-theoretic technique, we give an alternative simple proof to this theorem.

The following is a key lemma.

Lemma 3.4 ([18]). *Let κ be an uncountable cardinal and define $A(\kappa) = D(\kappa) \cup \{\infty\}$ is the one-point compactification of the discrete space $D(\kappa)$ of cardinality κ. Put $X = A(\kappa) \times A(\kappa) \setminus \{\langle \infty, \infty \rangle\}$ and $Y = (D(\kappa) \times \{\infty\}) \cup (\{\infty\} \times D(\kappa))$. Then Y is strongly pseudocompact in X.*

Proof. Let $\mathcal{U}$ be a collection of open subsets of X which is locally finite at every $y \in Y$ in X and such that $U \cap Y \neq \emptyset$ for all $U \in \mathcal{U}$. Suppose $\mathcal{U}$ is infinite. Put $\mathcal{U}' = \{U \in \mathcal{U} \mid U \cap (D(\kappa) \times \{\infty\}) \neq \emptyset\}$. Without loss of generality, we may assume $\mathcal{U}'$ is countably infinite. For each $U \in \mathcal{U}'$, take $\langle d_U, \infty \rangle \in U \cap (D(\kappa) \times \{\infty\})$. Then, there is a finite subset F_U of $D(\kappa)$ such that $\langle d_U, \infty \rangle \in \{d_U\} \times (A(\kappa) \setminus F_U) \subset U$. Note that for each $d \in D(\kappa)$, the collection $\{U \in \mathcal{U}' \mid d = d_U\}$ is at most finite. Since $\bigcup \{F_U \mid U \in \mathcal{U}'\}$ is countable, we can pick a $d^* \in D(\kappa) \setminus \bigcup \{F_U \mid U \in \mathcal{U}'\}$. Then, $\mathcal{U}$ is not locally finite at $\langle \infty, d^* \rangle$, a contradiction. $\square$

Alternative proof of Theorem 3.3. Let $D(\kappa)$ be the discrete space of cardinality κ and let Y, Z be subspaces of $D(\kappa)$ satisfying $|Y| = |Z| = \kappa$ and $D(\kappa) = Y \oplus Z$. Let $X = A(Y) \times A(Z) \setminus \{\langle \infty_Y, \infty_Z \rangle\}$, where $A(Y) = Y \cup$

$\{\infty_Y\}$ and $A(Z) = Z \cup \{\infty_Z\}$ are the one-point compactifications of Y and Z, respectively. Since $D(\kappa)$ are homeomorphic to $E = (Y \times \{\infty_Z\}) \cup (\{\infty_Y\} \times Z)$, X is a larger Tychonoff space of $D(\kappa)$ (containing $D(\kappa)$ as a closed subspace). By Lemma 3.4, $D(\kappa)$ is strongly pseudocompact in X. □

Next we consider other applications of Lemma 3.4. First, let us recall Proposition 3.5 below which are relative versions of the Scott-Watson theorem; every pseudocompact metacompact Tychonoff space is compact ([30], [31]). In the Proposition 3.5, (*a*), (*b*) and (*c*) follow from [22], [3] and [34], respectively. Note that Theorem 3.5 (*c*) also follows from Kočinac [22, 1.5 Theorem]. Moreover, Theorem 3.5 (*c*) has been proved by Arhangel'skiĭ [2, Theorem 8.20] in case Y is 1-paracompact in X and X is regular. Notice that each of three facts does not cover the others.

Proposition 3.5 ([22], [3], [34]). *For a subspace Y of a space X, the following hold.*

(*a*) *If Y is countably compact (in itself) and 3-metacompact in X, then Y is compact in X.*

(*b*) *If Y is strongly pseudocompact in X and 2-paracompact in X, then Y is compact in X.*

(*c*) *If Y is countably compact in X and 1-metacompact in X, then Y is compact in X.*

In view of these results, it is natural to ask "if Y is strongly pseudocompact in X and 1-metacompact in X, then is Y compact in X?" The answer is no. Indeed, let $X = A(\omega_1) \times A(\omega_1) \setminus \{\langle\infty, \infty\rangle\}$ and $Y = (\{\infty\} \times D(\omega_1)) \cup (D(\omega_1) \times \{\infty\})$. Then by Lemma 3.4, Y is strongly pseudocompact in X. Moreover, Y is 1-metacompact in X but not compact in X. It should be noted that even if Y is 2-paracompact in X and countably compact in X, Y need not compact in X ([18]).

Here, the following slightly generalizes Proposition 3.5 (*c*).

Theorem 3.6 ([18]). *Let Y and Z be subspaces of a space X. If Y is countably compact in X and Z is 1-metacompact in X, then $Y \cap Z$ is compact in X.*

Proposition 3.5 (*c*) and Theorem 3.6 affirmatively answer to [2, Problem 8.21]. Moreover, Theorem 3.6 clearly contains the following fact [14, Corollary 23] that for subspaces Y and Z of a regular space X, if $\overline{Y}^X$ is countably compact and Z is 1-metacompact in X, then $Y \cap Z$ is compact in X. On the other hand, we cannot generalize either of Proposition 3.5 (*a*) and (*b*) in a similar manner (see [18]).

A space X is said to be *weakly-normal* if for every disjoint closed subsets A, B of X, one of which is countable and discrete, there exist disjoint open subsets U, V of X such that $A \subset U$ and $B \subset V$ (cf. [8]). It is known that a Tychonoff space X is countably compact if and only if X is weakly-normal and pseudocompact ([8]). In the following proposition, (a) and (b) were proved in Arhangel'skiĭ and Genedi [3] and Gordienko [12], respectively.

Proposition 3.7 **([3], [12]).** *For a subspace Y of a regular space X, the following hold.*

(a) *If Y is normal in X and strongly pseudocompact in X, then Y is countably compact in X.*

(b) *If Y is supernormal in X and pseudocompact in X, then Y is countably compact in X.*

Here, Y is said to be *supernormal in* X if for every disjoint closed subsets A, B of X, at least one of which is contained in Y, there exist disjoint open subsets U, V of X such that $A \subset U$ and $B \subset V$ ([12]).

To refine Proposition 3.7, the following notions of relative weak-normality were introduced in [18]. Y is *weakly-normal in* X if for every disjoint closed subsets A, B of X, one of which is countable and discrete, there exist disjoint open subsets U, V of X such that $A \cap Y \subset U$ and $B \cap Y \subset V$. Furthermore, Y is said to be *strongly weakly-normal in* X if for every disjoint closed subsets A, B of Y, one of which is countable and discrete, there exist disjoint open subsets U, V of X such that $A \subset U$ and $B \subset V$. We say that Y is *super-weakly-normal in* X if for every disjoint closed subsets of X, one of which is countable discrete in X and contained in Y, there exist disjoint open subsets U, V of X such that $A \subset U$ and $B \subset V$.

The proof in [3] of Proposition 3.7 (a) essentially shows that the theorem also holds if we replace "Y is normal in X"by "Y is weakly-normal in X".

Clearly, normality of Y in X implies its weakly-normality in X. It is also obvious that strong normality of Y in X implies its strong weakly-normality in X. Moreover, supernormality of Y in X implies its super-weakly-normality in X, and the latter implies its superregularity in X. Note that if Y is strongly weakly-normal in X or super-weakly-normal in X, then Y is weakly-normal in X ([18]). It is obvious that if a space Y is feebly compact (in itself), then Y is strongly pseudocompact in every space X which contains Y as a subspace ([4]).

Theorem 3.8 **([18]).** *Let Y be a subspace of a space X. Then, Y is strongly weakly-normal in X and strongly pseudocompact in X if and only if Y is regular in X and countably compact* (*in itself*).

Theorem 3.9 ([18]). *Let Y be a subspace of a space X. Then, Y is superweakly-normal in X and pseudocompact in X if and only if Y is superregular in X and countably compact in X.*

Notice that for a subspace Y of a space X, Y is countably compact (in itself) if and only if every collection $\mathcal{U}$ of (not necessarily open) subsets of X which is locally finite at every $y \in Y$ in X and such that $U \cap Y \neq \emptyset$ for all $U \in \mathcal{U}$ is finite. Hence, Theorems 3.8 and 3.9 extend Proposition 3.7 (a) and (b), respectively.

We conclude this note by showing some results on relative DFCC. Recall that a space X satisfies the *discrete finite chain condition* (DFCC, for short) if every discrete collection of non-empty open subsets of X is finite (see [26], for example). A subspace Y of a space X is said to be DFCC *in* X if every discrete collection of open subsets of X, which satisfies $U \cap Y \neq \emptyset$ for all $U \in \mathcal{U}$, is finite. It is known pseudocompactness of Y in X implies its DFCC-ness in X, and conversely for regular spaces X ([26]). More generally, we have

Theorem 3.10 ([18]). *Let Y be a subspace of a space X. Suppose that Y is superregular in X. Then Y is pseudocompact in X if and only if Y is DFCC in X.*

Remark 3.11. Notice that by Theorem 3.10, "Y is pseudocompact in X"can be replaced by "Y is DFCC in X"in Proposition 3.7 (b) and Theorem 3.9.

Remark 3.12. Consider that a Tychonoff space Y is strongly pseudocompact (equivalently, pseudocompact, DFCC) in every larger Tychonoff space. This means, however, nothing but that Y is pseudocompact.

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