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From small divisors to Brjuno functions

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CONTENTS

1. Introduction

Small divisor problems arise naturally when nonlinear quasiperiodic dynamical systems are considered. In the general case of multifrequency systems not much progress has been made beyond the celebrated Kolmogorov Arnol'd Moser theory. For example, restricting the attention to near to integrable Hamiltonian systems, or to perturbations of translations on tori, we still do not know how to characterize *exactly* the set of rotation vectors ω for which an invariant torus carrying quasiperiodic motions of frequency ω always persists under a (sufficiently small) analytic perturbation. However, for one–frequency systems, exploiting the geometric renormalization approach, some spectacular results have been obtained in the last 20 years. We will describe here some of these results and some open problems. In particular we will discuss the results obtained by Yoccoz [Yo2, Yo3] on the problem of linearization of one–dimensional germs of holomorphic diffeomorphisms in a neighborhood of a fixed point. Here the optimal set of rotation numbers for which an analytic linearization exists is known, and it is given by the set of *Brjuno numbers.* The same set plays an analogue role for some area–preserving maps [Ma1, Da1], including the standard family [Da2, BG1, BG2].

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $(p_n/q_n)_{n\geq 0}$ be the sequence of the convergents of its continued fraction expansion. A *Brjuno number* is an irrational number α such that $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty$. The set of Brjuno numbers is invariant under the action of the modular group PGL$\,(2,\mathbb{Z})$ and it can be characterized as the set where the *Brjuno function* $B : \mathbb{R} \setminus \mathbb{Q} \rightarrow \mathbb{R} \cup \{+\infty\}$ is finite. This arithmetical function is $\mathbb{Z}$–periodic and satisfies a remarkable functional equation which allows B to be interpreted as a *cocycle* under the action of the modular group. In the problem of linearization of the quadratic polynomial the Brjuno function gives the size (modulus continuous functions) of the domain of stability around the indifferent fixed point [BC1, BC2, Yo2]. Conjecturally it gives this size modulus Hölder continuous functions in this problem as well as in other small divisor problems (see [Ma1, MS, MY]).

Let us now briefly describe the contents of this article.

In Section 2 we introduce small divisor problems in their simplest form through the study of special flows over irrational translations on tori. The question of existence and regularity of the conjugacy with a suspension flow leads to a (linear) cohomological equation by means of which Diophantine vectors can be given a purely dynamical definition.

The analysis of return times is important for understanding quasiperiodic dynamics. Continued fractions (Section 3) provide an efficient algorithm for computing return times. In Section 4 we use the rate of growth of the partial fractions and of the denominators of the convergents of an irrational number to characterize several diophantine conditions. Beyond diophantine numbers one can introduce Brjuno numbers and the associated Brjuno function (Section 5). Section 6 is a short and elementary introduction to linearization problems. These are the simplest nonlinear small divisor problems (Section 7) and the quadratic polynomial (Section 8) plays here a distinguished role, both as the "worst possible nonlinear perturbation" and as the model for which the results are most complete and satisfactory.

The basic idea and implementation of geometric renormalization is illustrated in Section 9 with a sketchy summary of the theory developed by Yoccoz in [Yo2]. The results obtained are summarized in Section 10 whereas various related numerical results are discussed in Section 11. Here the most important open problem is the Hölder interpolation conjecture (Conjecture 11.1) [Ma1, MMY] and its analogue for area–preserving maps [Ma1,MS]. If true, the Brjuno function would give an a-priori purely arithmetical estimate of the "size" of the domains of stability of quasiperiodic orbits modulus an error with a regular (Hölder continuous) dependence on the rotation number. In the case of the quadratic polynomial it is now known, after the work of Buff and Chéritat [BC2], that this is true modulus a continuous function. Numerically this function seems to be Hölder continuous with an exponent $= 1/2$ [Ca].

This conjecture has been the main motivation of an in-depth investigation of the properties of the Brjuno function [MMY1,MMY2] whose results are summarized in Sections 12, 13 and 14.

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2. Quasiperiodic dynamics and the analysis of linear flows : diophantine and liouvillean vectors

Among all recurrent orbits, periodic orbits are the simplest : they are just closed orbits. Almost periodic orbits are those orbits which behave as periodic orbits if one looks at the phase space with a finite resolution. If the resolution is increased the orbit seems again periodic but with a longer period. *Quasiperiodic* orbits have the further property that the frequencies of the motion (which one can obtain by Fourier analysis) span a finite dimensional space.

The prototype of a quasiperiodic dynamical system is given by the (non–resonant) linear flow on the torus in the continuous case and by a translation on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ in the discrete time case. If α and x are now two points of $\mathbb{T}^n$ we define $R_\alpha x = x + \alpha (\bmod \mathbb{Z}^n)$. One sees immediately three important features of this example :

- from the *algebraic* point of view, the centralizer of R_α is the whole torus $\mathbb{T}^n$. The dynamics is homogenous and the group of symmetries acts transitively on the phase space ;
- from the *topological* point of view, the family of iterates $(R_\alpha^n)_{n\in\mathbb{Z}}$ is equicontinuous. The topological entropy of R_α is zero ;
- from the *measure-theoretical* point of view, the Haar measure on $\mathbb{T}^n$ is invariant under R_α and the unitary operator U_{R_α} on $L^2(\mathbb{T}^n, \mathbb{C})$ defined by $U_{R_\alpha} F = F \circ R_\alpha$ has discrete spectrum $\{e^{2\pi i k\cdot\alpha}\}_{k\in\mathbb{Z}^n}$.

The translation flow on the torus $\mathbb{T}^n$ of vector $\alpha \in \mathbb{R}^n$ is the flow arising from the constant vector field $X(x) = \alpha$. We denote this flow by $R_{t\alpha}$. When the vector α is non resonant, i.e. when $\alpha_1, \ldots, \alpha_n$ are rationally independent, the flow is minimal and has a unique invariant probability measure which is the Haar measure on $\mathbb{T}^n$. In this case we say it is an *irrational flow.* Note that one of the coordinates of the corresponding vector field might be rational. More specifically, given a minimal translation R_α on $\mathbb{T}^n$ then the flow $R_{t(1,\alpha)}$ on $\mathbb{T}^{n+1}$ is irrational.

One of the simplest examples of the connection between the study of quasiperiodic dynamics and arithmetic is provided by the study of reparametrizations of linear flows. Indeed one can equivalently define diophantine numbers by means of a purely dynamical property of these flows, as we will see below.

Given $\phi \in C^r(\mathbb{T}^{n+1}, \mathbb{R}_+^*)$, $r \geq 1$, we define the *reparametrization*, or smooth time change, of $R_{t(1,\alpha)}$ with *speed* $\frac{1}{\phi}$ to be the flow given by

$$\frac{d\theta}{dt} = \frac{\alpha}{\phi(\theta, s)}, \quad \frac{ds}{dt} = \frac{1}{\phi(\theta, s)},$$

where $\theta \in \mathbb{T}^n$ and $s \in \mathbb{T}^1$. The reparametrized flow is still minimal and uniquely ergodic (the invariant measure is $\phi(x)dx$, where dx denotes the Haar measure on $\mathbb{T}^{n+1}$) while more subtle asymptotic properties may change under time change as we will see later. Considering a cross-section, $R_{t(1,\alpha)}$ can be viewed as the time 1 suspension over R_α. In the same way, the reparametrized flow can be represented as a *special flow* over R_α, with a roof function φ having the same regularity as ϕ. A special flow over a map f of $\mathbb{T}^{n-1}$ is defined on the manifold obtained from $\{(t,y) \mid y \in T^{n-1}, t \in \mathbb{R}, 0 \leq t \leq \varphi(y)\} \subset \mathbb{R} \times \mathbb{T}^{n-1}$ after identifying pairs $(\varphi(y), y)$ and $(0, f(y))$. Of course when f is a translation the result is again the n-dimensional torus but the vertical vector field $\frac{\partial}{\partial t}$ now induces the reparametrized flow instead of the linear one.

Definition 2.1 *A vector $\alpha \in \mathbb{R}^m$ is* diophantine *if and only if there exist two constants $\gamma > 0$ and $\tau \geq m$ such that*

$$|\alpha \cdot k + p| \geq \gamma(|k| + |p|)^{-\tau} \ \forall k \in \mathbb{Z}^m \setminus \{0\} \ and \forall p \in \mathbb{Z}, \tag{2.1}$$

where $k = (k_1, \ldots k_m)$, $|k| = |k_1| + \ldots + |k_m|$.

The remarkable fact is that we can equivalently say that α is diophantine if and only if any smooth reparametrization of $R_{t(1,\alpha)}$ is C^∞ conjugate to a linear flow :

Proposition 2.2 *Let $m \geq 1$. A vector $\alpha \in \mathbb{R}^m$ is* diophantine *if and only if for all strictly positive C^∞ function $\varphi : \mathbb{T}^m \to (0, +\infty)$ the flow built over the translation R_α on $\mathbb{T}^m$ under the roof function φ is C^∞ conjugate to the suspension flow over R_α under the constant function $\hat{\varphi}_0 = \int_{\mathbb{T}^m} \varphi$.*

The two properties being equivalent, one could simply replace the arithmetical definition with the statement of the above proposition and introduce diophantine numbers by means of a purely dynamical criterion.

Proof. The special flow (or reparametrized flow) is smoothly conjugate to a linear flow if it admits a smooth cross–section for which the return time is constant. Looking for this section as a graph $t = \tau(y)$ we obtain from the definition that α is diophantine if and only if the coboundary equation

$$\tau(y + \alpha) - \tau(y) = \hat{\varphi}_0 - \varphi(y) \ . \tag{2.2}$$

has a C^∞ solution τ for any given C^∞ function φ.

Let $\tau(y) = \sum_{k \in \mathbb{Z}^m} \hat{\tau}_k e^{2\pi i k \cdot y}$, $\varphi(y) = \sum_{k \in \mathbb{Z}^m} \hat{\varphi}_k e^{2\pi i k \cdot y}$. Comparing the Fourier coefficients on both sides of (2.2) one has

$$(e^{2\pi i k \cdot \alpha} - 1)\hat{\tau}_k = \hat{\varphi}_0 \delta_{k,0} - \hat{\varphi}_k \ . \tag{2.3}$$

The Fourier coefficients of φ are completely arbitrary (except for the constraint of being rapidly decreasing as $|k| \to \infty$) thus (2.3) has a C^∞ solution if and only if $|e^{2\pi i k \cdot \alpha} - 1|^{-1}$ grows at most as a power of $|k|$ as $|k| \to \infty$, i.e. (2.1). □

For time–reparametrizations of flows the coboundary equation (2.2) becomes a constant coefficients linear partial differential equation on $\mathbb{T}^n$

$$D_\alpha u := \alpha \cdot \partial u = v \ , \tag{2.4}$$

where $\alpha \in \mathbb{R}^n$, $\partial u = (\partial_1 u, \ldots, \partial_n u)$, is the gradient of u, $v \in C^{0,\infty}(\mathbb{T}^n, \mathbb{R}^m)$ (i.e. $v \in C^\infty(\mathbb{T}^n, \mathbb{R}^m)$ *and* $\int_{\mathbb{T}^n} v(x)dx = 0$). Indeed (2.2) is just the discrete analogue of (2.4) obtained replacing the directional derivative $\alpha \cdot \partial$ with a first order finite difference. Note that D_α is hypoelliptic if and only if α is diophantine.

Being diophantine is a generic property from the point of view of measure theory : almost all $\alpha \in \mathbb{R}^n$ is diophantine of exponent $\tau > n$. It is not very difficult to construct explicit examples of diophantine vectors : for example, one can use the following easy argument taken from the book of Y. Meyer [Me] (Proposition 2, p. 16). Let $\mathcal{R}$ be a real algebraic number field and let n be its degree over $\mathbb{Q}$. Let σ be the $\mathbb{Q}$–isomorphism of $\mathcal{R}$ such that $\sigma(\mathcal{R}) \subset \mathbb{R}$ and let $\alpha_1, \ldots \alpha_n$ be any basis of $\mathcal{R}$ over $\mathbb{Q}$. Then $(\sigma(\alpha_1), \ldots, \sigma(\alpha_n)) \in \mathbb{R}^n$ is diophantine of exponent $\tau = n - 1$.

One can gain some more insight on the nature of the problems associated to the analysis of quasiperiodic motions considering a solution u of the cohomological equation (2.2) or (2.4) and the bounds of its C^k norm $\| \ \|_k$. If α is diophantine with exponent τ then for all $r > \tau + n - 1$ and for all $i \in \mathbb{N}$ there exists a positive constant A_i such that $\|u\|_i \le A_i \|v\|_{i+r}$.

The fact that one needs r more derivatives to bound the norms of u in terms of those of v is what is called the "loss of differentiability". This is not an artefact of the methods used but a concrete manifestation of the unboundedness of the linear operator D_α^{-1}. The main consequence of this fact is that one cannot use Banach spaces techniques to study semilinear equations like $D_\alpha u = v + \varepsilon f(u)$, where ε is some small parameter. These semilinear equations are however typical of perturbation theory and arise naturally in the study of the stability of quasiperiodic motions under small perturbations (see [Mar2], [Yo1], [DLL] for an introduction).

When an irrational flow is reparametrized only the most robust of its asymptotic properties (like ergodicity, topological transitivity, minimality, the vanishing of its topological entropy) are preserved. Other important properties, studied by ergodic theory can be sensitive to time change. However when the time reparametrization function is a coboundary, i.e. (2.2) (or (2.4)) has a regular solution, the reparametrized flow is conjugate to the initial flow. This is always the case as we have seen for diophantine frequencies. This set has full Lebesgue measure but it is meagre in the sense of Baire category and the numbers that are not diophantine, the so called Liouvillean numbers, are therefore abundant from the topological point of view.

For a Liouvillean α the reparametrization of $R_{t(1,\alpha)}$ can have asymptotic properties that are much different from the initial flow. For instance the reparametrized flow can be weakly mixing, i.e. has no eigenfunctions at all. Specifically, M.D. Šklover [Sk] proved existence of analytic weakly mixing reparametrizations for some Liouvillean linear flows on $\mathbb{T}^2$; his result for special flows on which this is based is optimal in that he showed that for any analytic roof function φ other than a trigonometric polynomial there is α such that the special flow under the rotation R_α with the roof function φ is weakly mixing. At about the same time A. Katok found a general criterion for weak mixing. B. Fayad [Fay1] showed that for any Liouville translation R_α on the torus $\mathbb{T}^n$ the special flow under a generic C^∞ function φ is weak-mixing. Still a linear flow of $\mathbb{T}^2$ cannot become mixing under smooth time change, not even under a Lipschitz one [Ko]. The argument is based on Denjoy–Koksma type estimates which fail in higher dimension. Indeed, Fayad [Fay2] showed that there exist $\alpha \in \mathbb{R}^2$ and analytic functions φ for which the special flow over the translation R_α and under the function φ is mixing.

3. Rotations, return times and continued fractions

From now on we will concentrate on single–frequency quasiperiodic systems (1–frequency maps or linear flows on 2–tori). As we have seen, for a map f being quasiperiodic means that for a suitably chosen sequence $n_k \to \infty$ of *return times* one has $f^{n_k} \to$identity, i.e. $f^{n_k+1} \to f$. This remark is the starting point of the renormalization approach to the study of quasiperiodic dynamics [CJ, MK, Yo2, Yo3]. In order to be able to exploit it, it is of fundamental importance to have an efficient algorithm for choosing return times. The classical continued fraction algorithm generated by the Gauss map is the natural way to analyze and to define the return times and the (diophantine) approximation properties of the frequency of the motion.

The modular group $\mathrm{GL}\,(2,\mathbb{Z})$ is here of fundamental importance. It appears both as the group of isotopy classes of diffeomorphisms of the two–torus and as the group associated to the continued fraction algorithm (more on this connection will be explained later, in Section 13). To better understand the action of $\mathrm{GL}\,(2,\mathbb{Z})$ on $\mathbb{R}\setminus\mathbb{Q}$ we can introduce a fundamental domain $[0,1)$ for one of the two generators (the translation) and restrict our attention to the inversion $\alpha \mapsto 1/\alpha$ restricted to $[0,1)$. This gives us a "microscope" since $\alpha \mapsto 1/\alpha$ is expanding on $[0,1)$, i.e. its derivative is always greater than 1. Our microscope magnifies more and more as $\alpha \to 0+$ and leads to the introduction of continued fractions. These arise constructing the symbolic dynamics of the Gauss map (as well as they can be obtained considering symbolic dynamics for the linear flow on the two–dimensional torus or for the geodesic flow on the modular surface).

Let $\{x\}$ denote the fractional part of a real number x : $\{x\} = x - [x]$, where $[x]$ is the integer part of x. Here we will consider the iteration of the Gauss map $A : (0,1) \mapsto [0,1]$, defined by

$$A(x) = \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left[\frac{1}{x} \right] . \tag{3.1}$$

To each $x \in \mathbb{R} \setminus \mathbb{Q}$ we associate a continued fraction expansion by iterating A as follows. Let

$$\begin{aligned} x_0 &= x - [x] , \\ a_0 &= [x] , \end{aligned} \tag{3.2}$$

then $x = a_0 + x_0$. We now define inductively for all $n \geq 0$

$$\begin{aligned} x_{n+1} &= A(x_n) , \\ a_{n+1} &= \left[\frac{1}{x_n} \right] \geq 1 , \end{aligned} \tag{3.3}$$

thus

$$x_n^{-1} = a_{n+1} + x_{n+1} . \tag{3.4}$$

Therefore we have

$$x = a_0 + x_0 = a_0 + \frac{1}{a_1 + x_1} = \ldots = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + x_n}}} , \tag{3.5}$$

and we will write

$$x = [a_0, a_1, \ldots, a_n, \ldots] . \tag{3.6}$$

The nth-convergent is defined by

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}} . \tag{3.7}$$

The numerators p_n and denominators q_n are recursively determined by

$$p_{-1} = q_{-2} = 1 \ , \quad p_{-2} = q_{-1} = 0 \ , \tag{3.8}$$

and for all $n \geq 0$

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} , \\ q_n &= a_n q_{n-1} + q_{n-2} . \end{aligned} \tag{3.9}$$

Moreover

$$x = \frac{p_n + p_{n-1}x_n}{q_n + q_{n-1}x_n} , \tag{3.10}$$

$$x_n = -\frac{q_n x - p_n}{q_{n-1}x - p_{n-1}} , \tag{3.11}$$

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n . \tag{3.12}$$

Let

$$\beta_n = \Pi_{i=0}^{n} x_i = (-1)^n (q_n x - p_n) \quad \text{for } n \geq 0, \quad \text{and } \beta_{-1} = 1 \tag{3.13}$$

and let

$$G = \frac{\sqrt{5}+1}{2} , \; g = G^{-1} = \frac{\sqrt{5}-1}{2} . \tag{3.14}$$

The following proposition is an easy consequence of the previous formulas.

Proposition 3.1 *For all $x \in \mathbb{R} \setminus \mathbb{Q}$ and for all $n \geq 1$ one has*
(i) $\quad |q_n x - p_n| = \frac{1}{q_{n+1} + q_n x_{n+1}}$, *so that* $\frac{1}{2} < \beta_n q_{n+1} < 1$ *;*
(ii) $\beta_n \leq g^n$ *and* $q_n \geq \frac{1}{2} G^{n-1}$ *.*

Proof. Using (3.10) one has

$$\begin{aligned} |q_n x - p_n| &= \left| q_n \frac{p_{n+1} + p_n x_{n+1}}{q_{n+1} + q_n x_{n+1}} - p_n \right| = \frac{|q_n p_{n+1} - p_n q_{n+1}|}{q_{n+1} + q_n x_{n+1}} \\ &= \frac{1}{q_{n+1} + q_n x_{n+1}} \end{aligned}$$

by (3.12). This proves (i).

Let us now consider $\beta_n = x_0 x_1 \dots x_n$. If $x_k \geq g$ for some $k \in \{0, 1, \dots, n-1\}$, then, letting $m = x_k^{-1} - x_{k+1} \geq 1$,

$$x_k x_{k+1} = 1 - m x_k \leq 1 - x_k \leq 1 - g = g^2 .$$

This proves (ii). □

Remark 3.2 Note that from *(ii)* it follows that $\sum_{k=0}^{\infty} \frac{\log q_k}{q_k}$ and $\sum_{k=0}^{\infty} \frac{1}{q_k}$ are always convergent and their sum is uniformly bounded.

For all integers $k \geq 1$, the iteration of the Gauss map k times leads to the following partition of $(0, 1)$; $\sqcup_{a_1, \dots, a_k} I(a_1, \dots, a_k)$, where $a_i \in \mathbb{N}$, $i = 1, \dots, k$, and

$$I(a_1, \dots, a_k) = \begin{cases} \left(\frac{p_k}{q_k}, \frac{p_k + p_{k-1}}{q_k + q_{k-1}} \right) & \text{if } k \text{ is even} \\ \left(\frac{p_k + p_{k-1}}{q_k + q_{k-1}}, \frac{p_k}{q_k} \right) & \text{if } k \text{ is odd} \end{cases}$$

is the branch of A^k determined by the fact that all points $x \in I(a_1, \dots, a_k)$ have the first $k+1$ partial quotients exactly equal to $\{0, a_1, \dots, a_k\}$. Thus

$$I(a_1, \dots, a_k) = \left\{ x \in (0,1) \mid x = \frac{p_k + p_{k-1}y}{q_k + q_{k-1}y}, \ y \in (0,1) \right\}.$$

Note that $\frac{dx}{dy} = \frac{(-1)^k}{(q_k+q_{k-1}y)^2}$ is positive (negative) if k is even (odd). It is immediate to check that any rational number $p/q \in (0,1)$, $(p,q) = 1$, is the endpoint of exactly two branches of the iterated Gauss map. Indeed p/q can be written as $p/q = [\bar{a}_1, \dots, \bar{a}_k]$ with $k \geq 1$ and $\bar{a}_k \geq 2$ in a unique way and it is the left (right) endpoint of $I(\bar{a}_1, \dots, \bar{a}_k)$ and the right (left) endpoint of $I(\bar{a}_1, \dots, \bar{a}_k - 1, 1)$ if k is even (odd).

The intimate connection between the modular group and the Gauss map appears also through the fact that two points $x, y \in \mathbb{R} \setminus \mathbb{Q}$ have the same $\mathrm{SL}(2,\mathbb{Z})$–orbit *if and only if* $x = [a_0, a_1, \dots, a_m, c_0, c_1, \dots]$ and $y = [b_0, b_1, \dots, b_n, c_0, c_1, \dots]$.

In most cases, the analysis of return times can be reduced to the study of the sequence (q_n) thanks to the two following results (see [HW], respectively Theorems 182, p. 151 and 184, p. 153).

Theorem 3.3 (Best approximation) *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and let p_n/q_n denote its n–th convergent. If $0 < q < q_{n+1}$ then $|qx - p| \geq |q_n x - p_n|$ for all $p \in \mathbb{Z}$ and equality can occur only if $q = q_n$, $p = p_n$.*

Theorem 3.4 *If $\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is a convergent of x.*

4. Classical Diophantine Conditions

Let $\gamma > 0$ and $\tau \geq 0$ be two real numbers. We recall that an irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ is diophantine of exponent τ and constant γ if and only if for all $p, q \in \mathbb{Z}$, $q > 0$, one has $\left| x - \frac{p}{q} \right| \geq \gamma q^{-2-\tau}$. Here the choice of the exponent of q is such that τ is always non–negative and can attain the value 0 (e.g. on quadratic irrationals). We denote $\mathrm{CD}(\gamma, \tau)$ the set of all diophantine x of exponent τ and constant γ. $\mathrm{CD}(\tau)$ will denote the union $\cup_{\gamma>0} \mathrm{CD}(\gamma, \tau)$ and $\mathrm{CD} = \cup_{\tau \geq 0} \mathrm{CD}(\tau)$. The complement in $\mathbb{R} \setminus \mathbb{Q}$ of CD is called the set of *Liouville numbers.*

Applying Proposition 3.1 it is easy to see that

$$\begin{aligned} \mathrm{CD}(\tau) &= \{x \in \mathbb{R} \setminus \mathbb{Q} \mid q_{n+1} = \mathrm{O}(q_n^{1+\tau})\} = \{x \in \mathbb{R} \setminus \mathbb{Q} \mid a_{n+1} = \mathrm{O}(q_n^{\tau})\} \\ &= \{x \in \mathbb{R} \setminus \mathbb{Q} \mid x_n^{-1} = \mathrm{O}(\beta_{n-1}^{-\tau})\} = \{x \in \mathbb{R} \setminus \mathbb{Q} \mid \beta_n^{-1} = \mathrm{O}(\beta_{n-1}^{-1-\tau})\} \end{aligned} \tag{4.1}$$

Liouville proved that if x is an algebraic number of degree $n \geq 2$ then $x \in \mathrm{CD}\,(n-2)$. Thue improved this result in 1909 showing that $x \in \mathrm{CD}\,(\tau-1+n/2)$ for all $\tau > 0$. In the early fifties Roth showed that algebraic numbers belong to the set $\mathrm{RT} = \cap_{\tau>0}\mathrm{CD}\,(\tau)$, nowadays called the set of numbers of *Roth type*. Again from Proposition 3.1 one obtains two further (equivalent) arithmetical characterizations of Roth type irrationals :

- in terms of the growth rate of the denominators of the continued fraction : $q_{n+1} = \mathrm{O}\,(q_n^{1+\varepsilon})$ for all $\varepsilon > 0$;
- in terms of the growth rate of the partial quotients : $a_{n+1} = \mathrm{O}\,(q_n^{\varepsilon})$ for all $\varepsilon > 0$.

Clearly, RT, CD and $\mathrm{CD}\,(\tau)$ for all $\tau \geq 0$ are $\mathrm{SL}\,(2,\mathbb{Z})$–invariant. The set $\mathrm{CD}\,(0)$ is also called the set of numbers of *constant type*, since $x \in \mathrm{CD}\,(0)$ if and only if the sequence of its partial fractions is bounded. $\mathrm{CD}\,(0)$ has Hausdorff dimension 1 and zero Lebesgue measure, whereas RT and $\mathrm{CD}\,(\tau)$, $\tau > 0$, have full Lebesgue measure.

In addition to these purely arithmetical characterizations, equivalent definitions of diophantine and Roth type numbers arise naturally in the study of the cohomological equation

$$\Psi - \Psi \circ R_\alpha = \Phi$$

associated to the rotation $R_\alpha : x \mapsto x+\alpha$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. As we have seen in Section 2, α being diophantine is equivalent to the fact that each C^∞ function Φ with zero mean $\int_{\mathbb{T}} \Phi dx = 0$ on the circle is the coboundary of a C^∞ function Ψ. One can prove that α is of Roth type if and only if for all non integer $r, s \in \mathbb{R}$ with $r > s+1 \geq 1$ and for all functions Φ of class C^r on $\mathbb{T}$ with zero mean there exists a unique function Ψ of class C^s on $\mathbb{T}$ and with zero mean such that $\Psi - \Psi \circ R_\alpha = \Phi$.

5. Brjuno Numbers and the real Brjuno Function

A more general class than diophantine numbers will appear in the context of stability of quasiperiodic orbits under analytic perturbations : the Brjuno numbers. These have been introduced by A.D. Brjuno [Br] in the late Sixties and have become more important after the celebrated results of Yoccoz [Yo2, Yo3] on the Siegel problem and on linearizations of analytic circle diffeomorphisms.

Definition 5.1 *x is a* Brjuno number *if $B(x) := \sum_{n=0}^{\infty} \beta_{n-1} \log x_n^{-1} < +\infty$. The function $B : \mathbb{R}\setminus\mathbb{Q} \to (0,+\infty]$ is called the* Brjuno function.

By (4.1) all diophantine numbers are Brjuno numbers but also "many" Liouville numbers are Brjuno numbers : for example $\sum_{n\geq 1} 10^{-n!}$ is a Brjuno number. It is easy to prove that there exists $C>0$ such that for all Brjuno numbers x one has

$$\left| B(x) - \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} \right| \leq C\,. \tag{5.1}$$

A slightly different version of the Brjuno function has been first introduced by Yoccoz [Yo2] : the difference is that it is based on a variant of the continued fraction expansion which makes use of the distance to the nearest integer instead of the fractional part in the definition (3.1) of the Gauss map. In both cases the Brjuno function satifies a remarkable functional equation under the action of the generators of the modular group. Adopting the standard continued fraction algorithm (described Section 3) in the definition of the Brjuno function leads to the equations :

$$\begin{aligned} &B(x) = B(x+1)\,, \quad \forall x \in \mathbb{R}\setminus\mathbb{Q} \\ &B(x) = -\log x + xB\left(\frac{1}{x}\right)\,, \quad x \in \mathbb{R}\setminus\mathbb{Q}\cap(0,1) \end{aligned} \tag{5.2}$$

This makes clear that the set of Brjuno numbers is $\mathrm{SL}\,(2,\mathbb{Z})$–invariant. Moreover, since quadratic irrationals have an eventually periodic continued fraction expansion, for each of them one can compute the Brjuno function *exactly* with finitely many iterations of (5.2). Thus B is known exactly on a countable but dense set of irrationals. In Figure 1 one can see a plot of the Brjuno function at 10000 random values of α uniformly distributed in the interval $(0,1)$. Note the logarithmic singularities associated to each rational number.

6. Linearization of germs of analytic diffeomorphisms

Let $\mathbb{C}[[z]]$ denote the ring of formal power series and $\mathbb{C}\{z\}$ denote the ring of convergent power series.

Let G denote the group of germs of holomorphic diffeomorphisms of $(\mathbb{C},0)$ and let $\hat{G}$ denote the group of formal germs of holomorphic diffeomorphisms of $(\mathbb{C},0)$: $G=\{f\in z\mathbb{C}\{z\}\,,f'(0)\neq 0\}$, $\hat{G}=\{\hat{f}\in z\mathbb{C}[[z]]\,,\hat{f}_1\neq 0\}$. One has the

Small divisors and Brjuno functions

trivial fibrations

$$
\begin{array}{ccc}
G = \cup_{\lambda \in \mathbb{C}^*} G_\lambda & \hookrightarrow & \hat{G} = \cup_{\lambda \in \mathbb{C}^*} \hat{G}_\lambda \\
\pi \downarrow & & \hat{\pi} \downarrow \\
\mathbb{C}^* & & \mathbb{C}^*
\end{array}
\tag{6.1}
$$

where

$$
\hat{G}_\lambda = \{\hat{f}(z) = \sum_{n=1}^{\infty} \hat{f}_n z^n \in \mathbb{C}[[z]] \, , \, \hat{f}_1 = \lambda\} \, , \tag{6.2}
$$

$$
G_\lambda = \{f(z) = \sum_{n=1}^{\infty} f_n z^n \in \mathbb{C}\{z\} \, , \, f_1 = \lambda\} \, . \tag{6.3}
$$

Let R_λ denote the germ $R_\lambda(z) = \lambda z$. This is the simplest element of G_λ. It is easy to check that, if λ is not a root of unity, its centralizer is $\text{Cent}\,(R_\lambda) = \{R_\mu \, , \, \mu \in \mathbb{C}^*\}$.

Definition 6.1 *A germ $f \in G_\lambda$ is* linearizable *if there exists $h_f \in G_1$ (a linearization of f) such that $h_f^{-1} f h_f = R_\lambda$, i.e. f is conjugate to (its linear part) R_λ. f is* formally linearizable *if there exists $\hat{h}_f \in \hat{G}_1$ such that $\hat{h}_f^{-1} f \hat{h}_f = R_\lambda$ (note that in this case this is a functional equation in the ring $\mathbb{C}[[z]]$ of formal power series).*

When λ is a root of unity it is not difficult to prove the following Proposition (see, e.g. [Ma2])

Proposition 6.2 *Assume λ is a primitive root of unity of order q. A germ $f \in G_\lambda$ is linearizable if and only if $f^q = id$. The same holds for a formal germ $\hat{f} \in \hat{G}_\lambda$.*

When λ is not a root of unity the linearization (if it exists) is unique and one can recursively determine the coefficients h_n of the power series expansion of $h_f(z) = \sum_{n=1}^{\infty} h_n z^n$. Indeed from the linearization equation $f h_f = h_f R_\lambda$ we get, for $n \geq 2$ (remember that we want $h \in G_1$, thus $h_1 = 1$) :

$$
h_n = \frac{1}{\lambda^n - \lambda} \sum_{j=2}^{n} f_j \sum_{n_1 + \ldots + n_j = n} h_{n_1} \cdots h_{n_j} \, . \tag{6.4}
$$

In the holomorphic case the problem of a *complete* classification of the conjugacy classes is open, formidably complicated and perhaps unreasonable [Yo2, PM2, PM3]. The first important result in the holomorphic case is the classical Koenigs–Poincaré Theorem which gives a complete solution to the problem of conjugacy classes in the *hyperbolic* case, i.e. when $|\lambda| \neq 1$.

Theorem 6.3 (Koenigs–Poincaré) *If $|\lambda| \neq 1$ then G_λ is a conjugacy class, i.e. all $f \in G_\lambda$ are linearizable.*

Proof. Since f is holomorphic around $z = 0$ there exists $c_1 > 1$ and $r \in (0,1)$ such that $|f_j| \leq c_1 r^{1-j}$ for all $j \geq 2$. Since $|\lambda| \neq 1$ there exists $c_2 > 1$ such that $|\lambda^n - \lambda|^{-1} \leq c_2$ for all $n \geq 2$.

Let $(\sigma_n)_{n\geq 1}$ be the following recursively defined sequence :

$$\sigma_1 = 1\ , \quad \sigma_n = \sum_{j=2}^{n} \sum_{n_1+\ldots+n_j=n} \sigma_{n_1}\cdots\sigma_{n_j}\ . \tag{6.5}$$

The generating function $\sigma(z) = \sum_{n=1}^{\infty} \sigma_n z^n$ satisfies the functional equation

$$\sigma(z) = z + \frac{\sigma(z)^2}{1-\sigma(z)}\ , \tag{6.6}$$

thus $\sigma(z) = \frac{1+z-\sqrt{1-6z+z^2}}{4}$ is analytic in the disk $|z| < 3 - 2\sqrt{2}$ and bounded and continuous on its closure. By Cauchy's estimate one has $\sigma_n \leq c_3(3-2\sqrt{2})^{1-n}$ for some $c_3 > 0$.

Since λ is not a root of unity, f is formally linearizable and the power series coefficients of its formal linearization $\hat{h}_f$ satisfy (6.4). By induction one can check that $|\hat{h}_n| \leq (c_1c_2r^{-1})^{n-1}\sigma_n$, thus $\hat{h}_f \in \mathbb{C}\{z\}$. □

Remark 6.4 Since the bound $|\lambda^n - \lambda|^{-1} \leq c_2$ is uniform w.r.t $\lambda \in D(\lambda_0, \delta)$, where $\lambda_0 \in \mathbb{C}^* \setminus \mathbb{S}^1$ and $\delta < \mathrm{dist}\,(\lambda_0, \mathbb{S}^1)$, the above given proof of the Poincaré–Koenigs Theorem shows that the map

$$\begin{aligned} \mathbb{C}^* \setminus \mathbb{S}^1 &\to G_1 \\ \lambda &\mapsto h_{\tilde{f}}(\lambda) \end{aligned}$$

is analytic for all $\tilde{f} \in z^2\mathbb{C}\{z\}$, where $h_{\tilde{f}}(\lambda)$ is the linearization of $\lambda z + \tilde{f}(z)$. This notion needs a little comment since $\mathbb{C}\{z\}$ is a rather wild space : it is an inductive limit of Banach spaces, thus it is a locally convex topological vector space and it is complete but it is not metrisable, thus it is not a Fréchet space.

Here we simply mean that if λ varies in some relatively compact open connected subset of $\mathbb{C}^* \setminus \mathbb{S}^1$ then $h_{\bar{f}}(\lambda)$ belongs to some fixed Banach space of holomorphic functions (e.g. the Hardy space $H^\infty(\mathbb{D}_r)$ of bounded analytic functions on the disk $\mathbb{D}_r = \{z \in \mathbb{C}\,,\ |z| < r\}$, where $r > 0$ is fixed and small enough) and depends analytically on λ in the usual sense.

In the next Sections we will concentrate on the study of the problem of existence of linearizations of germs of holomorphic diffeomorphisms. To this purpose the following "normalization" will be useful.

Let us note that there is an obvious action of $\mathbb{C}^*$ on G by homotheties :

$$(\mu, f) \in \mathbb{C}^* \times G \mapsto \mathrm{Ad}_{R_\mu} f = R_\mu^{-1} f R_\mu \,. \tag{6.7}$$

Note that this action leaves the fibers G_λ invariant. Also, $f \in G_\lambda$ is linearizable if and only if $\mathrm{Ad}_{R_\mu} f$ is also linearizable for all $\mu \in \mathbb{C}^*$ (indeed if h_f linearizes f then $\mathrm{Ad}_{R_\mu} h_f$ linearizes $\mathrm{Ad}_{R_\mu} f$). Therefore, in order to study the problem of the existence of a linearization, it is enough to consider $G/\mathbb{C}^*$, i.e. we identify two germs of holomorphic diffeomorphisms which are conjugate by a homothety.

Consider the space $\mathcal{S}$ of univalent maps $F : \mathbb{D} \to \mathbb{C}$ such that $F(0) = 0$ and the projection

$$\begin{aligned} &G \to \mathcal{S} \\ &f \mapsto F = \begin{cases} f & \text{if } f \text{ is univalent in } \mathbb{D} \\ \mathrm{Ad}_{R_r} f & \text{if } f \text{ is univalent in } \mathbb{D}_r \end{cases} \end{aligned}$$

This map is clearly onto and two germs have the same image only if they coincide or if they are conjugate by some homothety. Thus this projection induces a bijection from $G/\mathbb{C}^*$ onto $\mathcal{S}$.

In what follows we will always consider the topological space $\mathcal{S}$ of germs of holomorphic diffeomorphisms $f : \mathbb{D} \to \mathbb{C}$ such that $f(0) = 0$ and f is univalent in $\mathbb{D}$. We will denote

- $\mathcal{S}_\lambda$ the subspace of f such that $f'(0) = \lambda$;
- $\mathcal{S}_\mathbb{T}$ the subspace of f such that $|f'(0)| = 1$.

Clearly the projection above induces a bijection between $G_\lambda/\mathbb{C}^*$ and $\mathcal{S}_\lambda$.

To each germ $f \in \mathcal{S}$, $|f'(0)| \le 1$, one can associate a natural f–invariant compact set

$$0 \in K_f := \bigcap_{n \ge 0} f^{-n}(\mathbb{D}) \,. \tag{6.8}$$

Let U_f denote the connected component of the interior of K_f which contains 0. Then 0 is stable if and only if $U_f \neq \emptyset$, i.e. if and only if 0 belongs to the interior of K_f. Clearly, if $f \in S$ and $|f'(0)| < 1$ then 0 is stable.The extremely remarkable fact is that stability, which is a topological property, is equivalent to linearizability, which is an analytic property.

Theorem 6.5 *Let $f \in S$, $|f'(0)| \leq 1$. 0 is stable if and only if f is linearizable.*

Proof. The statement is non–trivial only if $\lambda = f'(0)$ has unit modulus. If f is linearizable then the linearization h_f maps a small disk $\mathbb{D}_r$ around zero conformally into $\mathbb{D}$. Since $h_f(0) = 0$ and $|f^n(z)| < 1$ for all $z \in h_f(\mathbb{D}_r)$ one sees that 0 is stable.

Conversely assume now that 0 is stable. Then $U_f \neq \emptyset$ and one can easily see that it must also be simply connected (otherwise, if it had a hole V, surrounding it with some closed curve γ contained in U_f since $|f^n(z)| < 1$ for all $z \in \gamma$ and $n \geq 0$ the maximum principle leads to the same conclusion for all the points in V thus $V \subset U_f$). Applying the Riemann mapping theorem to U_f one sees that by conjugation with the Riemann map f induces a univalent map g of the disk into itself with the same linear part λ. By Schwarz' Lemma one must have $g(z) = \lambda z$ thus f is analytically linearizable. □

When $\lambda = f'(0)$ has modulus one, it is not a root of unity and 0 is stable then U_f is conformally equivalent to a disk and is called the *Siegel disk* of f (at 0). Thus the Siegel disk of f is the maximal connected open set containing 0 on which f is conjugated to R_λ. The conformal representation $\tilde{h}_f : \mathbb{D}_{c(f)} \to U_f$ of U_f which satisfies $\tilde{h}_f(0) = 0$, $\tilde{h}_f'(0) = 1$ linearizes f thus the power series of $\tilde{h}_f$ and h_f coincide. If $r(f)$ denotes the radius of convergence of the linearization h_f (whose power series coefficients are recursively determined as in (6.4)), we see that $c(f) \leq r(f)$. One can prove that $c(f) = r(f)$ when at least one of the two following conditions is satisfied :

(i) U_f is relatively compact in $\mathbb{D}$;

(ii) each point of $\mathbb{S}^1$ is a singularity of f.

7. Small divisors

When $|\lambda| = 1$ and λ is not a root of unity we can write

$$\lambda = e^{2\pi i\alpha} \text{ with } \alpha \in \mathbb{R} \setminus \mathbb{Q} \cap (-1/2, 1/2) ,$$

and whether $f \in G_\lambda$ is linearizable or not depends crucially on the arithmetical properties of α. R_α itself is the prototype of quasiperiodic dynamics, thus we can look at the linearization problem as the problem of deciding if quasiperiodic orbits are preserved (locally) under analytic perturbation. This is not always the case as the following simple Theorem shows :

Theorem 7.1 (Cremer) *If* $\limsup_{n\to+\infty} |\{n\alpha\}|^{-1/n} = +\infty$ *then there exists* $f \in G_{e^{2\pi i\alpha}}$ *which is not linearizable.*

Proof. First of all note that $\limsup_{n\to+\infty} |\{n\alpha\}|^{-1/n} = +\infty$ if and only if

$$\limsup_{n\to+\infty} |\lambda^n - 1|^{-1/n} = +\infty$$

since

$$|\lambda^n - 1| = 2|\sin(\pi n\alpha)| \in (2|\{n\alpha\}|, \pi|\{n\alpha\}|) .$$

Then we construct f in the following manner : for $n \geq 2$ we take $|f_n| = 1$ and we choose inductively $\arg f_n$ such that

$$\arg f_n = \arg \sum_{j=2}^{n-1} f_j \sum_{n_1+\ldots+n_j=n} \hat{h}_{n_1} \cdots \hat{h}_{n_j} , \tag{7.1}$$

(recall the induction formula (6.4) for the coefficients of the formal linearization of f and note that the r.h.s. of (7.1) is a polynomial in $n-2$ variables $f_2, \ldots, f_{n-1}$ with coefficients in the field $\mathbb{C}(\lambda)$). Thus

$$|\hat{h}_n| \geq \frac{|f_n|}{|\lambda^n - 1|} = \frac{1}{|\lambda^n - 1|}$$

and $\limsup_{n\to+\infty} |\hat{h}_n|^{1/n} = +\infty$: the formal linearization $\hat{h}$ is a divergent series. □

Clearly the set of irrational numbers satisfying the assumption of Cremer's Theorem is a dense G_δ with zero Lebesgue measure.

After this negative result, it was pretty clear that the problem of the existence of analytic linearizations was not an easy one. The main difficulty is given by the unavoidable presence of *small divisors* in the recurrence (6.4). This difficulty was first overcome by Siegel in 1942 [S] but it was clearly well–known among mathematicians at the end of the 19th and at the beginning of the 20th century.

Assume that $\alpha \in \mathrm{CD}(\tau)$ for some $\tau \geq 0$. Recalling the recurrence (6.4) for the power series coefficients of the linearization one sees that h_n is a polynomial in $f_2, \ldots, f_n$ with coefficients which are rational functions of λ : $h_n \in \mathbb{C}(\lambda)[f_2, \ldots, f_n]$ for all $n \geq 2$.

Let us compute explicitely the first few terms of the recurrence

$$\begin{aligned} h_2 &= (\lambda^2-\lambda)^{-1} f_2 \,, \\ h_3 &= (\lambda^3-\lambda)^{-1}[f_3 + 2f_2^2(\lambda^2-\lambda)^{-1}] \,, \\ h_4 &= (\lambda^4-\lambda)^{-1}[f_4 + 3f_3f_2(\lambda^2-\lambda)^{-1} + 2f_2f_3(\lambda^3-\lambda)^{-1} \\ &\qquad 4f_2^3(\lambda^3-\lambda)^{-1}(\lambda^2-\lambda)^{-1} + f_2^3(\lambda^2-\lambda)^{-2}] \,, \end{aligned} \tag{7.2}$$

and so on. It is not difficult to see that among all contributes to h_n there is always a term of the form

$$2^{n-2} f_2^{n-1} [(\lambda^n-\lambda)\dots(\lambda^3-\lambda)(\lambda^2-\lambda)]^{-1} \,. \tag{7.3}$$

If one then tries to estimate $|h_n|$ by simply summing up the absolute values of each contribution then one term will be

$$2^{n-2}|f_2|^{n-1}[|\lambda^n-\lambda|\dots|\lambda^3-\lambda||\lambda^2-\lambda|]^{-1} \le 2^{n-2}|f_2|^{n-1}(2\gamma)^{(n-1)\tau}[(n-1)!]^\tau \tag{7.4}$$

if $\alpha \in \mathrm{CD}\,(\gamma,\tau)$ and one obtains a divergent bound. Note the difference with the case $|\lambda| \neq 1$: in this case the bound would be $|\lambda|^{-(n-1)}2^{n-2}|f_2|^{n-1}c^{n-1}$ for some positive constant c independent of f. Thus one must use a more subtle majorant series method.

The key point is that the estimate (7.4) is far too pessimistic : indeed if one considers the generating series associated to the small denominators appearing in the terms (7.3) (i.e. the series $\sum_{n=1}^{\infty} \frac{z^n}{(\lambda^n-1)\dots(\lambda-1)}$) one can even prove [HL] that it has positive radius of convergence whenever $\limsup_{n\to\infty} \frac{\log q_{k+1}}{q_k} < +\infty$.

8. The quadratic polynomial and Yoccoz's function

In this Section we will study in detail the linearization problem for the quadratic polynomial

$$P_\lambda(z) = \lambda\left(z - \frac{z^2}{2}\right) \,. \tag{8.1}$$

Apart from being the simplest nonlinear map, one good reason for starting our investigations from P_λ is provided by a theorem of Yoccoz [Yo2, pp.59–62] which shows how the quadratic polynomial is the "worst possible perturbation of the linear part R_λ" as the following statement makes precise :

Theorem 8.1 *Let $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. If P_λ is linearizable then every germ $f \in G_\lambda$ is also linearizable.*

The previous Theorem shows that the linearizability of the quadratic polynomial for a certain λ implies that G_λ is a conjugacy class. On the other hand one can prove the following

Theorem 8.2 *Let $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. For almost all $\lambda \in \mathbb{T}$ the quadratic polynomial P_λ is linearizable.*

In 1942 C.L. Siegel [S] proved that all analytic germs $f \in G_\lambda$ with $\alpha \in \mathrm{CD}$ are analytically linearizable, thus showing a more precise and more general result than Theorem 8.2. Siegel's result, later improved by Brjuno [Br], is based on a very clever and careful control of the accumulation of small denominators in the nonlinear recurrence (6.4). Here however we want to follow a different approach and we will sketch an argument, again due to Yoccoz, which does not make use of any small denominators estimates.

The quadratic polynomial P_λ has a unique critical point $c = 1$ with corresponding critical value $v_\lambda = P_\lambda(c) = \lambda/2$. If $|\lambda| < 1$ by Koenigs–Poincaré theorem we know that there exists a unique analytic linearization H_λ of P_λ and that it depends analytically on λ as λ varies in $\mathbb{D}$. Let $r_2(\lambda)$ denote the radius of convergence of H_λ. One has the following

Proposition 8.3 *Let $\lambda \in \mathbb{D}$. Then :*

(1) $r_2(\lambda) > 0$;

(2) $r_2(\lambda) < +\infty$ and H_λ has a continuous extension to $\overline{\mathbb{D}_{r_2(\lambda)}}$. Moreover the map $H_\lambda : \overline{\mathbb{D}_{r_2(\lambda)}} \to \mathbb{C}$ is conformal and verifies $P_\lambda \circ H_\lambda = H_\lambda \circ R_\lambda$.

(3) On its circle of convergence $\{z\,,\,|z| = r_2(\lambda)\}$, H_λ has a unique singular point which will be denoted $u(\lambda)$.

(4) $H_\lambda(u(\lambda)) = 1$ and $(H_\lambda(z) - 1)^2$ is holomorphic in $z = u(\lambda)$.

Proof. The first assertion is just a consequence of Koenigs–Poincaré theorem.

The functional equation $P_\lambda(H_\lambda(z)) = H_\lambda(\lambda z)$ is satisfied for all $z \in \mathbb{D}_{r_2(\lambda)}$. Moreover $H_\lambda : \mathbb{D}_{r_2(\lambda)} \to \mathbb{C}$ is univalent (if one had $H_\lambda(z_1) = H_\lambda(z_2)$ with $z_1 \neq z_2$ and $z_1, z_2 \in \mathbb{D}_{r_2(\lambda)}$ one would have $H_\lambda(\lambda^n z_1) = H_\lambda(\lambda^n z_2)$ for all $n \geq 0$ which is impossible since $|\lambda| < 1$ and $H'_\lambda(0) = 1$). Thus $r_2(\lambda) < +\infty$. On the other hand if H_λ is holomorphic in $\mathbb{D}_r$ for some $r > 0$ and the critical value $v_\lambda \notin H_\lambda(\mathbb{D}_r)$ the functional equation allows to continue analytically H_λ to the disk $\mathbb{D}_{|\lambda|^{-1}r}$.

Therefore there exists $u(\lambda) \in \mathbb{C}$ such that $|u(\lambda)| = r_2(\lambda)$ and $H_\lambda(\lambda u(\lambda)) = v_\lambda$. Such a $u(\lambda)$ is unique since H_λ is injective on $\mathbb{D}_{r_2(\lambda)}$. If $|w| = |\lambda| r_2(\lambda)$ and $w \neq \lambda u(\lambda)$ one has $H_\lambda(w) = P_\lambda(H_\lambda(\lambda^{-1}w))$ and

$$H_\lambda(\lambda^{-1}w) = 1 - \sqrt{1 - 2\lambda^{-1}H_\lambda(w)} \tag{8.2}$$

which shows how to extend continuously and injectively H_λ to $\overline{\mathbb{D}_{r_2(\lambda)}}$. By construction the functional equation is trivially verified. This completes the proof of (2).

To prove (3) and (4) note that from $H_\lambda(\lambda u(\lambda)) = P_\lambda(H_\lambda(u(\lambda)))$ it follows that $H_\lambda(u(\lambda)) = 1$. Formula (8.2) shows that all points $z \in \mathbb{C}$, $|z| = r_2(\lambda)$ are regular except for $z = u(\lambda)$. Finally one has $(H_\lambda(z) - 1)^2 = 1 - 2\lambda^{-1}H_\lambda(\lambda z)$ which is holomorphic also at $z = u(\lambda)$. □

The fact that H_λ is injective on $\overline{\mathbb{D}}_{r_2(\lambda)}$ implies that $r_2(\lambda) < +\infty$. One can easily obtain a more precise upper bound by means of Koebe 1/4–Theorem and prove that $r_2(\lambda) \leq 2$. The use of further standard distorsion estimates for univalent functions allows to prove that the sequence of polynomials $u_n(\lambda) = \lambda^{-n}P_\lambda^n(1)$ converges uniformly to u on compact subsets of $\mathbb{D}$. Thus $u : \mathbb{D}^* \to \mathbb{C}$ has a bounded analytic extension to $\mathbb{D}$ and $u(0) = 1/2$. The polynomials u_n verify the recurrence relation

$$u_0(\lambda) = 1\,, \quad u_{n+1}(\lambda) = u_n(\lambda) - \frac{\lambda^n}{2}(u_n(\lambda))^2\,. \tag{8.3}$$

The function $u : \mathbb{D} \to \mathbb{C}$ will be called *Yoccoz's function.* It has many remarkable properties and it is the object of various conjectures (see Section 11). Using (8.3) it is easy to check that $u(\lambda) \in \mathbb{Q}\{\lambda\}$ and all the denominators are a power of 2. Moreover $u(\lambda) - u_n(\lambda) = \mathrm{O}(\lambda^n)$ thus one can also compute the first terms of the power series expansion of u :

$$u(\lambda) = \frac{1}{2} - \frac{\lambda}{8} - \frac{\lambda^2}{8} - \frac{\lambda^3}{16} - \frac{9\lambda^4}{128} - \frac{\lambda^5}{128} - \frac{7\lambda^6}{128} + \frac{3\lambda^7}{256} - \frac{29\lambda^8}{1024} - \frac{\lambda^9}{256} + \frac{25\lambda^{10}}{2048} + \frac{559\lambda^{11}}{32768} + \dots\,.$$

It is easy and fast to compute on a personal computer the first 2000 terms of the series expansion of u : in the Figures 2a to 2d one can see the images through u of the circles of radii 0.8, 0.9, 0.99 and 0.999. Figures 3 and 4 show respectively the graph of the function $x \mapsto |u(0.999e^{2\pi i x})|$ as x varies in the interval $[0, 1/2]$ and the graph of $x \mapsto \arg u(0.999e^{2\pi i x})$ as x varies in the interval $[-1/2, 1/2]$. Note that the argument of the function u seems to have a decreasing jump of approximately π/q at each rational number p/q. We refer to [BHH, Ca] for a detailed numerical study of this function.

We can now conclude the proof of Theorem 8.2. Let $\lambda_0 \in \mathbb{T}$ and assume that λ_0 is not a root of unity. Then on can check easily that

$$r_2(\lambda_0) \geq \limsup_{\mathbb{D}\ni\lambda\to\lambda_0} |u(\lambda)| \tag{8.4}$$

but much more can be proved [Yo2, pp. 65-69]

Proposition 8.4 *For all $\lambda_0 \in \mathbb{T}$, $|u(\lambda)|$ has a non–tangential limit in λ_0 which is equal to the radius of convergence $r_2(\lambda_0)$ of H_{λ_0}.*

Of course, if λ_0 is a root of unity then P_{λ_0} is not even formally linearizable and one poses $r_2(\lambda_0) = 0$.

Applying Fatou's Theorem on the existence and almost everywhere non–vanishing of non–tangential boundary values of bounded holomorphic functions on the unit disk to $u : \mathbb{D} \to \mathbb{C}$ one finds that there exists $u^* \in L^\infty(\mathbb{T}, \mathbb{C})$ such that for almost all $\lambda_0 \in \mathbb{T}$ one has $|u^*(\lambda_0)| > 0$ and $u(\lambda) \to u(\lambda_0)$ as $\lambda \to \lambda_0$ non tangentially. From (8.4) one concludes that for almost all $\lambda_0 \in \mathbb{T}$ one has $r_2(\lambda_0) > 0$. □

9. Renormalization and quasiperiodic orbits : Yoccoz's theorems

The idea of renormalization has been extremely successful in dynamics. Its origin is in statistical physics and quantum field theory.

In physics two approaches to renormalization coexist : one (and the oldest) is perturbative the other is non–perturbative. The latter grew up from the study of critical phenomena (ferromagnetism, superfluidity, polymers, conductivity of random media, etc.) and statistical physics. Here one observes that very different systems are surprisingly similar in a quantitative way, since they have the same critical exponents scaling laws (universality). At the critical transition these systems are somehow dominated by large distance correlations which are not sensitive to the details of the microscopic interactions.

The applications in dynamical systems of this second approach have been most successful. In this Section we will illustrate how one can rigorously prove optimal results for the problem of linearization of holomorphic germs using a (geometrical) renormalization approach. Other rigorous results have been obtained for the local and global conjugacy problems for analytic diffeomorphsism of the circle [Yo3]. In addition a number of heuristic results [CJ, Da2, MK] have been obtained for Hamiltonian systems with one frequency (area–preserving maps of the plane or of the annulus or time–dependent one degree of freedom Hamiltonian flows).

The basic idea of (non–perturbative) renormalization in studying quasiperiodic orbits is as follows : given our dynamical system $f \in \mathrm{End}_*(X)$ if it has some region X_0 of phase space filled by quasiperiodic orbits then its iterates form an equicontinuous family on it and $f^{n_j+1} \approx f$ for some sequence $n_j \to \infty$. Thus one can look for a region $U_0 \subset X_0$ which returns (at least approximately) to itself under n_1 iterations of f. If we then construct the quotient $X_1 = U_0/f$ identifying $x \in U_0$ with $f(x)$ the renormalized map $\mathcal{R}_{n_1} f \in \mathrm{End}_*(X_1)$ will be induced by the first return map to U_0 (under the iteration of f). Then f will have quasiperiodic orbits of frequency $\alpha_0 \in (0,1)$ if and only if $\mathcal{R}_{n_1} f$ has quasiperiodic orbits of frequency α_1 with $\alpha_0^{-1} = \alpha_1 + n_1$.

The process can be iterated. If one can control the sequence $\mathcal{R}_{n_j} \cdots \mathcal{R}_{n_1} f \in \mathrm{End}_*(X_j)$ (and the geometric limit $X_0 \to X_1 \to X_2 \to \ldots$) proving its convergence to some fixed point $f_\infty \in \mathrm{End}_*(X_\infty)$, where $X_\infty = \lim X_j$ then one has proved that the dynamics of f on X_0 is the same as the dynamics of f_∞ on X_∞. In the quasiperiodic case f_∞ typically is a linear automorphism of X_∞ (trivial fixed point) but other possibilities are conceivable.

In what follows we will very briefly and approximately describe Yoccoz's analysis of the renormalization of quasiperiodic orbits in the simplest case of Siegel domains.

A *qualitative* analysis of the basic construction (first return map and geometric quotient) already gives some non–trivial information on the dynamics of germs of holomorphic diffeomorphisms with an indifferent fixed point.

Let $\mathcal{Y}$ denote the set of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that all holomorphic germs of diffeomorphisms of $(\mathbb{C}, 0)$ with linear part $e^{2\pi i\alpha}$ are linearizable. The results of the previous Section show that $\mathcal{Y}$ has full measure (combine Theorems 8.1 and 8.2) but its complement in $\mathbb{R}\setminus\mathbb{Q}$ is a G_δ–dense (Theorem 7.1). Further information on the structure of $\mathcal{Y}$ is provided by the following

Theorem 9.1 (Douady–Ghys) *$\mathcal{Y}$ is $SL(2,\mathbb{Z})$–invariant.*

Proof. (sketch). $\mathcal{Y}$ is clearly invariant under T, thus we only need to show that if $\alpha \in \mathcal{Y}$ then also $U \cdot \alpha = -1/\alpha \in \mathcal{Y}$.

Let $f(z) = e^{2\pi i\alpha} z + O(z^2)$ and consider a domain V' bounded by

1) a segment l joining 0 to $z_0 \in \mathbb{D}^*$, $l \subset \mathbb{D}$;
2) its image $f(l)$;
3) a curve l' joining z_0 to $f(z_0)$.

We choose l' and z_0 (sufficiently close to 0) so that l, l' and $f(l)$ do not intersect except at their extremities. Note that l and $f(l)$ form an angle of $2\pi\alpha$ at 0.

Then glueing l to $f(l)$ one obtains a topological manifold $\overline{V}$ with boundary which is homeomorphic to $\overline{\mathbb{D}}$. With the induced complex structure its interior is biholomorphic to $\mathbb{D}$. Let us now consider the *first return map* $g_{V'}$ to the domain V' (this is well defined if z is choosen with $|z|$ small enough) : if $z \in V'$ (and $|z|$ is small enough) we define $g_{V'}(z) = f^n(z)$ where n (depends on z) is defined asking that $f(z), \dots, f^{n-1}(z) \notin V'$ and $f^n(z) \in V'$, i.e. $n = \inf\{k \in \mathbb{N}, k \geq 1, f^k(z) \in V'\}$. Then it is easy to check that $n = \left[\frac{1}{\alpha}\right]$ or $n = \left[\frac{1}{\alpha}\right] + 1$. The first return map $g_{V'}$ induces a map $g_{\overline{V}}$ on a neighborhood of $0 \in \overline{V}$ and finally a germ g of holomorphic diffeomorphism at $0 \in \mathbb{D}$ ($gp = pg_{\overline{V}}$, where p is the projection from $\overline{V}$ to the disk $\mathbb{D}$). It is easy to check that $g(z) = e^{-2\pi i/\alpha}z + O(z^2)$ (note that in the passage from V' to $\mathbb{D}$ through $\overline{V}$ the angle $2\pi\alpha$ at the origin is mapped in 2π).

To each orbit of f near 0 corresponds an orbit of g near 0. In particular

- f is linearizable if and only if g is linearizable ;
- if f has a periodic orbit near 0 then also g has a periodic orbit ;
- if f has a point of instability (i.e. a point whose iterates leave a neighborhood of 0) then also g has a point of instability (which will escape even more rapidly).

In particular these statements show that $\alpha \in \mathcal{Y}$ if and only if $-1/\alpha \in \mathcal{Y}$. □

We will now briefly describe how to turn the above construction into a quantitative construction and how to use it to give a artihmetical characterization of the set $\mathcal{Y}$: it will turn out that it coincides with the set of Brjuno numbers.

Let $\mathcal{S}(\alpha)$ denote the universal cover of $\mathcal{S}_{e^{2\pi i\alpha}}$. An element $F \in \mathcal{S}(\alpha)$ is a univalent function $F : \mathbb{H} \to \mathbb{C}$ and $F(z) = z + \alpha + \varphi(z)$ where φ is $\mathbb{Z}$–periodic and $\lim_{\Im m\, z \to +\infty} \varphi(z) = 0$. Let $E : \mathbb{H} \to \mathbb{D}^*$ be the exponential map $E(z) = e^{2\pi i z}$: each function $f \in \mathcal{S}_{e^{2\pi i\alpha}}$ lifts to such a map F and $E \circ F = f \circ E$.

Let $r > 0$, $\mathbb{H}_r = \mathbb{H} + ir$. It is clear that if $F \in \mathcal{S}(\alpha)$ and r is sufficiently large then F is very close to the translation $z \mapsto z + \alpha$ for $z \in \mathbb{H}_r$. Indeed using the compactness of the space $\mathcal{S}(\alpha)$ and the distorsion estimates for univalent functions one can prove the following :

Proposition 9.2 *Let $\alpha \neq 0$. There exists a universal constant $c_0 > 0$ (i.e. independent of α) such that for all $F \in \mathcal{S}(\alpha)$ and for all $z \in \mathbb{H}_{t(\alpha)}$ where*

$$t(\alpha) = \frac{1}{2\pi} \log \alpha^{-1} + c_0 , \tag{9.1}$$

one has

$$|F(z) - z - \alpha| \leq \frac{\alpha}{4} . \tag{9.2}$$

Given F, the lowest admissible value $t(F,\alpha)$ of $t(\alpha)$ such that (9.2) holds for all $z \in \mathbb{H}_{t(F,\alpha)}$ represents the height in the upper half plane $\mathbb{H}$ at which the strong nonlinearities of F manifest themselves. When $\Im m\, z > t(F,\alpha)$, F is very close to the translation $T_\alpha(z) = z + \alpha$. This is equivalent to say that f is very close to the rotation by $2\pi\alpha$ when $z \in \mathbb{D}$ is sufficiently small.

An example of strong nonlinearity is of course a fixed point : if $F(z) = z+\alpha+\frac{1}{2\pi i}e^{2\pi i z}$, $\alpha > 0$, then $z = -\frac{1}{4}+\frac{i}{2\pi}\log(2\pi\alpha)^{-1}$ is fixed and $t(F,\alpha) \geq \frac{1}{2\pi}\log(2\pi\alpha)^{-1}$.

Following the construction in the proof of Douady–Ghys theorem we can now consider the first return map in the "strip" B delimited by $l = [it(\alpha), +i\infty[$, $F(l)$ and the segment $[it(\alpha), F(it(\alpha))]$. Given z in B we can iterate F until $\Re e\, F^n(z) > 1$. If $\Im m\, z \geq t(\alpha) + c$ for some $c > 0$ then $z' = F^n(z) - 1 \in B$ and $z \mapsto z'$ is the first return map in the strip B. Glueing l and $F(l)$ by F one obtains a Riemann surface S corresponding to $\operatorname{int} B$ and biholomorphic to $\mathbb{D}^*$. This induces a map $g \in S_{e^{2\pi i/\alpha}}$ which lifts to $G \in \mathcal{S}(\alpha^{-\infty})$.

As we can see we have three steps :

(g) *glueing* l and $F(l)$ following F we get a topological manifold with boundary whose interior is biholomorphic to the standard half–cylinder ;

(u) *uniformization* of the manifold obtaining a standard cylinder ;

(d) *developing* the standard cylinder on the plane.

The biholomorphism $H = dug$ which "glues, uniformizes and develops" the "strip" B into the strip of width 1 conjugates F with the translation by 1

$$H(F(z)) = H(z) + 1$$

and the renormalized map is

$$G = \mathcal{R}_{a_1}F = HF^{a_1}T_{-1}H^{-1} ,$$

where a_1 is the integer part of α^{-1}. It is immediate to check that if $\Im m\, z$ is large then $H(z) = \frac{z}{\alpha} + \dots$ and one can then show the following (see [Yo2], pp. 32–33)

Proposition 9.3 *Let $\alpha \in (0,1)$, $F \in \mathcal{S}(\alpha)$ and $t(\alpha) > 0$ such that if $\Im m\, z \geq t(\alpha)$ then $|F(z)-z-\alpha| \leq \alpha/4$. There exists $G \in \mathcal{S}(\alpha^{-1})$ such that if $z \in \mathbb{H}$, $\Im m\, z \geq t(\alpha)$ and $F^i(z) \in \mathbb{H}$ for all $i = 0, 1, \dots, n-1$ but $F^n(z) \notin \mathbb{H}$ then there exists $z' \in \mathbb{C}$ such that*

1. *$\Im m\, z' \geq \alpha^{-1}(\Im m\, z - t(\alpha) - c_1)$, where $c_1 > 0$ is a universal constant ;*
2. *There exists an integer m such that $0 \leq m < n$ and $G^m(z') \notin \mathbb{H}$.*

Using this proposition it is easy to show the following claim :

Small divisors and Brjuno functions

CLAIM : *If α is a Brjuno number the renormalization scheme converges and all maps $f \in \mathcal{S}_{e^{2\pi i\alpha}}$ are linearizable.*

If $F \in \mathcal{S}(\alpha)$ is a lift of $f \in \mathcal{S}_{e^{2\pi i\alpha}}$, let $K_F \subset \mathbb{C}$ be defined as the cover of K_f : $E(K_F) = K_f$. It is immediate to check that

$$d_F = \sup\{\Im m\, z \mid z \in \mathbb{C} \setminus K_F\} = -\frac{1}{2\pi} \log \operatorname{dist}(0, \mathbb{C} \setminus K_f) . \tag{9.3}$$

An upper bound of the form

$$\sup_{F\in\mathcal{S}(\alpha)} d_F \leq \frac{1}{2\pi} B(\alpha) + C \tag{9.4}$$

for some universal constant $C > 0$ is therefore enough to establish our claim.

Assume that (9.4) is not true and that there exist $\alpha \in \mathbb{R} \setminus \mathbb{Q} \cap (0, 1/2)$ with $B(\alpha) < +\infty$, $F \in \mathcal{S}(\alpha)$, $z \in \mathbb{H}$ and $n > 0$ such that

$$\begin{aligned} \Im m\, F^n(z) &\leq 0 , \\ \Im m\, z &\geq \frac{1}{2\pi} B(\alpha) + C . \end{aligned}$$

Let us choose α, F and z so that n is as small as possible. By Proposition 9.3, if $C > c_0$, one gets

$$\begin{aligned} \Im m\, z' &\geq \alpha^{-1}[\Im m\, z - t(\alpha) - c_1] \\ &\geq \alpha^{-1}\left[\frac{1}{2\pi}(B(\alpha) - \log \alpha^{-1}) + C - c_0 - c_1\right] . \end{aligned}$$

By the functional equation of B one gets

$$\Im m\, z' \geq \frac{1}{2\pi} B(\alpha^{-1}) + \alpha^{-1}[C - c_0 - c_1] \geq \frac{1}{2\pi} B(\alpha^{-1}) + C$$

provided that $C \geq 2(c_0 + c_1)$. But Proposition 9.3 shows that this contradicts the minimality of n and we must therefore conclude that (9.4) holds. □

Yoccoz has been able to establish also a lower bound

$$\inf_{F\in\mathcal{S}(\alpha)} d_F \geq \frac{1}{2\pi} B(\alpha) + C \tag{9.5}$$

again using the renormalization construction together with some analytic surgery so as to be able at each step of the renormalization construction to glue a fixed point exactly at the height provided by Proposition 9.3 A nice description of the proof of this lower bound can be found in the Bourbaki seminar of Ricardo Perez–Marco [PM1].

10. Stability of quasiperiodic orbits in one-frequency systems : rigorous results

The main result of the renormalization analysis made by Yoccoz of the problem of linearization of holomorphic germs of $(\mathbb{C},0)$ can be very simply stated as

$$\mathcal{Y} = \{\alpha \in \mathbb{R}\setminus\mathbb{Q} \mid B(\alpha) < +\infty\} = \text{Brjuno numbers},$$

but he proves much more than the above :

Theorem 10.1
(a) If $B(\alpha) = +\infty$ there exists a non–linearizable germ $f \in S_{e^{2\pi i\alpha}}$;
(b) If $B(\alpha) < +\infty$ then $r(\alpha) > 0$ and

$$|\log r(\alpha) + B(\alpha)| \le C, \tag{10.1}$$

where C is a universal constant (i.e. independent of α) ;
(c) Let $\lambda = e^{2\pi i\alpha}$ and consider the Yoccoz function u defined in Section 8. Recall that $|u(\lambda)| = r_2(\lambda)$, i.e. the radius of convergence of the linearization of the quadratic polynomial. There exists a universal constant $C_1 > 0$ such that for all Brjuno numbers α one has

$$B(\alpha) - C_1 \le -\log|u(\lambda)| \le B(\alpha) + C_1 . \tag{10.2}$$

Note that the upper bound in (c) was proved in [Yo1] together with a weaker lower bound : this version of (c) is actually due to to X. Buff and A. Chéritat [BC1]. Results similar to Theorem 10.1 hold for the local conjugacy of analytic diffeomorphisms of the circle [Yo1, Yo3] and for some area–preserving maps [Ma1,Da1], including the standard family [Da2, BG1, BG2].

The remarkable consequence of (10.1) and (10.2) is that the Brjuno function not only identifies the set $\mathcal{Y}$ but also gives a rather precise estimate of the size of the Siegel disks.

The first open problem we want to address is whether or not the infimum in (10.1) is attained by the quadratic polynomial $P_\lambda(z) = \lambda z\left(1 - \frac{z}{2}\right)$:

Question 10.2 Does $r(\alpha) = \inf_{f\in S_{e^{2\pi i\alpha}}} r(f) = r_2(e^{2\pi i\alpha})$, i.e. the radius of convergence of the quadratic polynomial ?

Beyond the analytic category not much is known. Between $\mathbb{C}[[z]]$ and $\mathbb{C}\{z\}$ one has many important algebras of "ultradifferentiable" power series (i.e. asymptotic expansions at $z=0$ of functions which are "between" C^∞ and $\mathbb{C}\{z\}$). Consider two subalgebras $A_1 \subset A_2$ of $z\mathbb{C}[[z]]$ closed with respect to the composition of formal series. For example Gevrey–s classes, $s>0$ (i.e. series $F(z)=\sum_{n\geq 0} f_n z^n$ such that there exist $c_1, c_2 > 0$ such that $|f_n| \leq c_1 c_2^n (n!)^s$ for all $n \geq 0$). Let $f \in A_1$ being such that $f'(0) = \lambda \in \mathbb{C}^*$. We say that f *is linearizable in* A_2 if there exists $h_f \in A_2$ tangent to the identity and such that $f \circ h_f = h_f \circ R_\lambda$. It is easy to check that if one requires $A_2 = A_1$, i.e. the linearization h_f to be as regular as the given germ f, once again the Brjuno condition is sufficient. It is quite interesting to notice that given any algebra of formal power series which is closed under composition (as it should if one whishes to study conjugacy problems) a germ in the algebra is linearizable *in the same algebra* if the Brjuno condition is satisfied. If the linearization is allowed to be less regular than the given germ (i.e. A_1 is a proper subset of A_2) one finds that new arithmetical conditions (see [CM]), weaker than the Brjuno condition, are sufficient. It is not known if the conditions stated in [CM] are also necessary.

11. Stability of quasiperiodic orbits in one-frequency systems : numerical results

In this Section we will first deduce a formula due to Michel Herman (see [He2]) which allows to compute numerically the radius of convergence of the linearization of a germ of holomophic diffeomorphism. Then we will briefly illustrate the numerical results obtained in the case of the quadratic polynomial.

Let $f \in G_\lambda$ be linearizable, $\lambda = e^{2\pi i\alpha}$. Let U_f be the Siegel disk of f, h_f be the linearization of f, $z \in U_f$, $z = h_f(w)$, where $w \in \mathbb{D}_{r(f)}$, $|w| = r < r(f)$. Since h_f conjugates f to R_λ one has $f^j(z) = f^j(h_f(w)) = h_f(\lambda^j w)$ for all $j \geq 0$ and $w \in \mathbb{D}_{r(f)}$, thus

$$\frac{1}{m}\sum_{j=0}^{m-1} \log|f^j(z)| = \frac{1}{m}\sum_{j=0}^{m-1} \log|h_f(\lambda^j w)|\,.$$

h_f has neither poles nor zeros but $w=0$ thus by the mean property of harmonic functions one has $\int_0^1 \log|h_f(re^{2\pi i x})|dx = \log r$ for all $r \leq r(f)$. Finally note that $w \mapsto \lambda w$ is uniquely ergodic on $|w| = r$, and in this case Birkhoff's ergodic theorem holds for all initial points, thus

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{j=0}^{m-1}\log|f^j(z)| = \lim_{m\to+\infty}\frac{1}{m}\sum_{j=0}^{m-1}\log|h_f(\lambda^j w)|$$
$$= \int_0^1 \log|h_f(re^{2\pi i x})|dx = \log r\,.$$

Taking a sequence of points $z_j \to z \in \partial U_f$, from the above argument one deduces that for almost every point $z \in \partial U_f$ with respect to the harmonic measure one has

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{j=0}^{m-1}\log|f^j(z)| = \log r(f)\,. \tag{11.1}$$

The above formula can be used to compute numerically the radius of convergence of the linearization of the quadratic polynomial P_λ. In order to apply (11.1) one needs to know that some point belongs to the boundary of the Siegel disk of P_λ (and hope ...). The critical point cannot be contained in U_{P_λ} because $f|_{U_{P_\lambda}}$ is injective, and from the classical theory of Fatou and Julia one knows that ∂U_{P_λ} is contained in the closure of the forward orbit $\{P_\lambda^k(1)\mid k\geq 0\}$ of the critical point $z=1$. Finally Herman proved if α verifies an arithmetical condition $\mathcal{H}$, weaker than the Diophantine condition but stronger than the Brjuno condition (see, for example, [Yo1] for its precise formulation) the critical point belongs to ∂U_{P_λ}.

If $\alpha \in \mathrm{CD}\,(0)$ Herman has also proved that ∂U_{P_λ} is a *quasicircle*, that is the image of $\mathbb{S}^1$ under a quasiconformal homeomorphism. In this case h_{P_λ} admits a quasiconformal extension to $|w| = r_2(\lambda)$ and therefore is Hölder continuous [Po] :

$$|h_{P_\lambda}(w_1) - h_{P_\lambda}(w_2)| \leq 4|w_1 - w_2|^{1-\chi} \tag{11.2}$$

for all $w_1\,, w_2 \in \partial\mathbb{D}_{r_2(\lambda)}$, where $\chi \in [0,1[$ depends on λ is the so-called Grunsky norm [Po] associated with the univalent function $g(z) = r_2(\lambda)/h_{P_\lambda}(r_2(\lambda)/z)$ on $|z|>1$. Using this information one can show that (see [Ma1])

$$|\frac{1}{q_k}\sum_{j=0}^{q_k-1}\log|P_\lambda^j(z)| - \log r_2(\lambda)| \leq \frac{8}{r_2(\lambda)}(\frac{2\pi}{q_k})^{1-\chi}\,, \tag{11.3}$$

where p_k/q_k is a convergent of the continued fraction expansion of α. Note that (11.2) implies convergence to $\log|u(\lambda)|$ *for all* $z \in \partial U_{P_\lambda}$, thus also for the critical point $z=1$.

In Figure 5 one can see the result of applying (11.1) to 5000 uniformly distributed random values of α and replacing the limit at the r.h.s. with the Birkhoff average over 10^6 iterations. Note the similarity with Figure 1. In Figure 6 one can see the graph of the function $\alpha \mapsto |u(\alpha)|\exp(B(\alpha))$ at the same 5000 values of α of the previous figure. It is quite striking how the singularities at all rationals seem to compensate and lead to a Hölder continuous function. This numerical observation was the main result of [Ma1] and lead to the the following conjecture (see [MMY1]) :

Conjecture 11.1 (Hölder interpolation) *The function defined on the set of Brjuno numbers by $\alpha \mapsto B(\alpha) + \log|u(e^{2\pi i\alpha})|$ extends to a 1/2–Hölder continuous function as α varies in $\mathbb{R}$.*

The most recent numerical study of this conjecture is due to T. Carletti [Ca] : using the complexified Brjuno function (see Section 14) and the Littlewood-Paley theorems relating the Hölder regularity with the decay rate of the Fourier dyadic blocks he is able to confirm the exponent 1/2.

Very recently Buff and Chéritat [BC2] have proved the following result :

Theorem 11.2 *The function $\alpha \mapsto B(\alpha) + \log|u(e^{2\pi i\alpha})|$ extends to a continuous function as α varies in $\mathbb{R}$.*

The numerical analysis which relates the radius of convergence of the linearization of the quadratic polynomial to the Brjuno function has been extended in [Ma1] and [MS] to some analytic area–preserving maps. The simplest case is provided by the biholomorphic symplectic mapping F of $\mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$ defined by :

$$F(x,y) = (x_1, y_1), \quad \begin{cases} x_1 = x + y + e^{ix}, \\ y_1 = y + e^{ix}. \end{cases} \tag{11.4}$$

This is the so-called *semi-standard map*, which has been studied by many authors [He2],[Ma1],[Da1],[Da2],[Laz],[GLST] as a simplified model-problem of symplectic twist map.

In particular it provides a simple model for the study of invariant circles of symplectic twist maps, with power series involved instead of trigonometric series (see [He2] p.173) : indeed, for $\Im x$ large, we may see F as a perturbation of the rotation $R(x,y) = (x+y, y)$ and ask whether the invariant curves y =constant of R have any counterpart for F ; i.e. we fix $\alpha \in \mathbb{R}$ and we look for an invariant curve parametrized by θ :

$$\begin{aligned} x &= \theta + \varphi(e^{i\theta}), \\ y &= 2\pi\alpha + \psi(e^{i\theta}), \end{aligned} \tag{11.5}$$

in such a way that φ and ψ are analytic, vanishing at the origin and conjugate F to a rotation of α. This can be seen as a conjugacy problem : find H of the form $H(x,y) = (x+\varphi(e^{ix}), y+\psi(e^{ix}))$ such that $F\circ H = H\circ R$. One can prove that this problem admits a solution if and only if ω is a Brjuno number [Ma1,Da1].

It is not difficult to find the recursive relation which allows to compute the coefficients of the power series expansion of φ (and ψ). Applying Hadamard's criterion to the coefficients of order 2000 one can numerically compute the radius of convergence $\rho(\alpha)$ of the linearization (see Figure 7 for a plot at 2400 random values of α). As first noted in [Ma1] the function $\alpha \mapsto \rho(\alpha)e^{2B(\alpha)}$ *seems to be uniformly bounded away from* 0 *and infinity and Hölder continuous* (see Figure 8 for a plot at the same 2400 values of α of Figure 7).

One can formulate an analogue conjecture for the celebrated standard map, obtained replacing e^{ix} with $\varepsilon \sin x$ in (11.4). In this case the dynamics preserve the real phase space $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ and for $\epsilon > 0$ one can look for invariant circles which are analytic deformations of the invariant circles $y = 2\pi\alpha$ corresponding to $\varepsilon = 0$ and on which the rotation number is fixed. In [MS] a conjecture analogue to the interpolation conjecture 11.1 has been formulated. This conjecture has stimulated various numerical and analytical investigations [BG1,BG2,BG3,BM1,BM2,BMS,CaL,CM,Da1,Da2,LFLM].

12. The real Brjuno functions and their regularity properties

In order to study the regularity properties of B in [MMY] a systematic use of the functional equation (5.2) satisfied by the Brjuno function has been made. Since the first equation simply states that the function must be periodic, one can introduce the linear operator

$$(Tf)(x) = xf\left(\frac{1}{x}\right)\ ,\ x\in(0,1)\ , \tag{12.1}$$

and consider its action on measurable periodic functions which belong to $L^p(0,1)$. One wants to solve the equation

$$(1-T)B_f = f\ , \tag{12.2}$$

so that

$$\begin{aligned} B_f(x+1) &= B_f(x) \quad \forall x\in\mathbb{R}\ , \\ B_f(x) &= f(x) + xB_f(1/x) \quad \forall x\in(0,1)\ . \end{aligned} \tag{12.3}$$

The choice $f(x) = -\log\{x\}$ (where as usual $\{\cdot\}$ denotes fractional part) leads to the Brjuno function B. For other choices of the singular behaviour of f at 0 the condition $B_f < +\infty$ leads to different diophantine conditions. The strength of the singularity of f at the origin is related to how much restrictive the corresponding diophantine condition will be.

An easy argument which makes use of the absolutely continuous invariant probability measure preserved by the Gauss map shows that T has spectral radius bounded by $g = \frac{\sqrt{5}-1}{2}$. Indeed replacing the Lebesgue measure dx with the invariant measure $\frac{dx}{(1+x)\log 2}$ for the Gauss map one obtains the same L^p spaces since the density is bounded below and above. But now

$$\begin{aligned}\|T^m f\|_{L^p}^p &= \int |(T^m f)(x)|^p \frac{dx}{(1+x)\log 2} = \int \beta_{m-1}^p |(f \circ A^m)(x)|^p \frac{dx}{(1+x)\log 2} \\ &\leq g^{p(m-1)} \int |f(x)|^p \frac{dx}{(1+x)\log 2} = g^{p(m-1)} \|f\|_{L^p}^p\end{aligned}$$

where the inequality is obtained applying Proposition 3.1 (ii) and the invariance of the measure by A allows one to replace $f \circ A^m$ with f.

Thus $(1-T)$ is invertible and its inverse is given by the norm convergent series $\sum_{k\geq 0} T^k$. Thus the Brjuno function is simply obtained as $\sum_{k\geq 0} T^k(-\log\{\cdot\})$. Since the function $x \mapsto -\log\{x\}$ belongs to L^p for all $1 \leq p < +\infty$ the Brjuno function $B \in \cap_{p\geq 1} L^p(\mathbb{T})$. Note that $B \notin L^\infty(\mathbb{T})$. A suitable (but not at all trivial) adaptation of this argument shows that the Brjuno function belongs to BMO $(\mathbb{T}^1)$ (bounded mean oscillation, see references [Ga, GCRF] for its definition and more information).

By Fefferman's duality theorem BMO is the dual of the Hardy space H^1 thus one can add an L^∞ function to B so that the harmonic conjugate of the sum will also be L^∞ (actually, it is proved in [MMY2] that the harmonic conjugate of B is bounded, see below). This suggests to look for an holomorphic function $\mathcal{B}$ defined on the upper half plane which is $\mathbb{Z}$–periodic and whose trace on $\mathbb{R}$ has for imaginary part the Brjuno function B. The function $\mathcal{B}$ will be called the *complex Brjuno function* and we will discuss its construction and its properties in Section 14.

A further remarkable property of the operator T appears when one considers its action on Hölder continuous function. To be precise one should consider the operator T_e defined replacing the standard continued fraction map (which has infinitely many discontinuities) with the nearest integer continued fraction map (which is continuous), i.e.

$$(T_e f)(x) = x f\left(\left\|\frac{1}{x}\right\|_{\mathbb{Z}}\right), \quad \forall x \in (0, 1/2), \tag{12.4}$$

where $\|x\|_{\mathbb{Z}} = \min_{p\in\mathbb{Z}} |x-p|$, acting on even periodic functions. In [MMY] it is proved that if the datum f of the functional equation

$$(1-T_e)B_f = f \tag{12.5}$$

is Hölder continuous then B_f is also Hölder continuous. The exponent $1/2$ plays here a crucial role : let η denote the Hölder exponent of f. One has :

- if $\eta > 1/2$ then B_f is $1/2$–Hölder continuous ;
- if $\eta < 1/2$ then B_f is also η–Hölder continuous ;
- if $\eta = 1/2$ then B_f admits $x^{1/2}\log x$ as continuity modulus.

The distinguished role played by the exponent $1/2$ is perhaps the most convincing theoretical argument in favour of the choice of this exponent in the formulation of the Hölder interpolation conjecture. The remark that the operator T applied to the logarithm of the radius of convergence of the linearization in the semistandard map case gives a Hölder continuous function is due to Buric, Percival and Vivaldi [BPV]. The results proved in [MMY], together with the interpolation conjecture, provide a (partial) justification of their numerical results.

13. Continued fractions, the modular group and the real Brjuno function as a cocycle

In this Section we want to describe more in detail the relationship between continued fractions and the modular group. Moreover we will briefly explain how to interpret the real Brjuno function as a cocycle under the action by homographies of $\mathrm{PGL}(2,\mathbb{Z})$ on $\mathbb{R}\setminus\mathbb{Q}$.

Let Γ denote the group $\mathrm{GL}(2,\mathbb{Z})$ of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{Z}$ and determinant $\varepsilon_g := ad-bc = \pm 1$. We denote H its subgroup of order 8 of matrices of the form $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}$ or $\begin{pmatrix} 0 & \varepsilon \\ \varepsilon' & 0 \end{pmatrix}$, where $\varepsilon,\varepsilon' \in \{-1,+1\}$. $\mathcal{M}$ is the monoid with unit $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ made of matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that, if $g \neq \mathrm{id}$, we have $d \geq b \geq a \geq 0$ and $d \geq c \geq a$ and Z denotes the subgroup of matrices of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$.

Let $g(m) = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}$, where $m \geq 1$. Clearly $g(m) \in \mathcal{M}$. Moreover, $\mathcal{M}$ is the *free monoid* generated by the elements $g(m)$, $m \geq 1$: each element g of $\mathcal{M}$ can be written as

$$g = g(m_1)\cdots g(m_r)\ , \quad r \geq 0\ , \ m_i \geq 1\ , \tag{13.1}$$

and this decomposition is unique (see [MMY2], Proposition A1.2 and also [LZ1]). It is not difficult to show that $\Gamma = Z \cdot \mathcal{M} \cdot H$, i.e. the application

$$Z \times \mathcal{M} \times H \to \Gamma\ , \quad (z, m, h) \mapsto g = z \cdot m \cdot h$$

is a *bijection.*

The subset $Z \cdot \mathcal{M}$ of Γ is made of matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $d \geq c \geq 0$ with the following additional restrictions : $a = 1$ if $c = 0$, and, $b = a + 1$ if $d = c = 1$.

Let us now consider the usual action of Γ on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by homographies : $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. The following facts are easy to check :

(a) equation (3.4) can be written $x_n = g(a_{n+1}) \cdot x_{n+1}$, thus if $x_0 \in (0, 1)$ we have $x_0 = \frac{p_n + p_{n-1} x_n}{q_n + q_{n-1} x_n} = g(a_1) g(a_2) \cdots g(a_n) \cdot x_n$.

(b) The application $g \mapsto g \cdot 1 = \frac{a+b}{c+d}$ is a *bijection* of $Z\mathcal{M}$ over $\mathbb{Q}$ which maps $\mathcal{M}$ onto $\mathbb{Q} \cap (0, 1]$.

(c) The application $g \mapsto g \cdot 0 = b/d$ maps $Z\mathcal{M}$ onto $\mathbb{Q}$ and each rational number has exactly *two inverse images.* The two elements which map 0 on 1 are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(d) The application $g \mapsto g \cdot [0, +\infty]$ is a *bijection* of $Z\mathcal{M}$ on the set of Farey intervals (the convention we adopt here implies that $[n, +\infty]$ is a Farey interval, but $[-\infty, n]$ is not). For the definition and properties of the Farey partition of $[0, 1]$ we refer the reader to [HW].

Property (a) above makes the relation between the continued fraction algorithm and the monoid $\mathcal{M}$ transparent : $\mathcal{M}$ is the semigroup obtained considering all possible $\mathrm{GL}(2, \mathbb{Z})$ matrices which appera in (3.10) when $x \in (0, 1)$. The same monoid arises also in the investigations of Lewis and Zagier ([LZ1],[LZ2]) concerning period functions and the Selberg zeta function for the Laplace–Beltrami operator on the modular surface.

The transformations $T(x) = x + 1$ and $S(x) = x^{-1}$ generate $\mathrm{PGL}(2, \mathbb{Z})$. One has the following more precise result :

Proposition 13.1 *Let $g \in PGL(2, \mathbb{Z})$ and let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. There exist $r \geq 0$ and elements $g_1, \ldots, g_r \in \{S, T, T^{-1}\}$ such that*

(i) $\because$ $g = g_r \ldots g_1$;

(ii) $\because$ let $x_i = g_i x_{i-1}$ for $1 \leq i \leq r$, then $x_{i-1} > 0$ if $g_i = S$.

Moreover one can require that $g_i g_{i-1} \neq 1$ for $0 < i \leq r$, and in this case $r, g_1, \ldots, g_r$ are uniquely determined.

We will now very briefly explain how to interpret the real Brjuno function as a cocycle under the action of the modular group. For this goal we need to introduce some elementary notions borrowed from group cohomology. A beautiful introduction to the cohomology of $\mathrm{SL}(2,\mathbb{Z})$ and its applications to the theory of periods of modular forms can be found in [Za].

The abstract setting for group cohomology (see [Se] and [Sh]) is given by a group G, an abelian group L (later on G will be the modular group $\mathrm{PGL}(2,\mathbb{Z})$ and L the real projective line $\mathbb{P}^1(\mathbb{R})$). To say that L is a G–set means that G acts on L, i.e. there is a homomorphism $G \mapsto \mathrm{Hom}\,(L, L)$ or, equivalently, one has a map $G \times L \to L$, $(g,l) \mapsto g \cdot l$, such that $e \cdot l = l$, $g_1 \cdot (g_2 \cdot l) = (g_1 g_2) \cdot l$ for all $g_1, g_2 \in G$, $l \in L$ and where e is the neutral element of G. If in addition one has that $g \cdot (l_1 + l_2) = g \cdot l_1 + g \cdot l_2$ we say that L is a G–module. This is equivalent to giving L the structure of a $\mathbb{Z}[G]$–module.

Let $r \in \mathbb{N}$.An element $\varphi \in C^r(G,L) = \mathrm{Map}(G^r, L)$ is called an *r–cochain.* There is a sequence

$$\ldots \to 0 \to 0 \to C^0(G,L) \xrightarrow{d} C^1(G,L) \xrightarrow{d} C^2(G,L) \xrightarrow{d} \ldots$$

where $C^0(G,L) = L$ and the *coboundary operator* d is defined as follows : let $\varphi_i \in C^i(G,L)$, then

$$\begin{aligned}
(d\varphi_0)(g) &= g \cdot \varphi_0 - \varphi_0 \,, \\
(d\varphi_1)(g_1,g_2) &= g_1 \cdot \varphi_1(g_2) - \varphi_1(g_1 g_2) + \varphi_1(g_1) \,, \\
(d\varphi_2)(g_1,g_2,g_3) &= g_1 \cdot \varphi_2(g_2,g_3) - \varphi_2(g_1 g_2, g_3) + \varphi_2(g_1, g_2 g_3) - \varphi_2(g_1,g_2) \,, \\
\ldots &= \ldots
\end{aligned}$$

It is easy to check that $d \circ d = 0$, i.e. $\mathrm{Im}\, d \subset \mathrm{Ker}\, d$.

As usual, r–cocycles will be the elements of $Z^r(G,L) = \mathrm{Ker}\, d|_{C^r(G.L)}$, r–coboundaries will be those of $B^r(G,L) = \mathrm{Im}\, d|_{C^{r-1}(G.L)}$ and the r–th cohomology group will be $H^r(G,L) = Z^r(G,L)/B^r(G,L)$.

Suppose now that one is given a group G, a *set* X on which G acts (i.e. a group homomorphism $G \to \mathrm{End}\,(X)$), a commutative ring A (with multiplicative group A^*) and an A–module M. Let $M^X = \mathrm{Map}\,(X, M)$. There are several ways of giving to M^X the structure of a G–module. They correspond to different choiches of an automorphic factor :

Definition 13.2*A function* $\chi : G \times X \to A$ *is an* automorphic factor *if the application* $G \times M^X \to M^X$ *given by* $(g,\varphi) \mapsto g \cdot \varphi$ *where* $(g \cdot \varphi)(x) =$

$\chi(g^{-1},x)\varphi(g^{-1}\cdot x)$ defines a left action of G on M^X, i.e. $\chi(g_0g_1,x)=\chi(g_0,g_1\cdot x)\chi(g_1,x)$.

The datum of an automorphic factor gives M^X the structure of a G–module and the previous considerations apply. In particular a 1–*cocycle* is a map $c : G\to M^X$ such that $g_0\cdot c(g_1)-c(g_0g_1)+c(g_0)=0$. If we let $\check{c}(g)=c(g^{-1})$, being a 1–cocycle means that

$$\check{c}(g_0g_1,x)=\chi(g_1,x)\check{c}(g_0,g_1\cdot x)+\check{c}(g_1,x)\ \forall x\in X\,. \tag{13.2}$$

Proposition 13.1 has an important consequence for the interpretation of the Brjuno functional equation : it is enough to prescribe an automorphic factor for the $\mathrm{PGL}(2,\mathbb{Z})$–action on functions on $\mathbb{R}\setminus\mathbb{Q}$ giving its values in correspondence of the inversion S just at points belonging to the interval $(0,1)$. The same property holds for cocycles : they just need to be prescribed, in correspondence of the inversion, on the interval $(0,1)$.

Proposition 13.3

(a) Given two functions $t:\mathbb{R}\setminus\mathbb{Q}\to\mathbb{R}^*$, $s:(0,1)\cap(\mathbb{R}\setminus\mathbb{Q})\to R^*$, there exists a unique automorphic factor χ such that $\chi(T,x)=t(x)$ for all $x\in\mathbb{R}\setminus\mathbb{Q}$ and $\chi(S,x)=s(x)$ for all $x\in(\mathbb{R}\setminus\mathbb{Q})\cap(0,1)$.

(b) Given two functions $\check{c}_T:\mathbb{R}\setminus\mathbb{Q}\to\mathbb{R}$ and $\check{c}_S:(\mathbb{R}\setminus\mathbb{Q})\cap(0,1)\to\mathbb{R}$ there exists a unique cocycle $\check{c}$ such that $\check{c}(T,x)=\check{c}_T(x)$ for all $x\in\mathbb{R}\setminus\mathbb{Q}$ and $\check{c}(S,x)=\check{c}_S(x)$ for all $x\in(\mathbb{R}\setminus\mathbb{Q})\cap(0,1)$.

For more details and for an explicit computation of the cocycles we refer the reader to [MMY3].

14. Complex Brjuno functions

The construction of the complexification of the Brjuno function, carried out in [MMY2], is based on the extension of the action of the operator T to complex functions. This is achieved as follows.

The operator T extends to the space $A'([0,1])$ of hyperfunctions u with support contained in $[0,1]$ (see [MMY2], section 1.4 for a proof of this fact). The space $A'([0,1])$ is canonically isomorphic to the space $\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$ of functions holomorphic on $\mathbb{C}\setminus[0,1]$ and vanishing at infinity. Using this isomorphism the formula for the extension of T reads

$$(T\varphi)(z)=-z\sum_{m=1}^{\infty}\left[\varphi\left(\frac{1}{z}-m\right)-\varphi(-m)\right]+\sum_{m=1}^{\infty}\varphi'(-m)\,. \tag{14.1}$$

Formally we have

$$(1-T)^{-1}\varphi(z)=\sum_{r\geq 0}(T^r\varphi)(z)=\sum_{g\in\mathcal{M}}(L_g\varphi)(z)\,, \tag{14.2}$$

where it appears the monoid $\mathcal{M}$ defined in the previous Section. It acts on $\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$ according to

$$(L_g\varphi)(z)=(a-cz)\left[\varphi\left(\frac{dz-b}{a-cz}\right)-\varphi\left(-\frac{d}{c}\right)\right]-\det(g)c^{-1}\varphi'\left(-\frac{d}{c}\right)\,. \tag{14.3}$$

The series (14.1) actually converges in $\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$ to a function $\sum_{\mathcal{M}}\varphi$. Moreover it is not difficult to show that the complexification of T given by (14.1) acting on the spaces $H^p(\overline{\mathbb{C}}\setminus[0,1])$ has the same spectral radius of the real version of T acting on the spaces $L^p(0,1)$.

To construct the complex analytic extension of the solutions B_f of the functional equations (12.2) our strategy is the following :

1) take the restriction of the periodic function f to the interval $[0,1]$;
2) consider its associated hyperfunction u_f and its holomorphic representative $\varphi\in\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$.
3) Recover a holomorphic periodic function on $\mathbb{H}$ by summing over integer translates :

$$\mathcal{B}_f(z)=\sum_{n\in\mathbb{Z}}\left(\sum_{\mathcal{M}}\varphi\right)(z-n)\,. \tag{14.4}$$

The above series converges to the complex extension $\mathcal{B}_f$ of the real function B_f. The main difficulty (unless f belongs to some L^p space, see [MMY2], Section 4.3) would be to recover B_f as non–tangential limit of the imaginary part of $\mathcal{B}_f$ as $\Im m\, z\to 0$.

To construct the complex Brjuno function one has to take $\varphi_0(z)=-\frac{1}{\pi}\mathrm{Li}_2\left(\frac{1}{z}\right)$ where Li_2 is the dilogarithm (see [O] for a review of the remarkable properties of this special function). The main result of [MMY2] is the following :

Theorem 14.1

(I) The complex Brjuno function is given by the series

$$\begin{aligned}\mathcal{B}(z)=-\frac{1}{\pi}\sum_{p/q\in\mathbb{Q}}&\left\{(p'-q'z)\left[Li_2\left(\frac{p'-q'z}{qz-p}\right)-Li_2\left(-\frac{q'}{q}\right)\right]\right.\\&\left.+(p''-q''z)\left[Li_2\left(\frac{p''-q''z}{qz-p}\right)-Li_2\left(-\frac{q''}{q}\right)\right]+\frac{1}{q}\log\frac{q+q''}{q+q'}\right\}\,,\end{aligned} \tag{14.5}$$

where $\left[\frac{p'}{q'}, \frac{p''}{q''}\right]$ *is the Farey interval such that* $\frac{p}{q} = \frac{p'+p''}{q'+q''}$ *(with the convention* $p' = p-1$, $q' = 1$, $p'' = 1$, $q'' = 0$ *if* $q = 1$*) and* $Li_2(z)$ *is the dilogarithm of* z.

(II) The real part of $\mathcal{B}$ *is bounded on the upper half plane and its trace (i.e. non–tangential limit) on* $\mathbb{R}$ *is continuous at all irrational points and has a decreasing jump of* π/q *at each rational point* $p/q \in \mathbb{Q}$.

(III) As one approaches the boundary the imaginary part of $\mathcal{B}$ *behaves as follows :*

(a) if α *is a Brjuno number then* $\Im m\, \mathcal{B}(\alpha + w)$ *converges to* $B(\alpha)$ *as* $w \to 0$ *in any domain with a finite order of tangency to the real axis ;*

(b) if α *is diophantine one can allow domains with infinite order of tangency.*

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Small divisors and Brjuno functions

FIGURE CAPTIONS

Figure 1 : The Brjuno function at 10000 uniformly distributed random values of α between 0 and 1.

Figure 2 : The image of the circles $|\lambda| = r$ of radii (a) 0.8, (b) 0.9, (c) 0.99 and (d) 0.999 through the map u (numerically computed truncating its power series at the order 2000).

Figure 3 : The graph of the function $x \mapsto |u(0.999e^{2\pi i x})|$ as x varies in the interval $[0, 1/2]$ (numerically computed truncating its power series at the order 2000).

Figure 4 : The graph of the function $x \mapsto \arg u(0.999e^{2\pi i x})$ as x varies in the interval $[-1/2, 1/2]$ (numerically computed truncating its power series at the order 2000).

Figure 5 : $-\log r_2(\alpha)$ at 5000 uniformly distributed random values of α between 0 and 1/2. It is numerically computed by truncating the Birkhoff average on the l.h.s. of (11.1) at order 10^6 and choosing $z = 1$, i.e. the critical point of P_λ.

Figure 6 : The graph of the function $\alpha \mapsto r_2(\alpha)e^{B(\alpha)}$ at the same 5000 values of α of Figure 5.

Figure 7 : The radius of convergence $\rho(\alpha)$ of the conjugacy of the semi–standard map at 2400 uniformly distributed random values of α between 0 and 1/2. It is computed applying Hadamard's criterion to the term of order 2000.

Figure 8 : The graph of the function $\alpha \mapsto \rho(\alpha)e^{2B(\alpha)}$ at the same 2400 values of α of Figure 7.

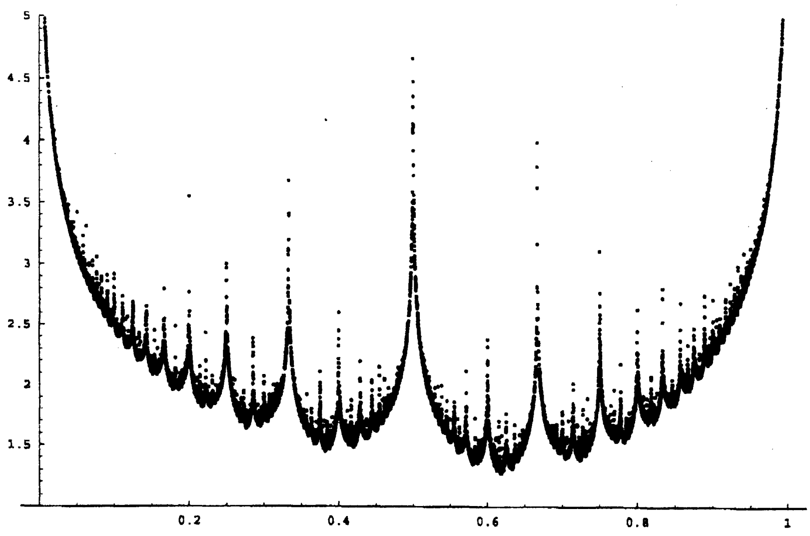


Figure 1: The Brjuno function at 10000 uniformly distributed random values of α between 0 and 1.

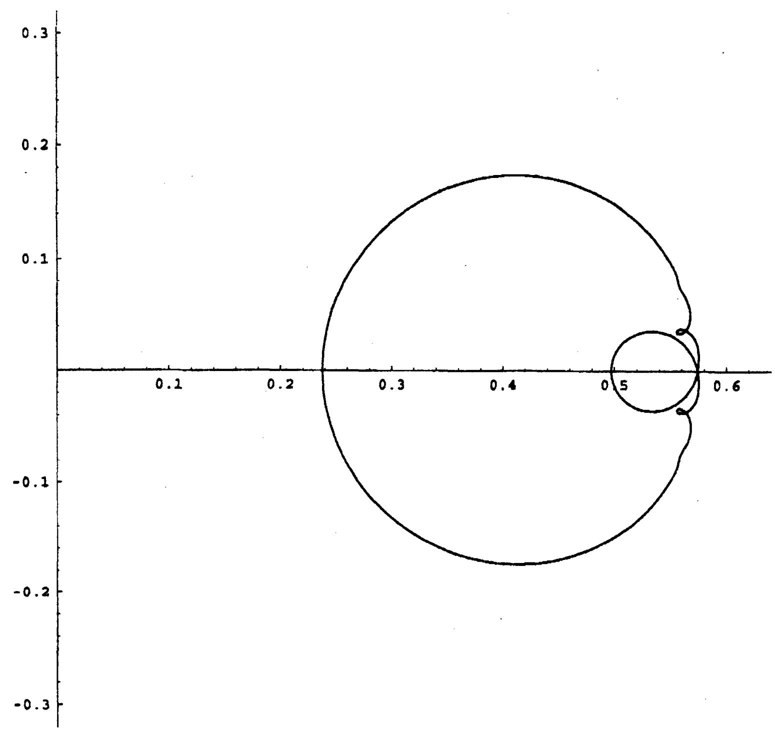


Figure 2(a): The image of the circles $|\lambda| = r$ of radii 0.8 through the map u (numerically computed truncating its power series at the order 2000).

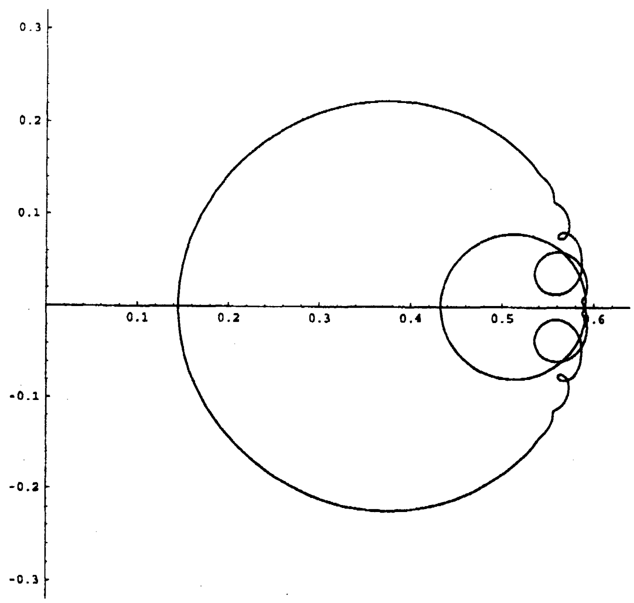


Figure 2(b): The image of the circles $|\lambda| = r$ of radii 0.9 through the map u (numerically computed truncating its power series at the order 2000).

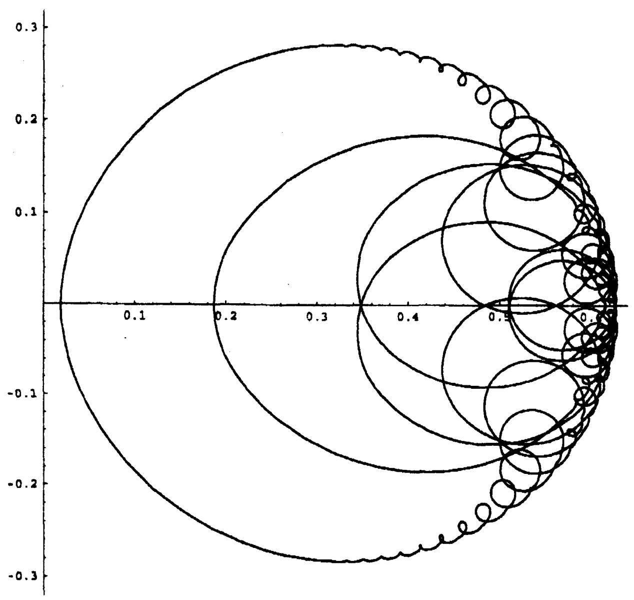


Figure 2(c): The image of the circles $|\lambda| = r$ of radii 0.99 through the map u (numerically computed truncating its power series at the order 2000).

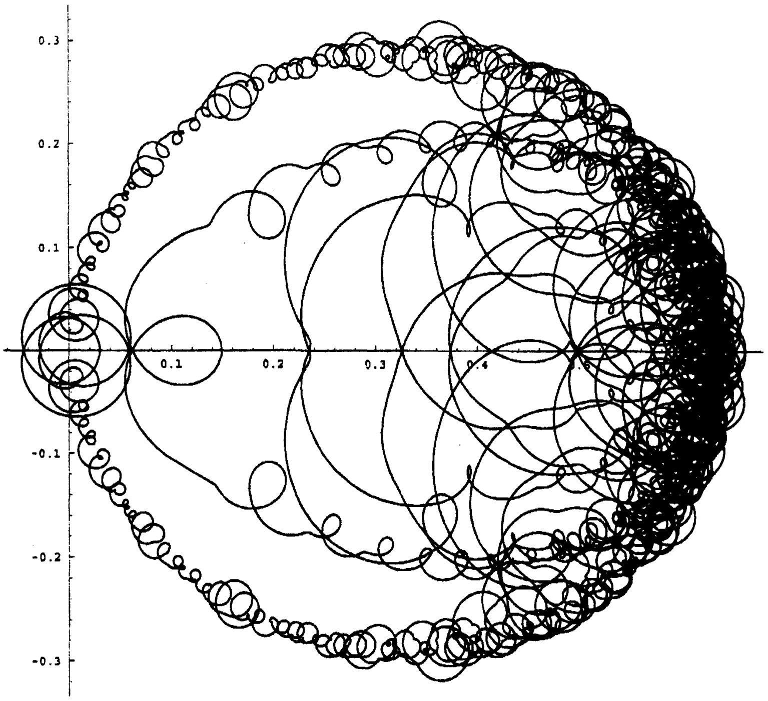


Figure 2(d): The image of the circles $|\lambda| = r$ of radii 0.999 through the map u (numerically computed truncating its power series at the order 2000).

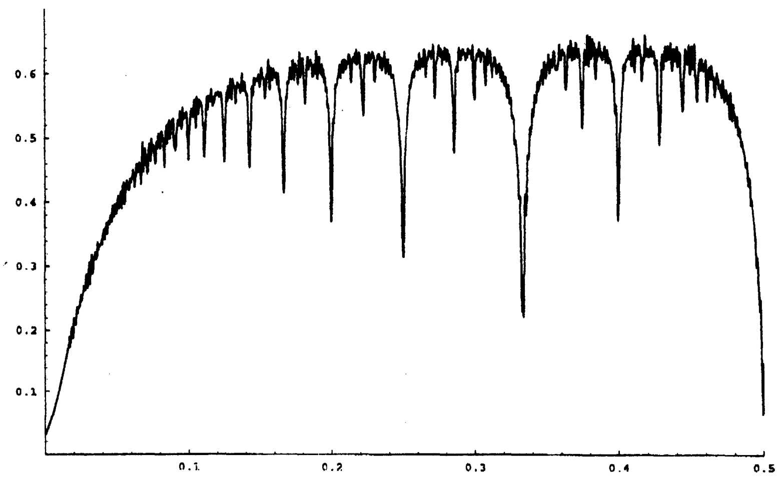


Figure 3: The graph of the function $x \mapsto |u(0.999e^{2\pi i x})|$ as x varies in the interval $[0, 1/2]$ (numerically computed truncating its power series at the order 2000).

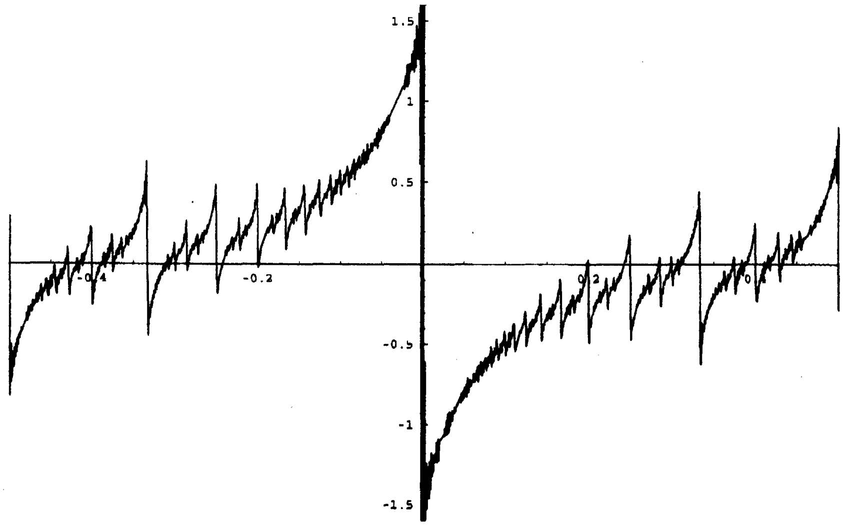
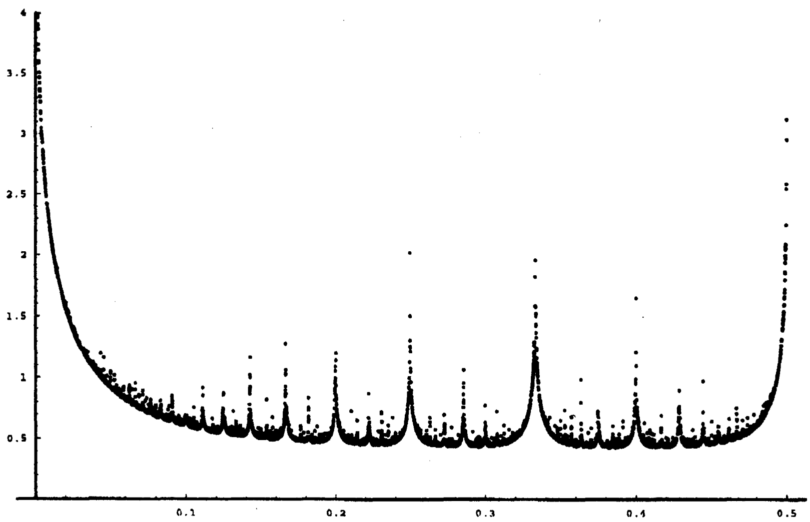


Figure 4: The graph of the function $x \mapsto \arg u(0.999e^{2\pi ix})$ as x varies in the interval $[-1/2, 1/2]$ (numerically computed truncating its power series at the order 2000).



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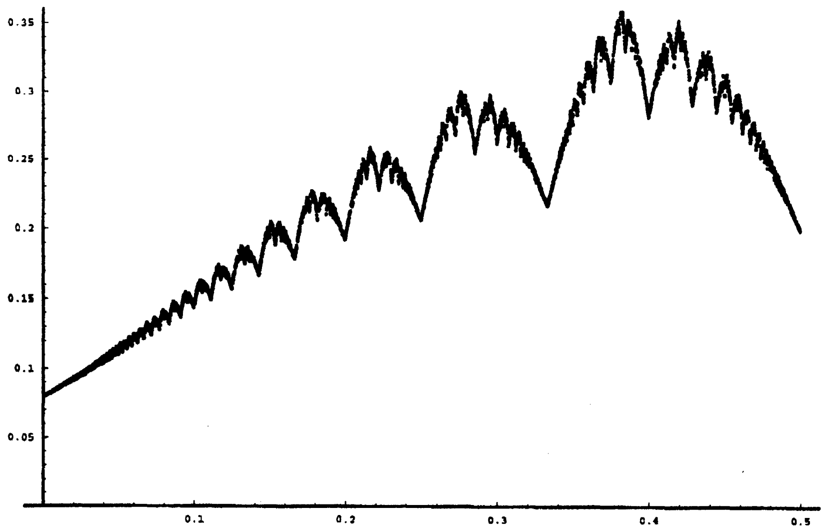

Figure 6: The graph of the function $\alpha \mapsto r_2(\alpha)e^{B(\alpha)}$ at the same 5000 values of α of Figure 5.

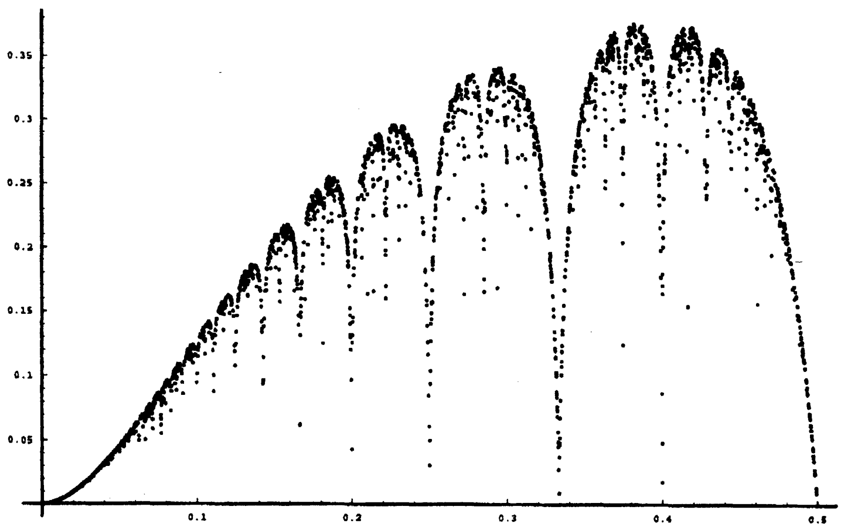

Figure 7: The radius of convergence $\rho(\alpha)$ of the conjugacy of the semi–standard map at 2400 uniformly distributed random values of α between 0 and 1/2. It is computed applying Hadamard's criterion to the term of order 2000.

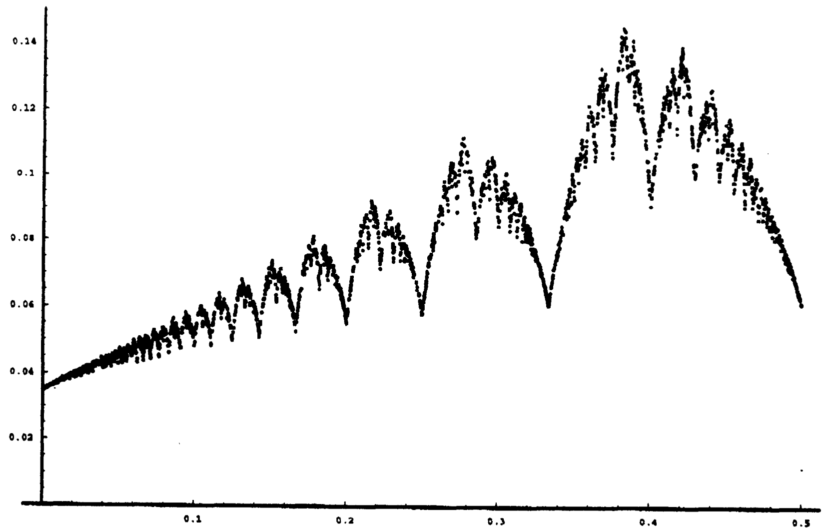

Figure 8: The graph of the function $\alpha \mapsto \rho(\alpha)e^{2B(\alpha)}$ at the same 2400 values of α of Figure 7.