

# Simultaneous linearization of hyperbolic and parabolic fixed points

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## 1 Statement of the result

This note is a summary of the preprint [8]. We will show that the Fatou coordinates (the solution to Abel equation) for a parabolic fixed point of holomorphic map of one variable can be obtained as a modified limit of the solution to Schröder equation for the perturbed hyperbolic maps. (An alternative proof is given by Kawahira [4].)

Let  $\{f_\tau\}_\tau$  be a family, depending on the parameter  $\tau$ , of holomorphic maps of the form

$$f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \dots$$

defined in a neighborhood of  $\infty$  of the Riemann sphere  $\widehat{\mathbb{C}}$ .

For each  $\tau$  with  $|\tau| > 1$ , we have a unique analytic function  $\chi_\tau(z)$  in a neighborhood of  $\infty$  satisfying the Schröder equation

$$\chi_\tau(f_\tau(z)) = \tau \chi_\tau(z)$$

and normalized so that

$$\lim_{z \rightarrow \infty} \frac{\chi_\tau(z)}{z} = 1.$$

We will show that, when  $\tau$  tends to 1 non-tangentially within the domain  $|\tau| > 1$ , the sequence

$$\chi_\tau(z) - \frac{1}{\tau - 1} - a_1(\tau) \log(\tau - 1)$$

converges to a solution to the Abel equation  $\varphi(z) \varphi(f_1(z)) = \varphi(z) + 1$ , on a half plane  $\{\operatorname{Re} z > R\}$  with sufficiently large  $R$ .

## 2 A family of linear maps

We begin with studying the family  $\{\ell_\tau\}_\tau$  of linear maps

$$\ell_\tau(z) = \tau z + 1 \quad (1)$$

on the Riemann sphere  $\widehat{\mathbb{C}}$  with a fixed point at  $\infty$ .

We will investigate the uniformity, with respect to the parameter  $\tau$ , of convergence of the sequence of the iterates  $\{f_\tau^n\}_{n=1}^\infty$ . Here, the parameter will be restricted in the closed sector

$$T_\alpha = \{\tau \in \mathbb{C} \mid \operatorname{Re} \tau - 1 \geq |\tau - 1| \cos \alpha\},$$

where  $\alpha$  is a real number with  $0 < \alpha < \pi/2$ .

To measure the rate of convergence to  $\infty$ , we define a function  $N : \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\} \rightarrow \mathbb{R} \cup \{\infty\}$  as follows.

$$\begin{aligned} N_\tau(z) &= \left| z - \frac{1}{1-\tau} \right| - \left| \frac{1}{1-\tau} \right| && \text{for } (z, \tau) \in \widehat{\mathbb{C}} \times (T_\alpha - \{1\}); \\ N_1(z) &= \sup_{|\theta| \leq \alpha} \operatorname{Re}(e^{i\theta} z) && \text{for } z \in \mathbb{C}. \end{aligned}$$

We will not define  $N_1(\infty)$ .

As is easily shown,  $N_\tau(z)$  is upper semi-continuous and

$$N_1(z) = \limsup_{T \ni \tau \rightarrow 1} N_\tau(z).$$

Further the inequality

$$|N_\tau(z) - N_\tau(w)| \leq |z - w| \quad z, w \in \mathbb{C}, \tau \in T_\alpha$$

and, in particular,

$$N_\tau(z) \leq |z|, \quad z \in \mathbb{C}, \tau \in T_\alpha.$$

hold.

For a real number  $R$ , let

$$\mathcal{V}_\alpha(R) = \{(z, \tau) \in \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\} \mid N_\tau(z) > R\}.$$

We note that  $\mathcal{V}_\alpha(R)$  is not open. Slices of  $\mathcal{V}_\alpha(R)$  by  $\tau = \text{const.}$  are open sets given by

$$\begin{aligned} V_\tau(R) &= \{z \in \widehat{\mathbb{C}} \mid N_\tau(z) > R\} \quad (\tau \neq 1); \\ V_1(R) &= \{z \in \mathbb{C} \mid N_1(z) > R\} = \bigcup_{|\theta| \leq \alpha} \{\operatorname{Re}(e^{i\theta} z) > 0\}. \end{aligned}$$

**Lemma 2.1** For  $(z, \tau) \in \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\}$ , we have

$$N_\tau(\ell_\tau(z)) \geq |\tau|N_\tau(z) + \cos \alpha.$$

If  $N_\tau(z) > 0$ , we have  $N_\tau(\ell_\tau(z)) \geq N_\tau(z) + \cos \alpha$ . So we have the following.

**Proposition 2.2** The sequence  $\{\ell_\tau^n(z)\}_n$  converges to  $\infty$  as  $n \rightarrow \infty$  uniformly on the set  $\mathcal{V}_\alpha(0)$ .

### 3 Families of maps with attracting/parabolic fixed points — Domain of convergence

Now we consider a family of holomorphic maps  $f_\tau : U \rightarrow \widehat{\mathbb{C}}$  of the form

$$f_\tau(z) = \tau z + 1 + \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \dots \quad (2)$$

defined on a neighborhood

$$U = \{z \in \widehat{\mathbb{C}} \mid R < |z| \leq \infty\}$$

of  $\infty \in \widehat{\mathbb{C}}$ . We suppose that  $f$  depends holomorphically on  $\tau \in \Delta_\rho(1) = \{\tau \in \mathbb{C} \mid |\tau - 1| < \rho\}$ . Let

$$A_\tau(z) = \frac{a_1(\tau)}{z} + \frac{a_2(\tau)}{z^2} + \dots$$

As in the previous section, we choose and fix  $\alpha$  so that  $0 < \alpha < \pi/2$  and let  $\delta = \frac{1}{2} \cos \alpha$ . By shrinking the neighborhoods  $U$  and  $W$ , we assume that there is a constant  $K_1$  such

$$|A_\tau(z)| < \frac{K_1}{|z|} < \delta \quad (3)$$

for  $(z, \tau) \in U \times W$ . Further we assume that  $f_\tau(z)$  is injective in  $z$  for every  $\tau \in \Delta_\rho(1)$

Since  $f_\tau(z)$  are approximated by linear maps  $\ell_\tau(z)$ , we have a result concerning the uniformity of convergence of  $\{f_\tau^n(z)\}$ . Let  $T_{\alpha, \rho} = T_\alpha \cap \Delta_\rho(1)$ .

**Lemma 3.1** For  $(z, \tau) \in U \times T_{\alpha, \rho}$  we have

$$N_\tau(f_\tau(z)) \geq |\tau|N_\tau(z) + \delta.$$

Now let  $\mathcal{V} = \mathcal{V}_{\alpha, \rho}(R) = \{(z, \tau) \in \mathcal{V}_\alpha(R) \mid \tau \in T_{\alpha, \rho}\}$ .

**Proposition 3.2** If  $(z, \tau) \in \mathcal{V}$ , then  $(f_\tau(z), \tau) \in \mathcal{V}$ . The sequence  $\{f_\tau^n(z)\}_n$  converges uniformly on  $\mathcal{V}$  to  $\infty$  as  $n \rightarrow \infty$ .

## 4 Schröder-Abel equation — special case

Here we consider the special case where the coefficient  $a_1(\tau)$  in (2) vanishes identically.

**Theorem 4.1** *There exists a function  $\varphi_\tau(z)$  continuous on  $\mathcal{V}$  such that*

(i)  $\varphi_\tau(z)$  satisfies the functional equation

$$\varphi_\tau(f_\tau(z)) = \tau\varphi_\tau(z) + 1; \quad (4)$$

(ii)  $\varphi_\tau(z)$  is injective in the variable  $z$  for each parameter  $\tau \in T_{\alpha,\tau}$ .

(iii)  $\lim_{z \rightarrow \infty} \varphi_\tau(z)/z = 1$  as  $z \rightarrow \infty$ , when  $|\tau| > 1$ .

In fact  $\varphi_\tau(z)$  is given by

$$\varphi_\tau(z) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{\tau^n} f^n(z) - \sum_{k=1}^n \frac{1}{\tau^k} \right\} \quad (5)$$

In the case where  $a_1(\tau)$  does not identically vanish, the expression in (5) is not convergent. So we have to modify (5) in order to yield convergence. For this purpose, we will introduce a function satisfying a difference equation in the next section.

## 5 Solution to a difference equation

We consider the difference equation

$$h_\tau(\ell_\tau(z)) - \tau h_\tau(z) = \frac{1}{z} + C_\tau. \quad (6)$$

where  $\ell_\tau(z) = \tau z + 1$  with  $|\tau| > 1$  or  $\tau = 1$ ; and  $C_\tau$  is a constant depending on  $\tau$ , which will be given later.

A solution to this equation is given by

$$h_\tau(z) = -\frac{1}{\tau z} + \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_\tau^n(0)} - \frac{1}{\ell_\tau^n(z)} \right\}. \quad (7)$$

**Proposition 5.1** *The function  $h_\tau(z)$  is continuous on  $\mathcal{V}_\alpha(0)$ .*

For a fixed  $\tau$  with  $|\tau| > 1$ , the function  $h_\tau(z)$  is meromorphic on  $\widehat{\mathbb{C}}$  except the essential singularity at  $1/(1-\tau)$ , and has poles at  $(1-\tau^{-n})/(1-\tau)$ , ( $n = 0, 1, 2, \dots$ ). This function  $h_\tau(z)$  is holomorphic at  $\infty$  and we write

$$H_\tau = h_\tau(\infty) = \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1} \ell_\tau^n(0)}. \quad (8)$$

For  $\tau = 1$ , we have  $\ell^n(z) = z + n$  and

$$h_1(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z+n} \right\}.$$

This function is meromorphic on  $\mathbb{C}$  and has poles at  $0, -1, -2, \dots$ . We note that

$$h_1(z) = \frac{\Gamma'(z)}{\Gamma(z)} + \gamma$$

where  $\Gamma(z)$  denotes the gamma function and  $\gamma$  denotes the Euler constant

$$\gamma = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \log n \right).$$

Now we study the dependence of  $h_\tau(z)$  on the parameter  $\tau$ .

**Corollary 1** *The constant  $C_\tau$  is a continuous function of  $\tau \in T_\alpha$ .*

The function  $h_\tau(z)$  satisfies the equation ( ) with

$$C_\tau = (1 - \tau)H_\tau. \quad (9)$$

for  $|\tau| > 1$  and with  $C_1 = 0$  for  $\tau = 1$ . We have  $C_\tau \rightarrow C_1 = 0$  ( $\tau \rightarrow 1$ ), since  $h_\tau(z)$  is continuous.

**Proposition 5.2** *For any  $\varepsilon > 0$ , there is a constant  $M$  such that*

$$|h'_\tau(z)| \leq \frac{M}{N_\tau(z)} \quad \text{on } \mathcal{V}_\alpha(\varepsilon)$$

## 6 Behavior of $H_\tau$

Now we look at the behavior of the function  $H_\tau$  defined by ( ), when  $\tau \rightarrow 1$  within the sector  $T$ . It is clear from the expression ( ) that  $H_\tau$  is unbounded, while  $C_\tau = (1 - \tau)H_\tau$  tends to 0 by the corollary to Proposition 2.4. Here we give a more precise description of its behavior.

**Proposition 6.1** *We have*

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(1)$$

as  $\tau \rightarrow 1$  within the sector  $T$ . Here  $\gamma$  denotes the Euler constant.

To show this, we write  $\lambda = 1/\tau$ . We have

$$H_{1/\lambda} = (1 - \lambda)L(\lambda) - \lambda.$$

Here  $L(\lambda)$  denotes the Lambert series defined by

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1 - \lambda^n}.$$

This series  $L(\lambda)$  defines a holomorphic function on  $|\lambda| < 1$ , and is developed into the power series

$$L(\lambda) = \sum_{n=1}^{\infty} d(n)\lambda^n = \lambda + 2\lambda^2 + 2\lambda^3 + 3\lambda^4 + \dots,$$

where  $d(n)$  denotes the number of divisors of  $n$ . Let

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n$$

with

$$D(n) = d(1) + \dots + d(n).$$

The asymptotic behavior of  $D(n)$  is given by a theorem of Dirichlet (see Apostol [1], Chandrasekharan [2]) :

$$D(n) = n \log n + (2\gamma - 1)n + O(\sqrt{n}) \quad (n \rightarrow \infty).$$

Using this estimate, we have

$$\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n = -\frac{\lambda \log(1 - \lambda)}{(1 - \lambda)^2} + \frac{\gamma\lambda}{(1 - \lambda)^2} + P(\lambda)$$

where  $P(\lambda) = \sum_{n=1}^{\infty} p_n \lambda^n$ . From the estimate of  $p_n$  we have

$$P(\lambda) = o((1 - \lambda)^{-2}) \quad \text{as } \lambda \rightarrow 1 \text{ non-tangentially}$$

Hence it follows that

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(\tau - 1)$$

## 7 Schröder-Abel equation — general case

Now we treat the general case where  $a_1(\tau)$  does not necessarily vanish. Let

$$B_\tau = 1 - a_1(\tau)C_\tau$$

we have the following result corresponding to Theorem ?

**Theorem 7.1** *There exists a function  $\varphi_\tau(z)$  continuous on  $\mathcal{V}$  such that*

(i)  $\varphi_\tau(z)$  satisfies the functional equation

$$\varphi_\tau(f_\tau(z)) = \tau\varphi_\tau(z) + B_\tau; \quad (10)$$

(ii)  $\varphi_\tau(z)$  is injective in the variable  $z$  for each parameter  $\tau \in T_{\alpha,\tau}$ .

(iii)  $\lim_{z \rightarrow \infty} \varphi_\tau(z)/z = 1$  as  $z \rightarrow \infty$ , when  $|\tau| > 1$ .

To define  $\varphi_\tau(z)$ , we let

$$\Phi_\tau(z) = z - a_1(\tau)h_\tau(z).$$

Then

$$\Phi_\tau(f_\tau(z)) = \tau\Phi_\tau(z) + B_\tau + \bar{A}(z).$$

From this we can define

$$\varphi_\tau(z) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{\tau^n} \Phi_\tau(f_\tau^n(z)) - B_\tau \sum_{k=1}^n \frac{1}{\tau^k} \right\} \quad (11)$$

## 8 Relation with the Schröder equation

When  $|\tau| > 1$ , the Schröder equation

$$\chi_\tau(f_\tau(z)) = \tau\chi_\tau(z).$$

has a unique solution  $\chi_\tau(z)$  of the form

$$\chi_\tau(z) = z + c_0 + \frac{c_1}{z} + \dots$$

in a neighbourhood of  $\infty$ .

**Theorem 8.1** *For  $\tau \in T_{\alpha,\rho} - \{1\}$  we have*

$$\varphi_\tau(z) = \chi_\tau(z) - \frac{B_\tau}{\tau - 1}.$$

**Proof** We can easily verify that  $\varphi(z) + B_\tau/(\tau - 1)$  satisfies the Schröder equation. The assertion follows from the uniqueness of the solution.  $\square$

Now recall that

$$\begin{aligned} \frac{B_\tau}{\tau - 1} &= \frac{1 - a_1 C_\tau}{\tau - 1} \\ &= \frac{1}{\tau - 1} - a_1 H_\tau \\ &= \frac{1}{\tau - 1} + a_1 \log(\tau - 1) + a_1(1 - \gamma) + o(1) \end{aligned}$$

Using this fact the theorem is reformulated as follows:

**Theorem 8.2** *Let*

$$\varphi(z) = \chi(z) - \frac{1}{\tau - 1} - a_1 \log(\tau - 1)$$

*for  $\tau \in T - \{1\}$ . Then  $\varphi(z)$  converges to a solution to the Abel equation.*

## References

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