An example of $J^+$ for complex Hénon mappings which is locally connected nowhere

Teisuke Jin

Abstract

It is known that $J^+$ for complex Hénon mappings is connected. We give a sufficient condition so that $J^+$ is locally connected nowhere.

1 Introduction

In this paper we denote $z = (x, y) \in \mathbb{C}^2$. Let $p_j(y)$ be monic polynomials of deg $g_j = d_j > 1$ for $j = 1, \ldots, m$. We call $g_j(x, y) = (y, p_j(y) - \delta_j x)$ generalized Hénon mappings, where $\delta_j \neq 0$. Moreover we define

$$f = f_m \circ \cdots \circ f_1, \quad \delta = \delta_1 \cdots \delta_m, \quad d = d_1 \cdots d_m.$$  

Friedland and Milnor [5] classified polynomial automorphisms of $\mathbb{C}^2$ into three types: affine mapping, elementary mapping, composite of generalized Hénon mappings. The last one has complicated dynamical structures.

We define $K^\pm = \{ z \in \mathbb{C}^2 \mid \{ f^{\pm n}(z) \mid n \in \mathbb{N} \} \text{ is bounded} \}, \ J^\pm = \partial K^\pm, \ K = K^+ \cap K^- \text{ and } J = J^+ \cap J^-$. They are closed invariant sets.

Let $d( \ ) \in \mathbb{C}$ be the Euclidean distance in $\mathbb{C}^2$. For $X \subset \mathbb{C}^2$, define the stable set $W^s(X)$ and the unstable set $W^u(X)$ as follows: $W^s(X) = \{ z \in \mathbb{C}^2 \mid d(f^n(z), f^n(X)) \to 0 (n \to \infty) \}, \ W^u(X) = \{ z \in \mathbb{C}^2 \mid d(f^n(z), f^n(X)) \to 0 (n \to -\infty) \}$.

Let $a$ be a periodic point with the period $l$ such that the eigenvalues of $D(f^l)(a)$ are $\lambda_s$ and $\lambda_u (|\lambda_u| < 1 < |\lambda_s|)$. Such a periodic point is called a saddle point. Then we call $W^s(a)$ a stable manifold and $W^u(a)$ an unstable manifold since there are non-singular bijective entire mappings $H_s : \mathbb{C} \to W^s(a)$ and $H_u : \mathbb{C} \to W^u(a)$ with $f \circ H_s(t) = H_s(\lambda_s t)$ and $f \circ H_u(t) = H_u(\lambda_u t)$. See [9] for example. Bedford and Smillie [2] showed $W^s(a) = J^+$ and $W^u(a) = J^-$. We call $\tilde{K}^s = H_s^{-1}(K)$ a stable slice and $\tilde{K}^u = H_u^{-1}(K)$ an unstable slice. We say $\tilde{K}^s$ is stably connected if $\tilde{K}^s$ has no compact connected components [4]. We say $\tilde{K}^s$ is bridged if the connected component of $\tilde{K}^s$ containing the origin is not a point [7]. An unstable connectivity and a bridgedness for $\tilde{K}^u$ are defined similarly. Note that a stable (unstable) connectivity implies a bridgedness and that the following are equivalent [7]:

- $\tilde{K}^s$ is bridged,
- the connected component of $\tilde{K}^s$ containing the origin is unbounded,
- $\tilde{K}^s$ has an unbounded connected component.

In particular $\tilde{K}^s$ is not bridged if and only if each component of $\tilde{K}^s$ is compact.
2 Main theorems

Theorem 2.1. If $\widetilde{K}^{u}$ is not unstably connected and $\widetilde{K}^{s}$ is not bridged then $J^{+}$ is not locally connected anywhere.

Theorem 2.2. Assume $\widetilde{K}^{u}$ is not unstably connected. Then there are at most finitely many periodic points $p_{1}, \ldots, p_{n}$ such that $J^{+}$ is locally connected only at the points.

Note that $\overline{W^{s}}(a) = J^{+}$ and hence $J^{+}$ is connected. It implies that Theorem 2.1 gives an example of a connected set which is not locally connected anywhere.

It was shown [7] if $\tilde{K}^{u}$ is bridged then the Yoccoz inequality holds. Therefore if $\tilde{K}^{u}$ does not satisfy the inequality then it is not unstably connected, and if $\tilde{K}^{s}$ does not then not bridged. Note that it is easy to give examples such that either $\tilde{K}^{u}$ or $\tilde{K}^{s}$ do not satisfy the inequality. It implies many Hénon mappings satisfy the assumptions of Theorem 2.1.

3 Proofs of the main theorems

In this section we assume the unstable slice $\tilde{K}^{u}$ is not unstably connected. For $X \subset \mathbb{C}^{2}$ we define $B(X, r) = \{z \in \mathbb{C}^{2} \mid d(z, X) < r\}$. Recall that the Green functions $G^{\pm}$ are defined [1] as:

$$G^{\pm}(z) = \lim_{n \to \infty} \frac{1}{d^{n}} \log^{+} ||f^{\pm n}(z)||$$

and have the following properties:

- $G^{\pm}$ are nonnegative continuous plurisubharmonic functions,
- $G^{\pm}(z) = 0$ if and only if $z \in K^{\pm}$,
- $G^{\pm}|_{C^{2} \setminus K^{\pm}}$ are positive plurisubharmonic functions,
- $G^{\pm} \circ f = d^{\pm 1} \cdot G^{\pm}$.

It is well-known [9] that in a neighborhood of saddle point $a$, $f^{t}$ is conjugate to

$$\tilde{f}(s, t) = (\lambda_{s}s + st\alpha(s, t), \lambda_{t}t + st\beta(s, t)),$$

(3.1)

where $\alpha, \beta$ are holomorphic functions defined in a bidisk $\tilde{\Delta}$ centered at the origin. We denote by $\Phi$ the conjugation mapping whose domain is $\tilde{\Delta}$. Define $\Delta = \Phi(\tilde{\Delta})$.

Proposition 3.1. Assume $J^{+}$ is locally connected at $z_{0} \in J^{+}$. Then for any $r > 0$, $H_{s}^{-1}(B(z_{0}, r))$ has an unbounded connected component. Moreover we have $z_{0} \notin \overline{W^{s}}(a)$.

Proof. The local connectivity implies there is an open neighborhood $V$ of $z_{0}$ in $\mathbb{C}^{2}$ such that $V \cap J^{+}$ is connected and $V \subseteq B(z_{0}, r)$. Let $\tilde{V}^{s}$ be a component of $H_{s}^{-1}(V)$ and $\tilde{B}^{s}$ the component of $H_{s}^{-1}(B(z_{0}, r))$ containing $\tilde{V}^{s}$. We assume $\tilde{B}^{s}$ is bounded and derive a contradiction.

We define $B^{s} = H_{s}(\tilde{B}^{s})$. Choose $n \geq 0$ so that $f^{in}(B^{s}) \subseteq \Phi(\{(s, 0) \in \tilde{\Delta}\})$ and define $B^{s}_{1} = f^{in}(B^{s})$, $\overline{B}^{s}_{1} = \Phi^{-1}(B^{s}_{1})$. Let $C$ be a simple closed curve in
$\tilde{\Delta} \cap \{t = 0\}$ which surrounds $\tilde{B}^s_1$ and does not intersect with $\Phi^{-1}(f^{\text{in}}(B(z_0, r)))$.

Choose $\varepsilon > 0$ so small and decrease $r > 0$ slightly if necessary so that $\tilde{C} = \{(s, t) \in \tilde{\Delta} \mid (s, 0) \in C, |t| < \varepsilon\}$ and $\Phi^{-1}(f^{\text{in}}(B(z_0, r)))$ do not intersect.

On the other hand, take a compact component $K_u^s$ of $H_u(\tilde{K}^u)$ contained in $\Phi\{(0, t) \in \tilde{\Delta}\}$ and define $\tilde{K}_1^u = \Phi^{-1}(K_u^s)$. Let $\Gamma$ be a closed curve in $\tilde{\Delta} \cap \{s = 0\}$ which surrounds $\tilde{K}_1^u$ and does not intersect with $\Phi^{-1}(H_u(\tilde{K}^u))$ [7, section 6]. Choose $\delta > 0$ so that $\hat{\Gamma} = \{(s, t) \mid (0, t) \in \Gamma, |s| < \delta\}$ does not intersect with $\Phi^{-1}(\Delta \cap K^+)$. By properties of the Green function $G^+$, for any $s_1$ with $|s_1| < \delta$, $\Phi^{-1}(K^+) \cap \{s = s_1\}$ is not empty inside of $\hat{\Gamma}$ [4].

By (3.1), $f^k(\hat{C})$ approaches $\{t = 0\}$ uniformly and expand along $\{t = 0\}$ uniformly. Therefore if we take $k$ large, $\hat{\Gamma}$ goes through $f^k(\hat{C})$.

Let us return to the starting point. Then $f^{-l(n+k)}(\Phi(\hat{\Gamma}))$ goes through $B(z_0, r)$ and $V$ if we take $k$ large if necessary. Since $K^+$ runs through inside of $f^{-l(n+k)}(\Phi(\hat{\Gamma}))$, we conclude that $V \cap J^+$ is not connected, which is a contradiction.

Let show the last statement of the theorem. Take $z_0 \in W^s(a)$. Since $W^s(a)$ is a 1-dimensional manifold, if we take $r > 0$ small, the connected component of $H_{s}^{-1}(B(z_0, r))$ containing $H_{s}^{-1}(z_0)$ is bounded. But an arbitrary open neighborhood $V$ of $z_0$ intersects with the component, which is a contradiction.

**Proof of Theorem 2.1.** By the assumption there is a closed curve $\gamma$ surrounding the origin and not intersecting with $K^+$ [7, section 6]. Since $f^{-n}$ diverges in $\mathbb{C}^2 \setminus K^+$ locally uniformly as $n \to +\infty$, $f^{-n}(H_{s}(\gamma)) = H_{s}(\lambda_{s}^{-n}\gamma)$ diverges uniformly.

Assume $J^+$ is locally connected at $z_0 \in J^+$. Then some component of $H_{s}^{-1}(B(z_0, r))$ is unbounded. But if we choose $n$ large, $f^{-n}(H_{s}(\gamma))$ is far from $B(z_0, r)$ and $\lambda_{s}^{-n}\gamma$ intersects $H_{s}^{-1}(B(z_0, r))$, which is a contradiction.

Let us proceed to prove Theorem 2.2. For $z_0 \in J^+ \setminus W^s(a)$ and $n \in \mathbb{Z}$, we define

$$u(t) = \log d(H_{s}(t), z_0), \quad u_{n}(t) = \max\{0, u(t) + n\}.$$ 

For a nonnegative subharmonic function $v$ on $\mathbb{C}$ we define the order of $v$ as follows:

$$\text{ord} v = \limsup_{r \to \infty} \frac{\log \max_{|t|=r} v(t)}{\log r}.$$ 

**Lemma 3.2.** The functions $u$ and $u_{n}$ are continuous subharmonic functions and we have

$$\rho = \text{ord} u_{n} = \frac{l \log d}{-\log |\lambda_{s}|}.$$ 

**Proof.** Since $\log ||z||$ is plurisubharmonic, $u, u_{n}$ are subharmonic functions.

If we set $(h_1, h_2) = H_{s}$, the orders of $h_1, h_2$ are [7]:

$$\text{ord} h_1 = \limsup_{r \to \infty} \frac{\log \log \max_{|t|=r} |h_1(t)|}{\log r} = \frac{l \log d}{-\log |\lambda_{s}|},$$ 

$$\text{ord} h_2 = \limsup_{r \to \infty} \frac{\log \log \max_{|t|=r} |h_2(t)|}{\log r} = \frac{l \log d}{-\log |\lambda_{s}|},$$

since the period of $a$ is $l$ and the degree of $f^{l}$ is $d^{l}$. It is easy to compute the order of $u_{n}$ using the above equations. 

\[\square\]
Lemma 3.3. Let $v$ be a nonnegative bounded subharmonic function in an unbounded open set $\Omega(\subset \mathbb{C})$ with an unbounded boundary. Let $c$ be a bounded subset of $\partial \Omega$. If $v \equiv 0$ on $\partial \Omega \setminus c$, then $v(t)$ converges $0$ uniformly as $|t| \to \infty$ with $t \in \Omega$.

Proof. We define

$$w(\tau) = \begin{cases} u(1/\tau) & \text{if } 1/\tau \in \Omega, \\ 0 & \text{if } 1/\tau \notin \Omega \cup \overline{\mathbb{C}}. \end{cases}$$

Then $w$ is a nonnegative bounded subharmonic function for $1/\tau \notin \overline{\mathbb{C}}$. Moreover since $w$ is bounded in a neighborhood of $\tau = 0$, Removable Singularity Theorem [8, p. 53] implies $w$ is subharmonic around the origin.

We may assume $w$ is non-constant for any neighborhood of the origin. Therefore we can apply Tsuji inequality [6, p. 548] to $w$. In fact, for $e^{-1} < \kappa < 1$ and $0 < r \leq \kappa^2 R$, we have

$$B(r) \leq C_2(\kappa)B(R) \exp\left\{-\int_{r/\kappa}^{\kappa R} \frac{\alpha(\rho)d\rho}{\rho}\right\},$$

where $B(r) = \max\{w(t) \mid |t| = r\}$, $C_2(\kappa) = 6(1 - \kappa)^{-3/2}$. In our case we can set $\alpha(\rho) = 1/2$ by the structure of $\Omega$. We have

$$B(r) \leq C_2(\kappa)B(R) \exp\left\{-\int_{r/\kappa}^{\kappa R} \frac{\rho}{2\kappa^2 R}\right\} \leq C_2(\kappa)B(R)\sqrt{\frac{r}{\kappa^{2}R}}.$$ 

Therefore $B(r) \to 0$ as $r \to 0$, i.e., $u(t) \to 0$ as $|t| \to \infty$.

Proof of Theorem 2.2. Assume $J^+$ is locally connected at $z_0 \in J^+$. The above proposition implies $z_0 \notin W^a(a)$. In the following we will show that $z_0$ is an asymptotic point of $H_2$. Once we obtain the fact, since each holomorphic function of finite order [7] has at most finitely many asymptotic values, the proof is completed.

In general, let $v$ be a nonnegative subharmonic function of complex one variable. Each connected component of $\{s \mid v(s) > 0\}$ is called tract. Then the number of tracts of $v$ is at most $\max\{1, 2\ord v\}$ [6, Chapter 8].

Therefore the number of tracts of $u_n$ is at most $\max\{1, 2\rho\}$. Take an appropriate $n_0 \in \mathbb{Z}$ such that the number of tracts of $u_{n_0}$ attains its maximum $q$. For each tract of $u_{n_0}$ choose an asymptotic path $\gamma_j : [0, \infty) \to \mathbb{C}$ ($0 \leq j \leq q$) with $u_{n_0}(\gamma(\xi)) > 0$ and $u_{n_0}(\gamma(\xi)) \to \infty$ as $\xi \to \infty$. Take sufficiently large $R > 0$ and we may assume all paths $\gamma_j$ intersect with $\{|t| = R\}$ only at their starting points. Then $\mathbb{C} \setminus (D_R \cup \gamma_1 \cup \cdots \cup \gamma_q)$ consists of $q$-unbounded connected components, where $D_R = \{|t| < R\}$.

Choose $U$ which is one of the components such that the infimum of $u$ is $-\infty$ in the domain. Moreover choose large $N$ so that

$$\min\{u_N(s) \mid t \in \overline{D_R \cup \gamma_1 \cup \cdots \cup \gamma_q}\} > 0.$$ 

For each $j = 1, 2, \ldots$, the above proposition implies we can take a point $s_j \in U$ such that the component of $\{s \in U \mid u(s) < -N - j\}$ containing $s_j$ is unbounded.
Let us show that we can draw a path joining $s_1$ and $s_2$ such that $u < -N$ on the path. By the construction, $s_1$ and $s_2$ are contained in the unbounded components $U_1$ and $U_2$ of $\{s \in U \mid u(s) < -N - 1\}$, resp. Draw a smooth curve $c_0$ in $U$ joining $s_1$ and $s_2$. We may assume $\overline{U_1} \cap \overline{U_2} \neq \emptyset$. Let us regard $U \setminus (\overline{U_2} \cup \overline{U_2} \cup c_0)$. Clearly the set is divided into two sides with respect to $c_j$: one can access $\partial U$, another cannot. We choose the open set which cannot and name it $\Omega$. Then $\partial \Omega$ consists of a part of $\partial U_1$ and $\partial U_2$ and $c_0$. Note that $u_{N+1} \equiv 0$ on $\partial U_1$ and $\partial U_2$, and that $\Omega$ is unbounded and that $u_{N+1}$ is bounded in $\Omega$. At this point, we can apply the above lemma, and obtain that $u_{N+1}$ decrease to 0 uniformly as $|s| \to \infty$ in $\Omega$. Therefore we can draw a path $\Gamma_1 : [0, 1] \to U$ joining $s_1$ and $s_2$ such that $u < -N$ on $\Gamma_1$.

Similarly we can draw paths $\Gamma_j : [0, 1] \to U$ joining $s_j$ and $s_{j+1}$ such that $u < -N - j + 1$ on $\Gamma_j$ for $j = 2, 3, \ldots$. If we define

$$\Gamma(\xi) = \Gamma_j(\xi - j + 1) \quad \text{for } j - 1 \leq \xi < j,$$

$\Gamma$ is an asymptotic path such that $u(\Gamma(\xi)) \to -\infty$ as $\xi \to \infty$, i.e., $H_s(\Gamma(\xi)) \to z_0$ as $\xi \to \infty$, which implies $z_0$ is an asymptotic point of $H_s$. □

References


